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THE INERTIA SET OF NONNEGATIVE SYMMETRIC SIGN PATTERN WITH ZERO DIAGONAL

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Abstract. The inertia set of a symmetric sign pattern A is the set $i(A) = \{i(B) \mid B = B^T \in Q(A)\}$, where $i(B)$ denotes the inertia of real symmetric matrix B, and $Q(A)$ denotes the sign pattern class of A. In this paper, a complete characterization on the inertia set of the nonnegative symmetric sign pattern A in which each diagonal entry is zero and all off-diagonal entries are positive is obtained. Further, we also consider the bound for the numbers of nonzero entries in the nonnegative symmetric sign patterns A with zero diagonal that require unique inertia.

Keywords: sign pattern, inertia, inertia set, unique inertia

MSC 2000: 15A18

Introduction

A matrix whose entries are from set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern). We denote the set of all $n \times n$ sign patterns by Q_n. For a real matrix B, by $\operatorname{sgn} B$ we mean the sign pattern in which each positive (respectively, negative, zero) entry of B is replaced by $+$ (respectively, $-$, 0). If $A \in Q_n$, then the *sign pattern class* of A is defined by

$$Q(A) = \{B \mid \operatorname{sgn} B = A\}.$$

The *inertia* of a real symmetric matrix B, written as $i(B)$, is the triple of integers $i(B) = (i_+(B), i_-(B), i_0(B))$, where $i_+(B)$ (respectively, $i_-(B), i_0(B)$) denotes the number of positive (respectively, negative, zero) eigenvalues of the matrix B counted

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with their algebraic multiplicities. Notice that the rank of a real symmetric matrix B is equal to $i_+(B) + i_-(B)$. For a symmetric sign pattern A, we define the *inertia set* of A to be $i(A) = \{i(B) \mid B = B^T \in Q(A)\}$. As a special case, if $i(B_1) = i(B_2)$ for all real symmetric matrices $B_1, B_2 \in Q(A)$, we say that the sign pattern A requires unique inertia. There is an extensive literature on inertias of matrices, see for instance the recent survey paper [1]. However, little was known about the inertia of a matrix solely on the bases of knowledge of the signs of the entries of the matrix. In [2], Drew et al. discussed the inertia set of a special tridiagonal sign pattern, and proved that it is inertially arbitrary for dimensions less than 8.

In this paper, we mostly restrict to nonnegative symmetric sign patterns with zero diagonal. In Section 2, we characterize the inertia set of such sign pattern in which all off-diagonal entries are positive. In Section 3, we consider the bound for the numbers of nonzero entries in such sign patterns A that require unique inertia.

In order to simplify our notation, we need the following concepts. A sign pattern $A \in Q_n$ is said to be *sign nonsingular* if every matrix $B \in Q(A)$ is nonsingular. It is well known that A is sign nonsingular if and only if $\det A = +$ or $\det A = -$, that is, in the standard expansion of $\det A$ into $n!$ terms, there is at least one nonzero term, and all nonzero terms have the same sign.

If A is a symmetric sign pattern, we defined $\mathrm{smr}(A)$, the *symmetric minimal rank* of A, by

$$\mathrm{smr}(A) = \min\{\mathrm{rank}(B) \mid B = B^T \in Q(A)\}.$$

Similarly, the *symmetric maximal rank* of A, $\mathrm{SMR}(A)$, is

$$\mathrm{SMR}(A) = \max\{\mathrm{rank}(B) \mid B = B^T \in Q(A)\}.$$

Clearly, if $A \in Q_n$ is symmetric sign nonsingular, then $\mathrm{smr}(A) = \mathrm{SMR}(A) = n$.

For any matrix A of order n and nonempty subsets α and β of $\{1, 2, \ldots, n\}$, we use $A[\alpha, \beta]$ to denote the submatrix of A determined by the rows whose index is in α and the columns whose index is in β. If $\alpha = \beta$, then we write $A[\alpha]$ instead of $A[\alpha, \beta]$. We also use I_n to denote the identity matrix of order n.

2. Inertia set

In this section, we consider the inertia set of the nonnegative symmetric sign pattern A of order n having the following form

$$(1)\qquad A=\begin{bmatrix} 0 & + & + & \dots & + & + \\ + & 0 & + & \dots & + & + \\ + & + & 0 & \dots & + & + \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ + & + & + & \dots & 0 & + \\ + & + & + & \dots & + & 0 \end{bmatrix}.$$

We obtain a complete characterization on the inertia set of such sign pattern A. In order to simplify our notation, in the remainder of this paper, we use NNS to denote the set of all sign patterns having the form (1).

Clearly, if $n=2$, then $i(A)=\{(1,1,0)\}$. Now we may assume that $n\geqslant 3$.

Lemma 2.1 ([3]). *Let B be an $n\times n$ real symmetric matrix. Then for any $n\times n$ nonsingular matrix P, $i(B)=i(PBP^T)$.*

Lemma 2.2. *Let $A\in\mathrm{NNS}$ be an $n\times n$ sign pattern. Then there exists a nonsingular symmetric matrix $B\in Q(A)$.*

Proof. Let

$$B=\begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix}.$$

It is clear that $B=B^T\in Q(A)$ and $\det B=(-1)^{n-1}(n-1)\neq 0$. Thus the lemma follows. $\square$

Lemma 2.3. *Let $A\in\mathrm{NNS}$ be an $n\times n$ sign pattern. Then*

(i) *for any integer $3\leqslant r<n$ and $r\times r$ nonsingular symmetric matrix D with $\operatorname{sgn} D\in\mathrm{NNS}$, there exists a symmetric matrix $B\in Q(A)$ having the following block form*

$$B=\begin{bmatrix} D & M \\ M^T & X \end{bmatrix}$$

such that $\operatorname{rank}(B)=r$.

(ii) $\operatorname{srm}(A)=3$ *and* $\operatorname{SRM}(A)=n$.

Proof. (i) Consider the following two cases.

Case 1. $r = 3$. Assume that

$$D = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix}$$

where $a_{12} > 0$, $a_{13} > 0$, $a_{23} > 0$. Then

$$D^{-1} = \begin{bmatrix} \frac{-a_{23}}{2a_{12}a_{13}} & \frac{1}{2a_{12}} & \frac{1}{2a_{13}} \\ \frac{1}{2a_{12}} & \frac{-a_{13}}{2a_{12}a_{23}} & \frac{1}{2a_{23}} \\ \frac{1}{2a_{13}} & \frac{1}{2a_{23}} & \frac{-a_{12}}{2a_{13}a_{23}} \end{bmatrix}.$$

Firstly, we prove that there exists a $3 \times (n-3)$ entrywise positive matrix M such that for each column vector x of M,

$$x^T D^{-1} x = 0. \tag{2}$$

Take $x = \left(a_{23}, k^2 a_{13}, (ka_{13} - a_{23})^2/a_{12}\right)^T$ with k is a positive number. Note that if $k > a_{23}/a_{13}$, then x is an entrywise positive vector and satisfies the condition (2). We now take $k > a_{23}/a_{13}$ and

$$M = (\beta_1, \beta_2, \ldots, \beta_{n-3})$$

where $\beta_i = \left(a_{23}, (k+i-1)^2 a_{13}, [(k+i-1)a_{13} - a_{23}]^2/a_{12}\right)^T$ for $i = 1, 2, \ldots, n-3$. Clearly, the matrix M satisfies the above conditions.

Next, we let

$$B = \begin{bmatrix} D & M \\ M^T & M^T D^{-1} M \end{bmatrix}.$$

Since D is a nonsingular symmetric matrix and

$$\begin{bmatrix} I_3 & 0 \\ -M^T D^{-1} & I_{n-3} \end{bmatrix} \begin{bmatrix} D & M \\ M^T & M^T D^{-1} M \end{bmatrix} = \begin{bmatrix} D & M \\ 0 & 0 \end{bmatrix},$$

we have $\operatorname{rank}(B)=\operatorname{rank}(D)=3$. For any $1\leqslant i,j\leqslant n-3$, using $(M^TD^{-1}M)_{ij}$ to denote the (i,j) entry of $M^TD^{-1}M$, it is easy to verify that

$$\begin{aligned}(M^TD^{-1}M)_{ij}&=\beta_i^TD^{-1}\beta_j\\&=\left(a_{23},(k+i-1)^2a_{13},\frac{[(k+i-1)a_{13}-a_{23}]^2}{a_{12}}\right)\\&\quad\times\begin{pmatrix}\dfrac{(k+j-1)[(k+j-1)a_{13}-a_{23}]}{a_{12}}\\ \dfrac{a_{23}-(k+j-1)a_{13}}{a_{12}}\\ k+j-1\end{pmatrix}\\&=\frac{(j-i)^2a_{13}a_{23}}{a_{12}}.\end{aligned}$$

Thus $B=B^T\in Q(A)$.

Case 2. $3<r<n$. Let $D_1=D[\{r-2,r-1,r\}]$ and $D_2=D[\{1,\ldots,r-3\},\{r-2,r-1,r\}]$. Thus D_1 is a nonsingular symmetric matrix of order 3. By Case 1, there exists an $(n-r+3)\times(n-r+3)$ symmetric matrix

$$B_1=\begin{bmatrix}D_1&M_1\\M_1^T&M_1^TD_1^{-1}M_1\end{bmatrix}$$

such that $\operatorname{sgn}B_1\in\mathrm{NNS}$ and $\operatorname{rank}(B_1)=3$. Letting

$$M=\begin{bmatrix}D_2D_1^{-1}M_1\\M_1\end{bmatrix},$$

since D_2 is an entrywise positive matrix, we can choose $k>0$ sufficiently large, where k is a variable in M_1, such that M is also an entrywise positive matrix. In this case, it is easily seen that the matrix

$$B=\begin{bmatrix}D&M\\M^T&M_1^TD_1^{-1}M_1\end{bmatrix}=B^T\in Q(A)$$

and $\operatorname{rank}(B)=\operatorname{rank}(D)=r$. So result (i) holds.

Result (ii) is clear from result (i). The lemma now follows. □

Lemma 2.4. *Let $A\in\mathrm{NNS}$ be an $n\times n$ sign pattern and $\operatorname{smr}(A)\geqslant 3$. Then, for any symmetric matrix $B\in Q(A)$, $i_-(B)\geqslant 2$.*

Proof. Noticing that B is a nonnegative matrix, by the nonnegative matrix theory, we have that the spectral radius $\varrho(B)$ of B is an eigenvalue of B. Since $\operatorname{tr}(B)=0$ and $\operatorname{rank}(B)\geqslant 3$, the lemma follows. □

Lemma 2.5. *Let B be an $r \times r$ ($r \geqslant 3$) nonsingular symmetric matrix with* $\operatorname{sgn} B \in \mathrm{NNS}$. *Then there exist nonsingular symmetric matrices B_1 and B_2 of order $r+1$ such that* $\operatorname{sgn} B_1 = \operatorname{sgn} B_2 \in \mathrm{NNS}$, *$i_-(B_1) = i_-(B)+1$ and $i_+(B_2) = i_+(B)+1$.*

Proof. Consider two cases.

Case 1. $r = 3$. Let

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{12} & 0 & b_{23} \\ b_{13} & b_{23} & 0 \end{bmatrix}$$

where $b_{12} > 0, b_{13} > 0, b_{23} > 0$. It is easily seen that

$$B^{-1} = \begin{bmatrix} \dfrac{-b_{23}}{2b_{12}b_{13}} & \dfrac{1}{2b_{12}} & \dfrac{1}{2b_{13}} \\ \dfrac{1}{2b_{12}} & \dfrac{-b_{13}}{2b_{12}b_{23}} & \dfrac{1}{2b_{23}} \\ \dfrac{1}{2b_{13}} & \dfrac{1}{2b_{23}} & \dfrac{-b_{12}}{2b_{13}b_{23}} \end{bmatrix}.$$

Taking

$$B_1 = \begin{bmatrix} B & x_1 \\ x_1^T & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} B & x_2 \\ x_2^T & 0 \end{bmatrix},$$

where $x_1 = \bigl(b_{23}, k^2 b_{13}, \frac{(kb_{13}-b_{23})^2+1}{b_{12}}\bigr)^T$ and $x_2 = \bigl(b_{23}, k^2 b_{13}, \frac{(kb_{13}-b_{23})^2-1}{b_{12}}\bigr)^T$, we may choose $k > 0$ sufficiently large so that $\operatorname{sgn} B_1 = \operatorname{sgn} B_2 \in \mathrm{NNS}$ and

$$x_1^T B^{-1} x_1 = \frac{4kb_{13}b_{23} - 1}{2b_{12}b_{13}b_{23}} > 0,$$

$$x_2^T B^{-1} x_2 = \frac{-4kb_{13}b_{23} - 1}{2b_{12}b_{13}b_{23}} < 0.$$

Let

$$P_i = \begin{bmatrix} I_3 & 0 \\ -x_i^T B^{-1} & 1 \end{bmatrix}, \quad i = 1, 2.$$

Then P_i is nonsingular and

$$P_i B_i P_i^T = \begin{bmatrix} B & 0 \\ 0 & -x_i^T B^{-1} x_i \end{bmatrix}.$$

By Lemma 2.1, we have $i_-(B_1) = i_-(B) + 1$ and $i_+(B_2) = i_+(B) + 1$.

Case 2. $r > 3$. Let $D = B[\{r-2, r-1, r\}]$. Since D is nonsingular and $\operatorname{sgn} D \in \mathrm{NNS}$, by Case 1, there exist two entrywise positive column vectors x_1 and x_2 of order 3 such that $x_1^T D^{-1} x_1 > 0$ and $x_2^T D^{-1} x_2 < 0$. Denote

$$B = \begin{bmatrix} Z & Y \\ Y^T & D \end{bmatrix} \text{ and } B_i = \begin{bmatrix} Z & Y & YD^{-1}x_i \\ Y^T & D & x_i \\ x_i^T D^{-1} Y^T & x_i^T & 0 \end{bmatrix}, \quad i = 1, 2.$$

It is not difficult to verify that we may choose $k > 0$ sufficiently large, where k is a variable in x_1 and x_2, such that the column vectors $YD^{-1}x_1$ and $YD^{-1}x_2$ are entrywise positive, that is, $\operatorname{sgn} B_1 = \operatorname{sgn} B_2 \in \mathrm{NNS}$. For $i = 1, 2$, taking

$$P_i = \begin{bmatrix} I_{r-3} & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & -x_i^T D^{-1} & 1 \end{bmatrix},$$

it is easy to verify that P_i is nonsingular and

$$P_i B_i P_i^T = \begin{bmatrix} Z & Y & 0 \\ Y^T & D & 0 \\ 0 & 0 & -x_i^T D^{-1} x_i \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & -x_i^T D^{-1} x_i \end{bmatrix}.$$

Thus B_i is nonsingular, $i_-(B_1) = i_-(B) + 1$ and $i_+(B_2) = i_+(B) + 1$. Now the lemma follows. $\square$

Combining all the lemmas above, we now obtain a complete characterization of the inertia set of a sign pattern $A \in \mathrm{NNS}$.

Theorem 2.6. *Let $A \in \mathrm{NNS}$ be an $n \times n$ $(n \geqslant 3)$ sign pattern. Then*

$$i(A) = \{(s, t, n - s - t) \mid s \geqslant 1,\ \ t \geqslant 2\}.$$

From above the theorem, we can obtain easily the following corollary which is a general result on the inertia set of a nonnegative symmetric sign pattern with zero diagonal. We omit the proof of it.

Corollary 2.7. *Let A be any $n \times n$ nonnegative symmetric sign pattern with zero diagonal. If $\operatorname{smr}(A) \geqslant 3$, then $i(A) \subseteq \{(s, t, n - s - t) \mid s \geqslant 1,\ \ t \geqslant 2\}$.*

3. Sign patterns that require unique inertia

Let $A \in \mathrm{NNS}$ be an $n \times n$ sign pattern. Then the number of nonzero entries of A is maximal in all $n \times n$ nonnegative symmetric sign patterns with zero diagonal. But, from Theorem 2.6, it is easy to see that A does not require unique inertia when $n \geqslant 4$. A natural question is: how many nonzero entries must there be in an $n \times n$ nonnegative symmetric sign pattern with zero diagonal in order that it require unique inertia? In this section, we consider the upper bound and the lower bound for it.

In order to simplify our notation, in this section, we use S_n to denote the set of all $n \times n$ nonnegative symmetric sign patterns with zero diagonal. For $A \in S_n$, $e(A)$ denotes the number of nonzero entries in A.

We first give five examples.

Example 3.1. Let $A \in S_n$ have the following form

$$A = \begin{bmatrix} 0 & + & & & & \\ + & 0 & + & & & \\ & + & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & + \\ & & & & + & 0 \end{bmatrix}.$$

Then $e(A) = 2(n-1)$ and A requires unique inertia.

Proof. It is easy to see that if n is odd, then $\det A = 0$ and $\mathrm{srm}(A) = \mathrm{SRM}(A) = n-1$, and that if n is even, then $\det A = (-)^{n/2}$, $\mathrm{srm}(A) = \mathrm{SRM}(A) = n$ and A is sign nonsingular. On the other hand, for any matrix $B \in Q(A)$, the characteristic polynomial of B is given by

$$f_B(\lambda) = \lambda^n - E_1(B)\lambda^{n-1} + E_2(B)\lambda^{n-2} - \ldots + (-1)^n E_n(B),$$

where $E_k(B)$ is the sum of all $k \times k$ principal minors of B. Noticing that $E_{2k-1}(B) = 0$, the characteristic polynomial of B becomes

$$f_B(\lambda) = \lambda^n + E_2(B)\lambda^{n-2} + E_4(B)\lambda^{n-4} + \ldots + E_{2m}(B)\lambda^{n-2m} = \lambda^{n-2m} f(\lambda^2),$$

for some m and some polynomial $f(\lambda)$. It follows that if $\lambda \neq 0$ is an eigenvalue of B, then so is $-\lambda$, and the algebraic multiplicities of λ and $-\lambda$ are the same, that is, $i_+(B) = i_-(B)$. From the fact that $\mathrm{rank}(B) = i_+(B) + i_-(B)$, we have $i(A) = \{(\frac{1}{2}n, \frac{1}{2}n, 0)\}$ if n is even, and $i(A) = \{(\frac{1}{2}(n-1), \frac{1}{2}(n-1), 1)\}$ if n is odd. Thus A requires unique inertia. $\square$

Example 3.2. Let $n = 2k$ and $A = A_1 \oplus A_2 \oplus \ldots \oplus A_k$, where $A_i = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ for $i = 1, 2, \ldots, k$. Then $e(A) = n$ and A requires unique inertia.

Proof. Clearly, $e(A) = n$. For any $B = B^T \in Q(A)$, B has the form $B = B_1 \oplus B_2 \oplus \ldots \oplus B_k$, where $B_i = B_i^T \in Q(A_i)$ for $i = 1, 2, \ldots, k$. It is clear that $i(B_i) = (1, 1, 0)$ for $i = 1, 2, \ldots, k$. Thus $i(B) = (k, k, 0)$ and $i(A) = \{(k, k, 0)\}$, that is, A requires unique inertia. $\square$

Example 3.3. Let $n = 2k+1$ and $A = A_1 \oplus A_1 \oplus \ldots \oplus A_k$, where $A_1 = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}$ and $A_i = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ for $i = 2, \ldots, k$. Then $e(A) = n+1$ and A requires unique inertia.

P r o o f. Clearly, $e(A) = n + 1$. For any $B = B^T \in Q(A)$, B has the form $B = B_1 \oplus B_2 \oplus \ldots \oplus B_k$, where $B_i = B_i^T \in Q(A_i)$ for $i = 1, 2, \ldots, k$. It is clear that $i(B_1) = (1, 1, 1)$ and $i(B_i) = (1, 1, 0)$ for $i = 2, \ldots, k$. Thus $i(B) = (k, k, 1)$ and $i(A) = \{(k, k, 1)\}$, that is, A requires unique inertia. $\square$

Example 3.4. Let $n = 2k \geqslant 4$ and $A = (a_{ij}) \in S_n$, where

$$a_{ij} = \begin{cases} 0, & i = j, \text{ or } i > 2 \text{ is even and } j < i - 1, \text{ or } j > 2 \text{ is even and } i < j - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $e(A) = \frac{1}{2}n^2$ and A requires unique inertia.

P r o o f. Clearly, $e(A) = \frac{1}{2}n^2$. For any $B = B^T \in Q(A)$, it is not difficult to verify that there is an $n \times n$ nonsingular matrix P such that

$$P^T BP = B_1 \oplus B_2 \oplus \ldots \oplus B_k$$

where $\operatorname{sgn} B_i = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ for $i = 1, 2, \ldots, k$. Thus $i(B) = (k, k, 0)$ and A requires unique inertia. $\square$

Example 3.5. Let $n = 2k + 1 \geqslant 5$ and $A = (a_{ij}) \in S_n$, where

$$a_{ij} = \begin{cases} 0 & i = j, \text{ or } i > 3 \text{ is odd and } j < i - 1, \text{ or } j > 3 \text{ is odd and } i < j - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $e(A) = \frac{1}{2}(n^2 + 3)$ and A requires unique inertia.

P r o o f. Clearly, $e(A) = \frac{1}{2}(n^2 + 3)$. For any $B = B^T \in Q(A)$, it is not difficult to verify that there is an $n \times n$ nonsingular matrix P such that

$$P^T BP = B_1 \oplus B_2 \oplus \ldots \oplus B_k$$

where $\operatorname{sgn} B_1 = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}$ and $\operatorname{sgn} B_i = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ for $i = 2, \ldots, k$. Thus $i(B) = (k, k + 1, 0)$ and A requires unique inertia. $\square$

From the above examples, we have the following two theorems, and we may omit the proofs.

Theorem 3.6. *Let $A \in S_n$ have no zero row and zero column and require unique inertia. Then $e(A) \geqslant n$ and equality may hold. In particular, if A is irreducible, then $e(A) \geqslant 2(n - 1)$ and equality may hold.*

Theorem 3.7. *For $n \geqslant 2$,*

$$\max\{e(A) \mid A \in S_n \textit{ requires unique inertia}\} \geqslant \begin{cases} \dfrac{n^2}{2}, & n \textit{ is even}, \\ \dfrac{n^2+3}{2}, & n \textit{ is odd}, \end{cases}$$

and there exists some $A \in S_n$ such that A requires unique inertia and

$$e(A) = \begin{cases} \dfrac{n^2}{2}, & n \textit{ is even}, \\ \dfrac{n^2+3}{2}, & n \textit{ is odd}. \end{cases}$$

It is not difficult to verify that when $n \leqslant 5$, Theorem 3.7 becomes Theorem 3.8, thus the upper bound is obtained for $n \leqslant 5$. For $n \geqslant 6$, we now do not know the value of the upper bound for the numbers of nonzero entries, but we can prove that each pair of off-diagonal zero entries $a_{ij} = a_{ji} = 0$ of A in Examples 3.4 and 3.5 is essential (i.e. upon replacing these two zero entries by positive entries we obtain a sign pattern A' which doesn't require unique inertia). We omit all proofs.

Theorem 3.8. *For $n = 2, 3, 4, 5$,*

$$\max\{e(A) \mid A \in S_n \textit{ requires unique inertia}\} = \begin{cases} \dfrac{n^2}{2}, & n \textit{ is even}, \\ \dfrac{n^2+3}{2}, & n \textit{ is odd}. \end{cases}$$

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