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MODULES COMMUTING (VIA Hom) WITH SOME COLIMITS

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Abstract. For every module M we have a natural monomorphism

$$\Psi : \coprod_{i\in I} \mathrm{Hom}_R(M, A_i) \to \mathrm{Hom}_R\Big(M, \coprod_{i\in I} A_i\Big)$$

and we focus our attention on the case when Ψ is also an epimorphism. Some other colimits are also considered.

Keywords: module, colimit, finitely presented module

MSC 2000: 16E30, 16B99, 18A35

0. Introduction

Let Δ be a diagram (i.e., a small category) of modules. Given a module M, we have a natural isomorphism

$$\lim(\mathrm{Hom}_R(\Delta, M)) \cong \mathrm{Hom}_R(\mathrm{colim}(\Delta), M)$$

and a natural (connecting) homomorphism

$$\Psi\colon\ \mathrm{colim}(\mathrm{Hom}_R(M, \Delta)) \to \mathrm{Hom}_R(M, \mathrm{colim}(\Delta)).$$

It may happen that Ψ is an isomorphism whenever Δ is a diagram of a certain type and, in such a case, we shall say that M commutes (via Hom) with colimits of the diagrams considered.

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The present short note is concerned with the most important colimits: direct sums, push-outs and colimits of upwards-directed spectra. The corresponding (commuting) modules are fully characterized in each of these cases. The more difficult (and less fashionable) limit case is not treated here (the reader is referred to [6]).

1. Preliminaries

Throughout the paper, R stands for a non-zero associative ring with unit and modules are unitary left R-modules. The category of modules and homomorphisms will be denoted by R-MOD (for various basic properties of this category, we refer e.g. to [7] or [15]). The category of abelian groups will be denoted by ABG.

Let S be a (non-empty) ordered set. By an *S-spectrum* (in a given category) we shall mean any diagram of the type $f_{r,s}\colon A_r \to A_s$, $r, s \in S$, $r \leqslant s$. An S-spectrum will be called *upwards-directed* if so is the ordered set S. As a special case, given a cardinal number $\mathfrak{a}$, we get *$\mathfrak{a}$-spectra* ($\mathfrak{a}$ with the usual order).

2. Pseudo-finitely related modules

A module M will be called *finitely related* if M has a projective presentation $0 \to K \to P \to M \to 0$ such that K is a finitely generated module. The following well known and easy facts will be useful in the sequel:

2.1. Proposition.

(i) *The class of finitely related modules is closed under extensions and finite direct sums.*

(ii) *The class of finitely related modules is closed under submodules if and only if every subprojective module is finitely related (e. g., R left hereditary).*

(iii) *The class of finitely related modules is closed under factormodules if and only if R is completely reducible.*

(iv) *Every projective module is finitely related and every finitely related flat module is projective.*

Let N be a submodule of a module M. We will say that N *satisfies the condition* (PFG) *in M* if the following is true:

If N_i, $i \in I$, is an upwards-directed family of submodules of N such that $\bigcup\limits_{i\in I} N_i = N$, then there exist $j \in I$ and a submodule K of M such that $N \cap K = N_j$ and $N + K = M$.

The following two lemmas are obvious:

2.2. Lemma. *N satisfies* (PFG) *in M in each of the following three cases:*
(1) *N is finitely generated;*
(2) *$N = M$;*
(3) *M is completely reducible.*

2.3. Lemma. *Let N satisfy* (PFG) *in M.*
(i) *If N is superfluous in M, then N is finitely generated.*
(ii) *If L is a finitely generated submodule of K, then $N \oplus L$ satisfies* (PFG) *in $M \oplus K$.*
(iii) *If M is finitely generated, then N is finitely generated.*

2.4. Example. Let N be a proper finitely generated submodule of a module M such that M is not completely reducible and let L be a submodule of a completely reducible module K such that L is not finitely generated. Then $N \oplus L$ satisfies (PFG) in $M \oplus K$, $N \oplus L \neq M \oplus K$, $N \oplus L$ is not finitely generated and $M \oplus K$ is not completely reducible (cf. 2.2, 2.3).

A module M will be called *pseudo-finitely related* if there exists a projective presentation $0 \to K \hookrightarrow P \to M \to 0$ of M such that K satisfies (PFG) in M.

2.5. Lemma. *Let $0 \to L \hookrightarrow Q \to M \to 0$ be a projective presentation of a pseudo-finitely related module M. Then L satisfies* (PFG) *in Q.*

Proof. We have the following (commutative) diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota_1"] \arrow[d] & P \arrow[r, "\pi"] \arrow[d, "\nu"'] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "\iota_2"] & Q \arrow[r, "\sigma"] & M \arrow[r] & 0 \end{tikzcd}$$

where ι_1 and ι_2 are natural injections, P is projective and K satisfies (PFG) in P. Let L_i, $i \in I$, be an upwards-directed family of submodules of L such that $\bigcup\limits_{i\in I} L_i = L$. Then $K = \bigcup\limits_{i\in I} K_i$, $K_i = \nu^{-1}(L_i)$, and $K \cap T = K_j$, $K + T = P$, for some $j \in I$ and a submodule T of P. Now, $Q = L + \nu(P) = L + \nu(T) = L + S$, $S = L_j + \nu(T)$, $L_j = L \cap S$. $\square$

2.6. Proposition. *The following conditions are equivalent for a module M:*
(i) *M is pseudo-finitely related.*
(ii) *M is a direct summand of a finitely related module.*
(iii) *M is a direct summand of a module that is a direct sum of a finitely presented module and a free module.*

P r o o f. (i) $\Rightarrow$ (ii). Let $0 \to K \hookrightarrow P \to M \to 0$ be a projective presentation of M such that K satisfies (PFG) in P and let K_i be the family of finitely generated submodules of K. Then $P/K_j \cong M \oplus K/K_j$.

(ii) $\Rightarrow$ (i). Consider the following (commutative) diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota_1"] \arrow[d] & P \arrow[r, "\pi"] \arrow[d, "\nu"'] & N \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & L \arrow[r, "\iota_2"] & Q \arrow[r, "\sigma"] & M \arrow[r] & 0
\end{tikzcd}$$

where ι_1 and ι_2 are natural injections, both P and Q are projective, K is finitely generated, M is a direct summand of N and μ is the corresponding natural projection. Let L_i be an upwards-directed family of submodules of L such that $\bigcup L_i = L$. Then $\nu(K) \subseteq L_j$ for some j and there are submodules P_1 and P_2 of P such that $P_1+P_2 = P$, $P_1 \cap P_2 = K$ and $\pi(P_1) = \operatorname{Ker}(\mu)$. Consequently, $N = \mu\pi(P_2) = \sigma\nu(P_2)$, $Q = L + \nu(P_2)$ and $Q = L + S$, $L_j = L \cap S$, where $S = L_j + \nu(P_2)$. □

Using 2.6 and some well known results, we come to the following observations:

2.7. Proposition.

(i) *The class of pseudo-finitely related modules is closed under direct summands and finite direct sums.*

(ii) *The class of pseudo-finitely related modules is closed under factormodules if and only if R is completely reducible.*

(iii) *Every finitely generated pseudo-finitely related module is finitely presented.*

(iv) *Every pseudo-finitely related module is a direct sum of countably generated pseudo-finitely related modules.*

(v) *Every pseudo-finitely related flat module is projective.*

(vi) *Every pseudo-finitely related module is projective if and only if R is (von Neumann) regular.*

2.8. Proposition. *Suppose that at least one of the following three cases takes place:*

(1) *R is left hereditary;*

(2) *R is left perfect;*

(3) *R is regular.*

Then every pseudo-finitely related module is finitely related.

P r o o f. Use 2.6 and 2.5, 2.3 (i), 2.7 (vi). □

3. Weakly pseudo-finitely related modules

A module M will be called $\cup$-*compact* if $M \neq \bigcup M_i$ whenever $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ is a sequence of proper submodules of M. These remarkable modules first appeared in [1, p. 54], [7], [11], [15, p. 74], [17] and [18] and they were and still are known under various names (e.g., small modules, $\coprod$-slender modules, dually slender modules, $\sum$-compact modules, $\sum$-modules or modules of type $\sum$).

The following observations are easy and well known:

3.1. Proposition.

(i) *The class of $\cup$-compact modules is closed under factormodules, extensions and finite direct sums.*

(ii) *Every (essential) submodule of a module M is $\cup$-compact if and only if M is noetherian.*

(iii) *If A_i is an infinite family of non-zero modules then $\coprod A_i$ is not $\cup$-compact.*

(iv) *Every finitely generated module is $\cup$-compact.*

(v) *Every projective $\cup$-compact module is finitely generated.*

(vi) *If N is a superfluous submodule of M such that M/N is $\cup$-compact, then M is $\cup$-compact.*

3.2. Remark. It seems to be an open problem whether there exists a ring R such that the $\cup$-compact modules are closed under direct products. If R is such a ring and if I is a maximal left ideal of R, then $I = I^2$ is not a two-sided ideal; in particular, R is not commutative. If R is a simple ring containing an infinite set of non-zero pairwise orthogonal idempotents, then injective modules are $\cup$-compact and, of course, injective modules are closed under direct products.

3.3. Remark. The ring R is said to be *left steady* (or *left poised*) if every $\cup$-compact module is finitely generated. Among such rings we shall certainly find all left noetherian rings, left perfect rings and left semiartinian rings with countable $\widehat{\mathrm{Soc}}$-length. These and other results on and examples of steady and non-steady rings can be found e.g. in [2]–[5], [8]–[10], [14], [16]–[22], [24] and [25].

A module M will be called *weakly finitely related* if M has a projective presentation $0 \to K \to P \to M \to 0$ such that K is $\cup$-compact.

3.4. Proposition.

(i) *The class of weakly finitely related modules is closed under extensions and finite direct sums.*

(ii) *Every $\cup$-compact weakly finitely related module is finitely generated.*

Let N be a submodule of a module M. We will say that N *satisfies the condition* (WPFG) *in* M if the following is true:

If $N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$ is a sequence of submodules of N such that $\bigcup N_i = N$, then there exist j and a submodule K of M such that $N \cap K = N_j$ and $N + K = M$.

3.5. Lemma. *N satisfies* (WPFG) *in M in each of the following two cases:*
(1) *N satisfies* (PFG) *in M;*
(2) *N is $\cup$-compact.*

3.6. Lemma. *Let N satisfy* (WPFG) *in M.*
(i) *If N is superfluous in M, then N is $\cup$-compact.*
(ii) *If M is $\cup$-compact, then N is $\cup$-compact.*
(iii) *If L satisfies* (WPFG) *in K, then $N \oplus L$ satisfies* (WPFG) *in $M \oplus K$.*

A module M will be called *weakly pseudo-finitely related* if there exists a projective presentation $0 \to K \hookrightarrow P \to M \to 0$ of M such that K satisfies (WPFG) in M.

3.7. Lemma. *Let $0 \to L \hookrightarrow Q \to M \to 0$ be a projective presentation of a weakly pseudo-finitely related module M. Then L satisfies* (WPFG) *in Q.*

Proof. Similar to that of 2.5. $\square$

3.8. Proposition. *The class of weakly-pseudo-finitely related modules is closed under direct summands and finite direct sums.*

Proof. The case of finite direct sums is clear from 3.6 (iii). Now, let $= M \oplus A$ be a weakly pseudo-finitely related module and let

$$0 \to K \hookrightarrow P \xrightarrow{\sigma} N \to 0 \quad \text{and} \quad 0 \to L \hookrightarrow Q \xrightarrow{\varrho} A \to 0$$

be projective presentations. Then

$$0 \longrightarrow K \oplus L \hookrightarrow P \oplus Q \xrightarrow{\pi} M \to 0$$

is a projective presentation of M, $\pi = \sigma \oplus \varrho$, and $K \oplus L$ satisfies (WPFG) in $P \oplus Q$ by 3.7. Further, let $\mu\colon P \oplus Q \to P$ and $\iota\colon P \to P \oplus Q$ denote the natural projection and injection, respectively, and let $K = \bigcup K_i$, $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$. Then there are j and a submodule V of $P \oplus Q$ such that $V + (K \oplus L) = P \oplus Q$ and $V \cap (K \oplus L) = K_j \oplus L$ and we have $P = K + Z$, $K_j = K \cap Z$ for $Z = \mu(V \cap (P \oplus L))$. $\square$

3.9. Proposition. *Suppose that R is left hereditary. Then:*

(i) *The classes of weakly finitely related and weakly pseudo-finitely related modules are closed under submodules.*

(ii) *Every weakly finitely related module is finitely related.*

(iii) *If M is a weakly pseudo-finitely related module with a projective presentation $0 \to K \to P \to M \to 0$ such that K is countably generated, then M is finitely related.*

3.10. Proposition. *If R is left perfect, then every weakly pseudo-finitely related module is finitely related.*

A module M will be called *weakly (pseudo-)finitely presented* if M is $\cup$-compact and weakly (pseudo-)finitely related.

3.11. Proposition.

(i) *The class of weakly finitely presented modules is closed under extensions and finite direct sums.*

(ii) *The class of weakly pseudo-finitely presented modules is closed under direct summands and finite direct sums.*

3.12. Proposition. *A module M is weakly finitely presented if and only if M is finitely generated and weakly pseudo-finitely presented.*

P r o o f. If M is weakly finitely presented, then M is finitely generated by 3.1 (v). The converse implication follows from 3.6 (ii) and 3.7. □

3.13. Corollary. *If R is left steady, then every weakly pseudo-finitely presented module is finitely presented.*

3.14. Example. Suppose that K is a $\cup$-compact submodule of a finitely generated projective module P such that K is not finitely generated (e.g., we can take $P = R$ to be a suitable valuation domain and K the maximal ideal of R). Then $M = P/K$ is a weakly finitely presented module that is not finitely presented.

3.15. Remark. It seems to be an open problem whether there exist non-finitely generated weakly pseudo-finitely presented modules at all.

4. Modules commuting with direct sums

Let I be a non-empty index set and A_i, $i \in I$, an indexed family of modules. We put $A = \coprod_{i\in I} A_i$ and $\mathrm{supp}_I(a) = \{i \in I\colon\ a(i) \neq 0\}$ for every $a \in A$. If J is a subset of I, then $A(J) = \{a \in A\colon\ \mathrm{supp}_I(a) \subseteq J\}$.

The following result is of folklore character:

4.1. Proposition. *The following conditions are equivalent for a module M:*

(i) *M commutes with direct sums.*

(ii) *M commutes with direct sums of countably many modules.*

(iii) *If $\varphi\colon\ M \to \coprod_{i\in I} A_i = A$ is a homomorphism, then there exists a finite subset J of I such that $\varphi(M) \subseteq A(J)$ (i.e., $\pi\varphi(M) = 0$, $\pi\colon\ A \to A/A(J)$ being the natural projection).*

(iv) *If Q is a cogenerator for R-MOD and if $\varphi\colon\ M \to Q^{(\aleph_0)} = P$ is a homomorphism, then there exists $n < \aleph_0$ with $\varphi(M) \subseteq P(n)$.*

(v) *M is $\cup$-compact.*

5. Modules commuting with push-outs

The following result is just a routine observation:

5.1. Proposition. *A module M commutes with push-outs if and only if M is projective.*

6. Modules commuting with colimits of upwards-directed spectra

The following three observations generalize (and partially repeat) some results and methods of H. Lenzing (see [13] and also [23; 24.9, 24.10, 25.2, 25.4]).

6.1. Observation. Let Δ be an upwards-directed spectrum of modules and let $\varphi_i\colon\ A_i \to A$ be a colimit of Δ. Further, let M be a module, $\mathrm{Hom}_R(M,\Delta)$ the corresponding Hom-spectrum in ABG and let $\alpha_i\colon\ \mathrm{Hom}_R(M,A_i) \to G$ be a colimit of $\mathrm{Hom}_R(M,\Delta)$. Finally, denote by $\Psi\colon\ G \to \mathrm{Hom}_R(M,A)$ the connecting homomorphism.

(i) If all homomorphisms of Δ are mono, then Ψ is a monomorphism.

(ii) If M is finitely generated, then Ψ is a monomorphism.

Let $u \in \mathrm{Ker}(\Psi)$. Then $u = \alpha_j(\mu)$ for some j and $\mu\colon\ M \to A_j$. Now, $0 = \Psi(u) = \Psi(\alpha_j(\mu)) = \mathrm{Hom}_R(\mathrm{id}_M, \varphi_j)(\mu) = \varphi_j\mu$, and so $\varphi_j\mu(M) = 0$. Since M is finitely generated and Δ is upwards-directed, it follows easily that there exists k such that $\Delta(f)(\mu(M)) = 0$, where $f\colon\ j \to k$. Then $0 = \Delta(f)(\mu) = \mathrm{Hom}_R(\mathrm{id}_M, \Delta(f))(\mu)$, and therefore $u = \alpha_j(\mu) = \alpha_k \,\mathrm{Hom}_R(\mathrm{id}_M, \Delta(f))(\mu) = 0$.

(iii) If all the modules A_i (of the spectrum) are noetherian, then Ψ is a monomorphism.

We can proceed in the same way as in (ii).

(iv) If all homomorphisms of Δ are mono and if M is finitely generated, then Ψ is an isomorphism.

By (i) (or (ii)), Ψ is mono. Now, let $\mu\colon\ M \to A$ be a homomorphism. Then $\mu(M) \subseteq \varphi_j(A_j)$ for some j. On the other hand, φ_j is mono, and so $\mu = \varphi_j\nu$ and $\mu = \mathrm{Hom}_R(\mathrm{id}_M, \varphi_j)(\nu) = \Psi(\alpha_j(\nu))$ for a homomorphism $\nu\colon\ M \to A_j$.

(v) If M is finitely presented, then Ψ is an isomorphism.

By (ii), Ψ is mono. Now, let $\mu\colon\ M \to A$ be a homomorphism. Then $\mu(M) \subseteq \varphi_j(A_j)$ and we get the following (commutative) diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota_1"] \arrow[d] & P \arrow[r, "\pi"] \arrow[d, "\nu"] & M \arrow[r] \arrow[d, "\mu_1"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\varphi_j) \arrow[r, "\iota_2"] & A_j \arrow[r, "\varphi_{j,1}"] \arrow[d, "\varphi_j"] & \varphi_j(A_j) \arrow[r] \arrow[d, "\iota"] & 0 \\
 & & A \arrow[r, equal] & A &
\end{tikzcd}$$

where ι_1, ι_2, ι are natural imbeddings, $\iota\mu_1 = \mu$, P is projective and both P and K are finitely generated. Then $\nu(K)$ is a finitely generated submodule of A_j and $\varphi_j(\nu(K)) = 0$. Consequently, $\Delta(f)(\nu(K)) = 0$ for some $f\colon\ j \to k$. Now, $\Delta(f)\nu = \sigma\pi$ for a suitable $\sigma\colon\ M \to A_k$ and we have $\varphi_k\sigma\pi = \varphi_k\Delta(f)\nu = \iota\varphi_{j,1}\nu = \iota\mu_1\pi = \mu\pi$. Since π is epi, it follows that $\mu = \varphi_k\sigma$.

(vi) If all homomorphisms of Δ are epi and if M is pseudo-finitely related, then Ψ is an epimorphism.

All $\varphi_i\colon\ A_i \to A$ are epi and we shall consider the following (commutative) diagram with exact rows:

$$\begin{tikzcd}
 & & & M \arrow[d, "\iota"] & \\
0 \arrow[r] & K \arrow[r, "\iota_1"] \arrow[d] & P \arrow[r, "\pi"] \arrow[d] & \overline{M} \arrow[r] \arrow[d, "\bar{\mu}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\varphi_j) \arrow[r, "\iota_2"] & A_j \arrow[r, "\varphi_j"] & A \arrow[r] & 0
\end{tikzcd}$$

where ι_1, ι_2, ι are natural imbeddings, $\bar{\mu}\iota = \mu$, P is projective, K is finitely generated, M is a direct summand of $\overline{M}$ and $\bar{\mu} = \mu \oplus 0$. Proceeding similarly as in (iv) we find $\sigma\colon \overline{M} \to A_k$ such that $\bar{\mu} = \varphi_k\sigma$. Then $\mu = \bar{\mu}\iota = \varphi_k\sigma\iota$, $\sigma\iota\colon M \to A_k$.

6.2. Observation. Let M be a module and $\mathscr{F}$ a (non-empty) upwards-directed set of submodules of M such that $\bigcup\mathscr{F} = M$. Then we get an upwards-directed spectrum $\Delta\colon (\mathscr{F}, \subseteq) \to R{-}\mathrm{MOD}$, where $\Delta(N_i) = N_i$, $N_i \in \mathscr{F}$, and homomorphisms are natural injections. Clearly, M (together with the natural injections) is a colimit of Δ.

Let $\alpha_i\colon \mathrm{Hom}_R(M, N_i) \to G$, $N_i \in \mathscr{F}$, be a colimit of $\mathrm{Hom}_R(M, \Delta)$ and let $\Psi\colon G \to \mathrm{Hom}_R(M, M)$ be the connecting homomorphism. Obviously, Ψ is a monomorphism.

Now, assume that Ψ is an epimorphism. We are going to show that $M = N_j \in \mathscr{F}$.

Indeed, there is $u \in G$ such that $\Psi(u) = \mathrm{id}_M$. Since $G = \bigcup \mathrm{Im}(\alpha_i)$, there exist $N_j \in \mathscr{F}$ and $\mu\colon M \to N_j$ such that $u = \alpha_j(\mu)$. Now, $\mathrm{id}_M = \Psi(\alpha_j(\mu)) = \mathrm{Hom}_R(\mathrm{id}_M, \varphi_j)(\mu) = \varphi_j\mu$, $\varphi_j\colon N_j \to M$ being the natural injection. Then $\varphi_j = \mu = \mathrm{id}_M$ and $M = N_j$.

6.3. Observation. Let A be a submodule of a module B and let $\mathscr{F}$ be an upwards-directed set of submodules of A such that $\bigcup\mathscr{F} = A$. We get an upwards-directed spectrum $\Delta\colon (\mathscr{F}, \subseteq) \to R - \mathrm{MOD}$, where $\Delta(A_i) = B/A_i$, $A_i \in \mathscr{F}$, and if $A_i \subseteq A_j$, then the corresponding homomorphism is the natural projection $B/A_i \to B/A_j$. Now, B/A (together with the natural projections $B/A_i \to B/A$) is a colimit of Δ.

Let $\alpha_i\colon \mathrm{Hom}_R(B/A, B/A_i) \to G$ be a colimit of $\mathrm{Hom}_R(B/A, \Delta)$ and let $\Psi\colon G \to \mathrm{Hom}_R(B/A, B/A)$ be the connecting homomorphism.

(i) Ψ is a monomorphism if and only if for every $A_j \in \mathscr{F}$ and every homomorphism $\mu\colon B/A \to B/A_j$ there is $A_k \in \mathscr{F}$ such that $A_j \subseteq A_k$ and $\mathrm{Im}(\mu) \subseteq A_k/A_j$.

(ii) If Ψ is a monomorphism and if there exist $A_j \in \mathscr{F}$ and an epimorphism $B/A \to B/A_j$, then $A = A_k \in \mathscr{F}$.

(iii) Suppose that $u \in G$ is such that $\Psi(u) = \mathrm{id}_{B/A}$. We have $u = \alpha_j(\mu)$ for some $A_j \in \mathscr{F}$ and a homomorphism $\mu\colon B/A \to B/A_j$. Now $\mathrm{id}_{B/A} = \Psi(u) = \Psi(\alpha_j(\mu)) = \mathrm{Hom}_R(\mathrm{id}_{B/A}, \varphi_j)(\mu) = \varphi_j\mu$, $\varphi_j\colon B/A_j \to B/A$ being the natural projection, and hence $B/A_j = \mathrm{Im}(\mu) \oplus A/A_j$, $\mathrm{Im}(\mu) \cong B/A$.

In particular, if B and all A_i belong to a class of modules that is closed under homomorphic images and extensions, then A is contained in the class.

(iv) If Ψ is an epimorphism and all B, A_i are finitely generated, then A is finitely generated.

6.4. Theorem.

(i) *For every module M and every upwards-directed monomorphic spectrum Δ, the connecting homomorphism Ψ is a monomorphism.*

(ii) *A module M is pseudo-finitely related if and only if the connecting homomorphism Ψ is an epimorphism for every upwards-directed epimorphic spectrum.*

(iii) *The following conditions are equivalent for a module M:*

  (iii1) *M is finitely generated;*

  (iii2) *The connecting homomorphism Ψ is a monomorphism for every upwards-directed spectrum;*

  (iii3) *Ψ is a monomorphism for every upwards-directed epimorphic spectrum;*

  (iii4) *Ψ is an epimorphism for every upwards-directed monomorphic spectrum.*

(iv) *A module M is finitely presented if and only if the connecting homomorphism Ψ is an epimorphism for every upwards-directed spectrum.*

P r o o f. Combine 6.1, 6.2 and 6.3. $\square$

6.5. Corollary.

(i) *A module M commutes with colimits of upwards-directed monomorphic spectra if and only if M is finitely generated.*

(ii) *A module M commutes with colimits of upwards-directed (epimorphic) spectra if and only if M is finitely presented.*

6.6. Observation. Let $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$ be an $\aleph_0$-spectrum of modules and $\varphi_i\colon A_i \to A$ a colimit of the spectrum. Further, let M be a module and $\alpha_i\colon \mathrm{Hom}_R(M, A_i) \to G$ a colimit of the corresponding Hom-spectrum. Finally, let $\Psi\colon G \to \mathrm{Hom}_R(M, A)$ denote the connecting homomorphism.

(i) If M is $\cup$-compact, then Ψ is a monomorphism.

We can proceed similarly as in 6.1 (ii).

(ii) If all f_i are mono and if M is $\cup$-compact, then Ψ is an isomorphism.

By (i), Ψ is mono. Now, let $\mu\colon M \to A$ be a homomorphism and $M_i = \mu^{-1}(\varphi_i(A_i))$. Then $\bigcup M_i = M$, and so $M = M_j$ and $\mu(M) \subseteq \varphi_j(A_j)$ for some j. The rest is clear (cf. 6.1 (iv)).

(iii) If M is weakly pseudo-finitely presented, then Ψ is an isomorphism.

By (i), Ψ is mono. Let $\mu\colon M \to A$ be a homomorphism. Then $\mu(M) \subseteq \varphi_j(A_j)$ for some j (see (ii)) and we get a diagram similar to that in 6.1 (v), where P is projective and K satisfies (WPFG) in P. Now, for $k < \aleph_0$, let $K_k = K \cap \mathrm{Ker}(f_{j+k} \dots f_j \nu) \subseteq K$. Then $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ and $K = \bigcup K_k$. Consequently, there are l and a submodule Q of P such that $K + Q = P$ and $K \cap Q = K_l$. We get the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\iota_1} & P & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \big\downarrow & & {\scriptstyle\varkappa}\big\downarrow & & {\scriptstyle\mu_2}\big\downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\varphi_q) & \xrightarrow{\iota_2} & A_q & \xrightarrow{\varphi_{q,1}} & \varphi_q(A_q) & \longrightarrow & 0 \\
 & & & & {\scriptstyle\varphi_q}\big\downarrow & & {\scriptstyle\iota}\big\downarrow & & \\
 & & & & A & = & A & &
\end{array}$$

where ι_1, ι_2, ι are natural imbeddings, $\iota\mu_2 = \mu$, $\varkappa = f_{j+l} \dots f_j \nu$ and $q = j + l + 1$. Further, $\varkappa(K_l) = 0$ and we can define a homomorphism $\sigma\colon M \to A$ by $\sigma\pi(x) = \varkappa(x)$ for every $x \in Q$. Now, $\mu = \varphi_q \sigma$.

(iv) If all the homomorphisms f_i are epi and if M is weakly pseudo-finitely related, then Ψ is an epimorphism.

Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\iota_1} & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow & & {\scriptstyle\nu}\big\downarrow & & {\scriptstyle\mu}\big\downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\varphi) & \xrightarrow{\iota_2} & A_0 & \xrightarrow{\varphi} & A & \longrightarrow & 0
\end{array}$$

where ι_1 and ι_2 are natural imbeddings, $\varphi = \varphi_0$, P is projective and K satisfies (WPFG) in P. We have $K = \bigcup\limits_k (K \cap \mathrm{Ker}(f_k \dots f_0 \nu))$ and we can proceed similarly as in (iii).

6.7. Theorem. (i) *A module M is weakly pseudo-finitely related if and only if the connecting homomorphism Ψ is an epimorphism for every epimorphic $\aleph_0$-spectrum.*

(ii) *The following conditions are equivalent for a module M:*

(ii1) *M is $\cup$-compact;*

(ii2) *the connecting homomorphism Ψ is a monomorphism for every $\aleph_0$-spectrum;*

(ii3) *Ψ is a monomorphism for every epimorphic $\aleph_0$-spectrum;*

(ii4) *Ψ is an epimorphism for every monomorphic $\aleph_0$-spectrum.*

(iii) *A module M is weakly pseudo-finitely presented if and only if the connecting homomorphism Ψ is an epimorphism for every $\aleph_0$-spectrum.*

Proof. Combine 6.2, 6.3 and 6.6. $\square$

6.8. Corollary.

(i) *A module M commutes with colimits of monomorphic $\aleph_0$-spectra if and only if M is $\cup$-compact.*

(ii) *A module M commutes with colimits of (epimorphic) $\aleph_0$-spectra if and only if M is weakly pseudo-finitely presented.*

7. Summary

7.1. Theorem.

(i) *The following conditions are equivalent for a module M:*
  - (i1) *M is finitely generated and projective;*
  - (i2) *M commutes with colimits of all diagrams;*
  - (i3) *M commutes with direct sums and push-outs.*

(ii) *A module M commutes with push-outs if and only if M is projective.*

(iii) *A module M commutes with direct sums if and only if M is $\cup$-compact.*

(iv) *A module M commutes with colimits of upwards-directed monomorphic spectra if and only if M is finitely generated.*

(v) *The following conditions are equivalent for a module M:*
  - (v1) *M is finitely presented;*
  - (v2) *M commutes with colimits of upwards-directed spectra;*
  - (v3) *M commutes with colimits of upwards-directed epimorphic spectra.*

(vi) *A module M commutes with colimits of monomorphic $\aleph_0$-spectra if and only if M is $\cup$-compact.*

(vii) *The following conditions are equivalent for a module M:*
  - (vii1) *M is weakly pseudo-finitely presented;*
  - (vii2) *M commutes with colimits of $\aleph_0$-spectra;*
  - (vii3) *M commutes with colimits of epimorphic $\aleph_0$-spectra.*

Proof. (i) The first implication is only a technical question, the second implication is trivial and the third implication follows by combination of 4.1 and 5.1.

(ii) See 5.1.

(iii) See 4.1.

(iv) and (v) See 6.5.

(vi) and (vii) See 6.8. □

7.2. Remark.

(i) The following conditions are equivalent for a module M:
  - (i1) M is $\cup$-compact;
  - (i2) M commutes with colimits of monomorphic $\aleph_0$-spectra;

(i3) M commutes with direct sums of countably many modules;

(i4) M commutes with arbitrary direct sums.

(ii) Dualizing the (equivalent) conditions of (i), we arrive at the following classes of modules (see [12] and [6]):
$\cap$-compact modules;
slender modules;
slender modules;
slim modules.

Now, every finitely cogenerated module is $\cap$-compact and it easily follows that there always exist (over any non-zero ring R) $\cap$-compact modules that are not slender. On the other hand, slender modules are known to be closed under direct sums and consequently, if M is a non-zero slender module, then $M^{(\aleph_0)}$ is slender but not $\cap$-compact. In particular, the class of slender modules is contained in the class of $\cap$-compact modules if and only if all slender modules are zero (such rings exist but are not too frequent).

Every slim module is slender, but the converse is true if and only if there exist no measurable cardinal numbers. Furthermore, there exists at least one non-zero slim module (over at least one ring) if and only if there are not too many measurable cardinals ([6, Theorem 8.2]).

It seems that the properties dual to those of $\cup$-compact modules are better reflected by $\cap$-compact modules than by the slender ones. In particular, it is more suitable to define dually steady rings as those where every $\cap$-compact module is finitely cogenerated (see [12]).

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