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ON TOTAL INCOMPARABILITY OF MIXED TSIRELSON SPACES

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Abstract. We give criteria of total incomparability for certain classes of mixed Tsirelson spaces. We show that spaces of the form $T[(\mathscr{M}_k, \theta_k)_{k=1}^{l}]$ with index $i(\mathscr{M}_k)$ finite are either c_0 or ℓ_p saturated for some p and we characterize when any two spaces of such a form are totally incomparable in terms of the index $i(\mathscr{M}_k)$ and the parameter θ_k. Also, we give sufficient conditions of total incomparability for a particular class of spaces of the form $T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$ in terms of the asymptotic behaviour of the sequence $\Big\|\sum\limits_{i=1}^{n} e_i\Big\|$ where (e_i) is the canonical basis.

Keywords: mixed Tsirelson spaces, totally incomparable spaces

MSC 2000: 46B03, 46B20

0. Introduction

Denote by c_{00} the vector space of all real valued sequences which are eventually zero and by $(e_i)_{i=1}^{\infty}$ its usual unit vector basis. For $E \subset \mathbb{N}$ and $x = \sum\limits_{i=1}^{\infty} a_i e_i \in c_{00}$ we denote $Ex = \sum\limits_{i \in E} a_i e_i$. Also, for finite subsets $E, F \subseteq \mathbb{N}$, we write $E < F$ (or $E \leqslant F$) if $\max E < \min F$ ($\max E \leqslant \min F$). For simplicity, we write $n \leqslant E$ instead of $\{n\} \leqslant E$.

Mixed Tsirelson spaces were introduced in full generality in [2]. We can define those spaces, denoted by $T[(\mathscr{M}_k, \theta_k)_{k \in I}]$, as the completion of c_{00} under a norm which satisfies an implicit equation of the following kind:

$$\|x\| = \max\Bigg\{\|x\|_\infty, \sup_{k \in I}\Bigg\{\theta_k \sup_{n \in \mathbb{N}}\Bigg\{\sum_{i=1}^{n} \|E_i x\| \mid (E_i)_{i=1}^{n}\ \mathscr{M}_k\text{-admissible}\Bigg\}\Bigg\}\Bigg\}, \quad x \in c_{00}$$

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where the $\mathscr{M}_k$'s are certain (see Definition 4 below) families of finite subsets of $\mathbb{N}$, $\theta_k \in (0,1]$ for all $k \in I \subseteq \mathbb{N}$ and $(E_i)_{i=1}^n$ is $\mathscr{M}_k$-admissible if there exists $\{m_1,\dots,m_n\} \in \mathscr{M}_k$ such that $m_1 \leqslant E_1 < m_2 \leqslant E_2 < \ldots < m_n \leqslant E_n$.

The first remarkable space in this class is the so called Tsirelson space, introduced by Figiel and Johnson [7] in 1974. (It is actually the dual of the space originally constructed by Tsirelson in [12].) In our notation this space is $T[\mathscr{S}, 1/2]$, where $\mathscr{S}$ is Schreier's class, that is, the set of subsets of $\mathbb{N}$ of cardinality smaller than their first element. Since its construction it was usually considered a "pathological space", a place to look for counterexamples to statements in the Banach space theory. In fact, the reason why it was constructed was to provide a counterexample to the assertion "every Banach space contains c_0 or ℓ_p for some $1 \leqslant p < \infty$".

The second space of the class is Tzafriri space, introduced in 1979 in [13] ($T[(\mathscr{A}_k, \gamma/\sqrt{k})_{k\in\mathbb{N}}]$, $0 < \gamma < 1$ in our notation where $\mathscr{A}_k$ is the set of subsets of $\mathbb{N}$ of at most k elements), also constructed as a counterexample to a statement in the Banach space theory. In 1991 a third example, namely the Schlumprecht space $T[(\mathscr{A}_k, 1/\log_2(1+k))_{k\in\mathbb{N}}]$, was considered, see [11], and with its help a fruitful period started when many "classical" problems in the infinite dimensional Banach space theory were solved, such as the distortion problem or the unconditional basic sequence problem.

A common feature of the three Banach spaces mentioned above is that they do not contain any ℓ_p, $1 \leqslant p < \infty$ or c_0. (Actually, in the case of Tzafriri spaces this has been proved, as far as we know, only for $0 < \gamma < 10^{-6}$, see [6].) Moreover, since ℓ_p, $1 \leqslant p < \infty$ and c_0 are minimal (recall that a Banach space X is minimal if every subspace of X contains a further subspace isomorphic to X) it easily follows that they are totally incomparable to any of the three examples above (recall that two Banach spaces X and Y are totally incomparable if no subspace of X is isomorphic to any of Y). We use the word "subspace" here and throughout the paper for "closed infinite dimensional subspace".

In 1986 Bellenot [3] showed that ℓ_p, $1 \leqslant p < \infty$ and c_0 are isomorphic to mixed Tsirelson spaces of the form $T[(\mathscr{A}_n, \theta)]$, $\theta \in (0,1]$. This was somewhat surprising as it showed that ℓ_p, $1 \leqslant p < \infty$ and c_0 belong to a class of spaces up to then considered pathological.

It is well known that ℓ_p, $1 \leqslant p < \infty$ and c_0 are totally incomparable to each other. Moreover, ℓ_p and c_0 and the three examples, with $0 < \gamma < 10^{-6}$ in the case of Tzafriri space, are all totally incomparable to each other (see [6] for the details and also use the minimality of the Schlumprecht space). This shows that, at least in the examples considered, the modification of the θ_k's or the $\mathscr{M}_k$'s produce totally incomparable spaces.

In the first section we discuss in full generality the case when $\theta_k = 1$ for some k. In this case, the spaces c_0 and ℓ_1 will play a crucial role.

In the second section we consider mixed Tsirelson spaces of the form $T[(\mathscr{M}_k, \theta_k)_{k=1}^{l}]$, $\theta_k \in (0,1)$, with index $i(\mathscr{M}_k)$, as defined in [2], finite and we characterize when any two spaces of such a form are totally incomparable. This is done by following the ideas in [4] and showing that every such space is either c_0 or ℓ_p saturated for some p. Recall that given a Banach space Y, a Banach space X is Y saturated if every subspace of X contains a further subspace isomorphic to Y.

In the third section we focus on spaces of the form $T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$, $\theta_k \in (0,1]$, such that ℓ_1 is finitely block represented in every block subspace. We give sufficient conditions of total incomparability in terms of the asymptotic behaviour of the sequence $\Bigl\|\sum\limits_{i=1}^{n} e_i\Bigr\|$ where (e_i) is the canonical basis. These conditions apply to cases different from those considered in [9].

Notation. If K is a subset of a Banach space X, $\overline{\mathrm{Span}}\{K\}$ denotes the closure of the algebraic linear span of K. If $x = \sum\limits_{i=1}^{\infty} a_i e_i \in c_{00}$, the support of x is the set $\mathrm{supp}(x) = \{i \in \mathbb{N} \mid a_i \neq 0\}$. For $x, y \in c_{00}$ we write $x < y$ if $\mathrm{supp}(x) < \mathrm{supp}(y)$. We say that $E_1, \ldots, E_n \subset \mathbb{N}$ are successive if $E_1 < E_2 < \ldots < E_n$. The vectors $x_1, \ldots, x_n$ are successive if their supports are. A block sequence (x_i) is a sequence of successive vectors. The cardinality of a set E is denoted by $|E|$. The standard norm of ℓ_p, $1 \leqslant p \leqslant \infty$ is denoted by $\|\cdot\|_p$. Other unexplained notation is standard and can be found for instance in [8].

Definition 1. Let $\mathscr{M}$ be a family of finite subsets of $\mathbb{N}$. We say that $\mathscr{M}$ is compact if the set $\{\aleph_E \mid E \in \mathscr{M}\}$ is a compact subset of the Cantor set $\{0,1\}^{\mathbb{N}}$ with the product topology.

Remark 1. In Definition 1, $\{0,1\}^{\mathbb{N}}$ is identified with the space of all mappings $f\colon \mathbb{N} \longrightarrow \{0,1\}$ and $\aleph_E$ is the characteristic function of E. In $\{0,1\}^{\mathbb{N}}$, the convergence under the product topology is the pointwise convergence. Therefore if $E \subseteq \mathbb{N}$ is a finite set and $\aleph_{E_k}$ converges to $\aleph_E$ pointwise, there exists $N \in \mathbb{N}$ such that $E \subseteq E_k$ for all $k \geqslant N$.

Definition 2. Let $\mathscr{M}$ be a family of finite subsets of $\mathbb{N}$. We say that $\mathscr{M}$ is hereditary if $E \in \mathscr{M}$ and $F \subseteq E$ implies that $F \in \mathscr{M}$.

Definition 3. Let $\mathscr{M}$ be a compact family of finite subsets of $\mathbb{N}$. We define a transfinite sequence $(\mathscr{M}^{(\lambda)})$ of subsets of $\mathscr{M}$ as follows:

1. $\mathscr{M}^{(0)} = \mathscr{M}$.
2. $\mathscr{M}^{(\lambda+1)} = \{E \in \mathscr{M} \mid \aleph_E$ is a limit point of the set $\{\aleph_E \mid E \in \mathscr{M}^{(\lambda)}\}\}$.

3. If λ is a limit ordinal then $\mathscr{M}^{(\lambda)} = \bigcap_{\mu<\lambda} \mathscr{M}^{(\mu)}$.

We call the least λ for which $\mathscr{M}^{(\lambda)} \subseteq \{\emptyset\}$ the index of $\mathscr{M}$ and denote it by $i(\mathscr{M})$.

Definition 4. Let $I \subseteq \mathbb{N}$. Let $(\mathscr{M}_k)_{k\in I}$ be a sequence of compact hereditary families of finite subsets of $\mathbb{N}$ and let $(\theta_k)_{k\in I} \subset (0,1]$. We denote by $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ the completion of c_{00} with respect to the norm defined by

$$\|x\| = \max\left\{ \|x\|_\infty, \sup_{k\in I}\left\{ \theta_k \sup_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} \|E_i x\| \mid (E_i)_{i=1}^n \ \mathscr{M}_k\text{-admissible} \right\}\right\}\right\}$$

and we call it the mixed Tsirelson space defined by the sequence $(\mathscr{M}_k,\theta_k)_{k\in I}$.

Remark 2. The existence of such a norm is shown, for instance, in [10]. It follows from the definition of the norm that the sequence $(e_i)_{i=1}^\infty$ is a normalized 1-unconditional basis for $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$.

Remark 3. There are two useful alternative ways to define the norm. Given $x = \sum_{n=1}^{\infty} a_n e_n \in c_{00}$,

(i) define a non decreasing sequence of norms on c_{00}:

$$|x|_0 = \max_{n\in\mathbb{N}} |a_n|,$$
$$|x|_{s+1} = \max\left\{ |x|_s, \sup_{k\in I}\left\{ \theta_k \sup_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} |E_i x|_s \mid (E_i)_{i=1}^n \ \mathscr{M}_k\text{-admissible} \right\}\right\}\right\}.$$

Then $\|x\| = \sup_{s\in\mathbb{N}\cup\{0\}} |x|_s$.

(ii) Let $K_0 = \{\pm e_n \mid n \in \mathbb{N}\}$. Given K_s, $s \in \mathbb{N}\cup\{0\}$, let

$$K_{s+1} = K_s \cup \{\theta_k \cdot (f_1 + \ldots + f_d) \mid k \in I,\ d \in \mathbb{N},\ f_i \in K_s,\ i = 1,\ldots,d$$
$$\text{are successive and } (\mathrm{supp}(f_1),\ldots,\mathrm{supp}(f_d))\ \mathscr{M}_k\text{-admissible}\}.$$

Let $K = \bigcup_{s=0}^{\infty} K_s$. Then $\|x\| = \sup\{f(x) \mid f \in K\}$.

The latter definition of the norm provides information about the dual space. Looking at the set K as a set of functionals it is not difficult to see that B_{X^*} is the closed convex hull of K, where the closure is taken either in the weak-$*$ topology or in the pointwise convergence topology.

1. The case $\theta_k = 1$

Let $J = \{k \in I \mid \theta_k = 1\}$. If J is not empty, we give information about the structure of $T[(\mathscr{M}_k, \theta_k)_{k\in I}]$ depending on the index $i(\mathscr{M}_k)$, $k \in J$. It is known that if $i(\mathscr{M}_k) \geqslant 2$ for some $k \in J$, then $T[(\mathscr{M}_k, \theta_k)_{k\in I}]$ contains an isomorphic copy of ℓ_1. Actually it is possible to say much more as our next proposition shows.

Proposition 1. *If $i(\mathscr{M}_{k_0}) \geqslant 2$ for some $k_0 \in J$, then $T[(\mathscr{M}_k, \theta_k)_{k\in I}]$ is ℓ_1-saturated.*

Proof. By the Bessaga-Pełczyński principle (see e.g. [6], p. 10), it suffices to show that every block subspace contains a further subspace isomorphic to ℓ_1. Recall that a block subspace is a space of the form $\overline{\mathrm{Span}}\{u_i,\ i \in \mathbb{N}\}$, with $(u_i)_{i=1}^{\infty}$ a block sequence.

Let $(u_i)_{i=1}^{\infty}$ be a block sequence. We are going to construct a subsequence $(u_{i_k})_{k=1}^{\infty}$ of $(u_i)_{i=1}^{\infty}$ equivalent to the ℓ_1 basis.

Let $\{p\} \in \mathscr{M}_{k_0}^{(1)}$. We can choose u_{i_1} such that $p < u_{i_1}$. Now, since $\{p\} \in \mathscr{M}_{k_0}^{(1)}$, there exists $n_1 \in \mathbb{N}$ such that $n_1 > u_{i_1}$ and $\{p, n_1\} \in \mathscr{M}_{k_0}$, so we can take u_{i_2} such that $n_1 < u_{i_2}$. Continuing in this manner, we can construct a subsequence $(u_{i_k})_{k=1}^{\infty}$ of $(u_i)_{i=1}^{\infty}$ such that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $u_{i_k} < n_k < u_{i_{k+1}}$ and $\{p, n_k\} \in \mathscr{M}_{k_0}$. It is now easy to see that $(u_{i_k})_{k=1}^{\infty}$ is equivalent to the ℓ_1 basis.

$\square$

The following example shows a Tsirelson type space ℓ_1-saturated but not isomorphic to ℓ_1. It was shown to us by I. Deliyanni.

Example 1. Let $\mathscr{M} = \{F \subseteq \mathbb{N} \mid \exists\, i \in \mathbb{N} \text{ such that } F \subseteq \{1, 2^i\}\}$ and $\theta = 1$.

It is clear that $i(\mathscr{M}) = 2$. If $T[\mathscr{M}, \theta]$ were isomorphic to ℓ_1 then since ℓ_1 has a unique—up to equivalence—normalized unconditional basis, there would exist a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C}\sum_{i=1}^{n} |a_i| \leqslant \Big\|\sum_{i=1}^{n} a_i e_i\Big\| \leqslant C \sum_{i=1}^{n} |a_i|.$$

Now taking $x = \sum\limits_{i=2^k+1}^{2^{k+1}} e_i$ we would obtain $2^k - 1 \leqslant C$ for all $k \in \mathbb{N}$.

We now examine $T[(\mathscr{M}_k, \theta_k)_{k\in I}]$ with $i(\mathscr{M}_k) = 1$, $k \in J$. We will find different subspaces depending on whether the set $\bigcup\limits_{k\in J} \mathscr{M}_k$ contains only a finite number of non singleton sets or not.

Proposition 2. *Let $I' \subseteq I$ be such that $\bigcup_{k\in I'} \mathscr{M}_k$ contains only a finite number of non singleton sets.*

(1) *If $I' \neq I$, then $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ is isomorphic to $T[(\mathscr{M}_k,\theta_k)_{k\in I\setminus I'}]$.*

(2) *If $I' = I$, then $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ is isomorphic to c_0.*

Proof. (1) Let $\|\cdot\|$ and $\|\cdot\|'$ be the norms of the spaces $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ and $T[(\mathscr{M}_k,\theta_k)_{k\in I\setminus I'}]$, respectively. We will see that they are equivalent. Clearly, $\|\cdot\|' \leqslant \|\cdot\|$.

For the other inequality let $M = \max\Big\{\max E \;\Big|\; E \in \bigcup_{k\in I'} \mathscr{M}_k,\ \text{non singleton}\Big\}$ and write $x = \sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{M} a_i e_i + \sum_{i=M+1}^{\infty} a_i e_i := x_1 + x_2$.

We have $\|x_1\| \leqslant M\|x\|'$ since $\|x_1\| = \Big\|\sum_{i=1}^{M} a_i e_i\Big\| \leqslant \sum_{i=1}^{M} |a_i| \leqslant \sum_{i=1}^{M} \|x\|_\infty \leqslant M\|x\|'$.

On the other hand, we show first by induction over s that $|x_2|_s \leqslant |x_2|'_s$. For $s=0$ it is clear. Suppose now that it is true for s and let $E_1,\dots,E_n$ be a sequence of finite subsets of $\mathbb{N}$, $\mathscr{M}_k$-admissible for some k. There are two possibilities, either $k \in I\setminus I'$ and then $\theta_k \sum_{i=1}^{n} |E_i x_2|_s \leqslant \theta_k \sum_{i=1}^{n} |E_i x_2|'_s \leqslant |x_2|'_{s+1}$, or $k \in I'$ and then, by hypothesis, $n=1$, E_1 is $\mathscr{M}_k$-admissible and $\theta_k |E_1 x_2|_s \leqslant \theta_k |x_2|_s \leqslant |x_2|'_s \leqslant |x_2|'_{s+1}$.

Therefore, $\|x_2\| \leqslant |x_2\|'$ and by 1-unconditionality, $\|x_2\|' \leqslant \|x\|'$. Thus, $\|x\|' \leqslant \|x\| \leqslant (M+1)\|x\|'$. □

For (2), it is easy to see that $T(\mathscr{M}_0,\theta_0)$ is isomorphic to c_0, where $\mathscr{M}_0 = \{\{i\} \mid i \in \mathbb{N}\}$, and $\theta_0 = 1$. Now use (1) to get that $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ is isomorphic to $T[(\mathscr{M}_k,\theta_k)_{k\in I\cup\{0\}}]$ and once again to see that the latter is isomorphic to $T(\mathscr{M}_0,\theta_0)$.

Proposition 2 for $I' = J$ yields

Proposition 3. *Let $J = \{k \in I \mid \theta_k = 1\}$.*

(1) *Let $\bigcup_{k\in J} \mathscr{M}_k$ contain only a finite number of non singleton sets.*

  1.1. *If $J = I$, then $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ is isomorphic to c_0.*

  1.2. *If $J \neq I$, then $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ is isomorphic to $T[(\mathscr{M}_k,\theta_k)_{k\in I\setminus J}]$.*

(2) *Let $\bigcup_{k\in J} \mathscr{M}_k$ contain an infinite number of non singleton sets.*

  Then $T[(\mathscr{M}_k,\theta_k)_{k\in I}]$ contains a subspace isomorphic to ℓ_1.

Proof. (1) follows from Proposition 2. For (2), we will construct a subsequence $(e_{n_i})_{i=1}^{\infty}$ of $(e_i)_{i=1}^{\infty}$ equivalent to the ℓ_1 basis.

Let $M_1 \in \bigcup_{k\in J} \mathscr{M}_k$ be a non singleton. Let $n_1 = \min M_1$. Having chosen n_i, we can take $M_{i+1} \in \bigcup_{k\in J} \mathscr{M}_k$ a non singleton such that $\min M_{i+1} > \max M_i$, and take $n_{i+1} = \min M_{i+1}$.

Consider the sequence $(e_{n_i})_{i=1}^{\infty}$ and let's show that it is equivalent to the ℓ_1 basis.

Let $x = \sum_{i=1}^{\infty} a_i e_{n_i}$. By the definition of the norm and the fact that for every $N \in \mathbb{N}$ and $i < N$, $(\{n_i\}, [n_{i+1}, n_N] \cap \mathbb{N}\})$ is $\mathscr{M}_k$-admissible for some $k \in J$ we have

$$\|x\| \geqslant |a_1| + \left\| \sum_{i=2}^{N} a_i e_{n_i} \right\| \geqslant \ldots \geqslant |a_1| + |a_2| + \ldots + |a_N|.$$

The proof is complete since always $\|x\| \leqslant \|x\|_1$. $\square$

Observe that in statement (2) of Proposition 3 we do not ensure ℓ_1 saturation. Actually, in some cases we can also find c_0 as a subspace. This is a consequence of the following general result.

Proposition 4. *Let $\mathscr{M}_k$ be compact and hereditary for all $k \in I \subseteq \mathbb{N}$, $\theta_k \in (0,1]$ for all $k \in I$. If for all $N \in \mathbb{N}$ there exists $n \geqslant N$ such that for all $M \in \bigcup_{k \in I} \mathscr{M}_k$ either $n < \min M$ or $n \geqslant \max M$, then $T[(\mathscr{M}_k, \theta_k)_{k \in I}]$ contains a subspace isomorphic to c_0. Moreover, if $\theta_k = 1$ for all $k \in I$, the converse is true.*

Proof. We will construct a subsequence $(e_{n_i})_{i=1}^{\infty}$ of the basis $(e_i)_{i=1}^{\infty}$ equivalent to the basis of c_0.

Let $N_1 = 1$. By hypothesis there exists $n_1 \geqslant N_1$ such that for all $M \in \bigcup_{k \in I} \mathscr{M}_k$, $n_1 < \min M$ or $n_1 \geqslant \max M$.

Suppose that n_i is chosen and write $N_{i+1} = n_i + 1$. Then there exists $n_{i+1} \geqslant N_{i+1}$ verifying the hypothesis. Now, consider the sequence $(e_{n_i})_{i=1}^{\infty}$.

Let $x = \sum_{i=1}^{\infty} a_i e_{n_i} \in c_{00}$ and write $|x|_0 = \|x\|_\infty$ as in Remark 3.

Let $(E_i)_{i=1}^{n}$ be a sequence of finite subsets of $\mathbb{N}$, $\mathscr{M}_k$-admissible for some $k \in I$. Then we have $\theta_k \sum_{i=1}^{n} |E_i x|_0 = \theta_k |E_{i_0} x|_0 \leqslant |x|_0$ and so $|x|_1 \leqslant |x|_0$. Indeed, the first equality is true since by the construction of (n_i), there exists at most one E_i such that $\operatorname{supp}(x) \cap E_i \neq \emptyset$ and the inequality is straightforward by 1-unconditionality. So we have proved that $|x|_1 = |x|_0$ and therefore $|x|_n = |x|_{n+1}$ and $\|x\| = \|x\|_\infty$. $\square$

The converse is a consequence of the following

Claim. *If there is an N_0 such that for all $n \geqslant N_0$, there exists $M \in \bigcup_{k \in I} \mathscr{M}_k$ such that $\min M \leqslant n < \max M$, then every normalized block sequence in $T[(\mathscr{M}_k, 1)_{k \in I}]$ has a subsequence equivalent to the canonical basis of ℓ_1 and in particular, $T[(\mathscr{M}_k, 1)_{k \in I}]$ is ℓ_1-saturated.*

Proof of Claim. Let $(x_i)_{i=1}^{\infty}$ be a normalized block sequence. Let i_1 be such that $N_0 \leqslant \min x_{i_1}$. We split $x_{i_1} = \sum\limits_{k=p_1+1}^{p_2} a_k e_k$ in the following manner:

Let $A^{(1)}(x_{i_1}) = \Big\{ j > \min x_{i_1} \;\Big|\; \{t, j\} \in \bigcup\limits_{k \in I} \mathscr{M}_k,\; t \leqslant \min x_{i_1} \Big\}$. By hypothesis $A^{(1)}(x_{i_1})$ is not empty and $j^{(1)}(x_{i_1}) := \min A^{(1)}(x_{i_1}) > \min x_{i_1}$.

Therefore,

$$x_{i_1} = \sum_{k=p_1+1}^{p_2} a_k e_k = \sum_{k=p_1+1}^{j^{(1)}(x_{i_1})-1} a_k e_k + \sum_{k=j^{(1)}(x_{i_1})}^{p_2} a_k e_k := x_{i_1}^{(1)} + u^{(1)}.$$

Let $y_{i_1}^{(1)} = x_{i_1}^{(1)} / \|x_{i_1}^{(1)}\|$. Suppose $y_{i_1}^{(l)}$ is defined and we have $x_{i_1} = x_{i_1}^{(1)} + \ldots + x_{i_1}^{(l)} + u^{(l)}$. If $u^{(l)} \neq 0$, define $x_{i_1}^{(l+1)} = (u^{(l)})^{(1)}$ and $y_{i_1}^{(l+1)} = x_{i_1}^{(l+1)} / \|x_{i_1}^{(l+1)}\|$ and keep going until we have $u^{(d_1)} = 0$ for some $d_1 \in \mathbb{N}$. Then we have $x_{i_1} = \sum\limits_{l=1}^{d_1} \|x_{i_1}^{(l)}\| y_{i_1}^{(l)}$.

Now, take i_2 such that $\operatorname{supp}(x_{i_2}) > j^{(d_1)}(x_{i_1})$ and split it as before. Continuing in this manner, we obtain a sequence

$$(y_{i_1}^{(1)}, y_{i_1}^{(2)}, \ldots, y_{i_1}^{(d_1)}, y_{i_2}^{(1)}, \ldots, y_{i_2}^{(d_2)}, \ldots, y_{i_n}^{(1)}, \ldots, y_{i_n}^{(d_n)}, \ldots) := (u_k)_{k=1}^{\infty}.$$

For this sequence we have

$$\left\| \sum_{k=1}^{n} a_k u_k \right\| = |a_1| + \left\| \sum_{i=2}^{n} a_k u_k \right\| = \ldots = \sum_{k=1}^{n} |a_k|,$$

that is, $(u_k)_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_1. But $(x_{i_k})_{k=1}^{\infty}$ is a block sequence of $(u_k)_{k=1}^{\infty}$ and therefore it is also equivalent to the canonical basis of ℓ_1. □

Remark 4.

1. Observe that, in particular, the hypothesis of Proposition 4 implies that $i(\mathscr{M}_k) = 1$ for all $k \in I$.
2. The proof of the converse of Proposition 4 states that either $T[(\mathscr{M}_k, 1)_{k \in I}]$ contains a subspace isomorphic to c_0 or $T[(\mathscr{M}_k, 1)_{k \in I}]$ is ℓ_1-saturated.

We now give an example of a Tsirelson type space which contains ℓ_1 and c_0.

Example 2. Let $\mathscr{M} = \{F \subseteq \mathbb{N} \mid \exists\, i \in \mathbb{N} \text{ such that } F \subseteq \{2i-1, 2i\}\}$. $T(\mathscr{M}, 1)$ contains ℓ_1 by Proposition 3 and c_0 by Proposition 4. Moreover, it is easy to see that the space is isomorphic to $\ell_1 \oplus c_0$.

2. The case $(\mathscr{M}_k, \theta_k)_{k=1}^l$

In view of the previous results, in this section we will consider Tsirelson type spaces defined by finite sequences $(\mathscr{M}_k, \theta_k)_{k=1}^l$, with $\theta_k \in (0,1)$ for all $k = 1, \ldots, l$. The main result of the section is

Theorem 1. *Let $i(\mathscr{M}_k) = n_k \in \mathbb{N}$ and $\theta_k \in (0,1)$ for all $k = 1, \ldots, l$.*
1. *If $\theta_k \leqslant 1/n_k$ for all k then $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ is c_0-saturated.*
2. *If $\theta_k > 1/n_k$ for some k then $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ is ℓ_p-saturated for some $p \in (1, +\infty)$.*

Our proof of this theorem is based on Theorem 2 below, proved in [4]. In order to state it we first need some definitions.

Definition 5. Let $m \in \mathbb{N}$ and $\varphi \in K_m \setminus K_{m-1}$. An analysis of φ is any sequence $\{K^s(\varphi)\}_{s=0}^m$ of subsets of K such that for every s,
1. $K^s(\varphi)$ consists of successive elements of K_s and $\bigcup\limits_{f \in K^s(\varphi)} \operatorname{supp}(f) = \operatorname{supp}(\varphi)$.
2. If $f \in K^{s+1}(\varphi)$ then either $f \in K^s(\varphi)$ or there exists k and successive $f_1, \ldots, f_d \in K^s(\varphi)$ with $(\operatorname{supp}(f_1), \ldots, \operatorname{supp}(f_d))$ $\mathscr{M}_k$-admissible and $f = \theta_k(f_1 + \ldots + f_d)$.
3. $K^m(\varphi) = \{\varphi\}$.

Definition 6.
1. Let $\varphi \in K_m \setminus K_{m-1}$ and let $\{K^s(\varphi)\}_{s=0}^m$ be a fixed analysis of φ. Then for a given finite block sequence $(x_k)_{k=1}^l$ we set for every $k \in \{1, \ldots, l\}$

$$s_k = \begin{cases} \max\{s \mid 0 \leqslant s < m, \text{ and there are at least two } f_1, f_2 \in K^s(\varphi) \\ \qquad \text{such that } |\operatorname{supp}(f_i) \cap \operatorname{supp}(x_k)| > 0,\ i = 1, 2\}, \\ \qquad \text{when this set is non-empty} \\ \\ 0 \ \text{ if } |\operatorname{supp}(x_k) \cap \operatorname{supp}(\varphi)| \leqslant 1. \end{cases}$$

2. For $k = 1, \ldots, l$ we define the initial part and the final part of x_k with respect to $\{K^s(\varphi)\}_{s=0}^m$, and denote them respectively by x_k' and x_k'', as follows: If $\{f \in K^{s_k}(\varphi) \mid \operatorname{supp}(f) \cap \operatorname{supp}(x_k) \neq \emptyset\} := \{f_1, \ldots, f_d\}$ with $f_1 < \ldots < f_d$, we set $x_k' = (\operatorname{supp}(f_1))x_k$ and $x_k'' = \Big(\bigcup\limits_{i=2}^d \operatorname{supp}(f_i)\Big)x_k$.

Notation. Let $m \in \mathbb{N}$, $\varphi \in K^m \setminus K^{m-1}$, let $\{K^s(\varphi)\}_{s=0}^m$ be an analysis of φ, $(v_i)_{i=1}^\infty$ a block sequence and $(x_j)_{j=1}^\infty$ a block sequence with $x_j \in \operatorname{Span}\{v_i \mid i \in \mathbb{N}\}$. Suppose that there exists n_φ such that $\operatorname{supp}(\varphi) \subseteq \bigcup\limits_{j=1}^{n_\varphi} \operatorname{supp}(x_j)$ and denote by x_j' and

x''_j the initial and the final part of x_j, $j \leqslant n_\varphi$. For all $f = \theta_k(f_1 + \ldots + f_d) \in K^s(\varphi)$ and $J \subseteq \{1, \ldots, n_\varphi\}$ we define the following sets for (x'_j):

$$I' = \{i \mid 1 \leqslant i \leqslant d \ \text{ and } \ \operatorname{supp}(f_i) \cap \operatorname{supp}(x'_j) \neq \emptyset \ \text{ for at least two different } j \in J\}$$

and for every $i \in I$,

$$\begin{aligned} D'_{f_i} = \{j \in J \mid \ & \operatorname{supp}(f_i) \cap \operatorname{supp}(x'_j) \neq \emptyset \\ & \text{and } \ (\operatorname{supp}(f) \cap \operatorname{supp}(x'_j)) \setminus \operatorname{supp}(f_i) \subseteq \operatorname{supp}(v_t) \ \text{ for some } t\} \end{aligned}$$

and

$$\begin{aligned} T' = \Big\{ j \in J \mid j \notin \bigcup_{i \in I'} D'_{f_i} \ & \text{ and } \exists\, t_1 \neq t_2 \\ & \text{such that } \ \operatorname{supp}(x'_j) \cap \Big(\bigcup_{i \notin I'} \operatorname{supp}(f_i) \Big) \cap \operatorname{supp}(v_{t_i}) \neq \emptyset, \ \ i = 1, 2 \Big\}. \end{aligned}$$

In the same manner we define sets I'', D''_{f_i}, T'' exchanging x'_j for x''_j.

Theorem 2 ([4]). *Given $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ with $l \in \mathbb{N}$, $\theta_k \in (0,1)$ and $i(\mathscr{M}_k) = n_k \in \mathbb{N}$, for all $k = 1, \ldots, l$, let $(v_i)_{i=1}^\infty$ be a normalized block sequence. If there exists a sequence $x_j = \sum_{i \in I_j} a_i v_i$ with $(a_i)_{i=1}^\infty \subset \mathbb{R}$ and $(I_j)_{j=1}^\infty \subset \mathbb{N}$ successive such that*

(a) *$1/2^{j+1} \leqslant |a_j| < 1/2^j$ and*
(b) *for all $m \in \mathbb{N}$, $\varphi \in K^m \setminus K^{m-1}$, each analysis $\{K^s(\varphi)\}_{s=1}^m$ of φ, all $f = \theta_k(f_1 + \ldots + f_d) \in K^s(\varphi)$, and all $J \subseteq \{1, \ldots, n_\varphi\}$, the inequalities $|I'| + |T'| \leqslant n_k$ and $|I''| + |T''| \leqslant n_k$ hold,*

then $(x_j)_{j=1}^\infty$ is equivalent to the canonical basis of $T[(\mathscr{A}_{n_k}, \theta_k)_{k=1}^l]$.

Recall, see [4], that the space $T[(\mathscr{A}_{n_k}, \theta_k)_{k=1}^l]$ is either isometrically isomorphic to c_0, when $n_k \cdot \theta_k \leqslant 1$ for all k, or isomorphic to ℓ_p, where $p = \min\Big\{ \frac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \ \Big| \ n_k \cdot \theta_k > 1 \Big\}$. So, to prove Theorem 1 we need to find the sequence $(x_j)_{j=1}^\infty$ and the next lemma will be useful for constructing it.

Lemma 1. *Let $l \in \mathbb{N}$, $\theta_k \in (0,1)$ and $\mathscr{M}_k$ be such that $i(\mathscr{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \ldots l$. Then for every block sequence $(u_i)_{i=1}^\infty$ in $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ there exists an infinite subset $\mathscr{P} = \{p_i\}_{i=1}^\infty$ of $\mathbb{N}$ and a subsequence $(v_i)_{i=1}^\infty$ of $(u_i)_{i=1}^\infty$ having the following properties:*

(a) $p_1 \leqslant \operatorname{supp}(v_1) < p_2 \leqslant \operatorname{supp}(v_2) < \ldots < p_i \leqslant \operatorname{supp}(v_i) < p_{i+1} \leqslant \ldots$

(b) *For every sequence* $E_1 < E_2 \ldots < E_{n_k}$ *of finite subsets of* $\mathscr{P}$*, where* $E_i = \{p_{l_1^i}, \ldots, p_{l_{t_i}^i}\}, i = 1, \ldots, n_k$*, the family*

$$\Biggl(\bigcup_{j=l_1^1}^{l_{t_1}^1} \operatorname{supp}(v_j), \ldots, \bigcup_{j=l_1^{n_k}}^{l_{t_{n_k}}^{n_k}} \operatorname{supp}(v_j)\Biggr)$$

is $\mathscr{M}_k$*-admissible.*

(c) *If* $r \geqslant n_k + 1$*,* $S = \{s_1, \ldots s_r\} \subseteq \mathbb{N}$ *is such that*

$$|\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \operatorname{supp}(v_j) \neq \emptyset\}| \geqslant 2$$

for all $i = 1, \ldots, r-1$*, then* $S \notin \mathscr{M}_k$*.*

P r o o f. The proof is based on the following result from [4]:

Lemma 2. *Let* $l, n_1, \ldots, n_l \in \mathbb{N}$*. Let* $\mathscr{M}_k, k = 1, \ldots, l$ *be such that* $i(\mathscr{M}_k) = n_k$*. Then there exists an infinite subset* Q *of* $\mathbb{N}$ *having the following properties:*

1. *Let* $k \in \{1, \ldots, l\}$*. Every sequence* $E_1 < E_2 \ldots < E_{n_k}$ *of length* n_k *of finite subsets of* Q *is* $\mathscr{M}_k$*-admissible.*
2. *Let* $k \in \{1, \ldots, l\}$*. If* $r \geqslant n_k + 1$*, then no sequence* $E_1 < E_2 \ldots < E_r$ *of finite subsets of* Q *with* $|E_i| \geqslant 2$ *for all* $i = 1, \ldots, r$*, is* $\mathscr{M}_k$*-admissible.*

Now, let $Q = \{k_i\}_{i=1}^{\infty}$ be the sequence in Lemma 2. Take $p_1 = k_1$, and $v_1 = u_l$ such that $p_1 \leqslant \operatorname{supp}(u_l)$. Having chosen p_i and v_i with $p_i \leqslant \operatorname{supp}(v_i)$, since $\{k_i\}_{i=1}^{\infty}$ is increasing, let k_{j_i} be such that $p_i \leqslant \operatorname{supp}(v_i) < k_{j_i}$, and take $p_{i+1} = k_{j_i+1}$ and $v_{i+1} = u_l$ such that $p_{i+1} \leqslant \operatorname{supp}(u_l)$.

The sequences $\{p_i\}_{i=1}^{\infty}$ and $(v_i)_{i=1}^{\infty}$ satisfy the assertions of Lemma 1:

(a) By construction.

(b) It is sufficient to see that $\bigcup\limits_{j=l_1^i}^{l_{t_i}^i} \operatorname{supp}(v_j) \subseteq [p_{l_1^i}, p_{l_{t_i}^i}]$ and, since the family $\{\{p_{l_1^i}, p_{l_{t_i}^i}\}\}_{i=1}^{n_k}$ is $\mathscr{M}_k$-admissible by Lemma 2, $\Biggl(\bigcup\limits_{j=l_1^i}^{l_{t_i}^i} \operatorname{supp}(v_j)\Biggr)_{i=1}^{n_k}$ is also $\mathscr{M}_k$-admissible.

(c) Let $S = \{s_1, \ldots, s_r\}$ be such that $|\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \operatorname{supp}(v_j) \neq \emptyset\}| \geqslant 2$ for all $i = 1, \ldots, r-1$, let $d_i = \min\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \operatorname{supp}(v_j) \neq \emptyset\}$. Then $k_{j_{d_i}}$ and $p_{d_i+1} \in [s_i, s_{i+1}] \cap Q$ for all $i = 1, \ldots, r-1$, and by the property (2) of Lemma 2, $S \notin \mathscr{M}_k$. □

P r o o f of Theorem 1. It suffices to show that c_0 or l_p is included in every block subspace of $T[(\mathscr{M}_k,\theta_k)_{k=1}^l]$.

Let $(u_i)_{i=1}^\infty$ be a normalized block sequence. Let $\mathscr{P}=\{p_i\}_{i=1}^\infty$ and $(v_i)_{i=1}^\infty$ be the sequences associated to $(u_i)_{i=1}^\infty$ from Lemma 1.

If $\sup\limits_{m\in\mathbb{N}}\Big\|\sum\limits_{i=1}^m v_i\Big\|$ is finite, then $(v_i)_{i=1}^\infty$ is equivalent to the canonical basis of c_0, and from Corollary 1 of [4] we have $n_k\cdot\theta_k\leqslant 1$.

Suppose now that $\lim\limits_{m\to\infty}\Big\|\sum\limits_{i=1}^m v_i\Big\|=\infty$. Then we can construct a sequence $(y_i)_{i=1}^\infty$ supported by the subsequence $(v_i)_{i=1}^\infty$ with the following properties: For every j, $y_j=\frac{1}{2^{j+1}}\sum\limits_{i\in I_j}v_i$, where

(i) I_j are successive intervals of $\mathbb{N}$, and

(ii) $1-\frac{1}{2^{j+1}}\leqslant\|y_j\|\leqslant 1$.

If $x_j=y_j/\|y_j\|$, the sequence x_j satisfies condition (a) of Theorem 2.

We prove condition (b) of Theorem 2 for the initial parts of (x_j) since for the final parts the proof is analogous. Suppose that φ, f and J are fixed. Let $m_1\leqslant\operatorname{supp}(f_1)<m_2\leqslant\operatorname{supp}(f_2)<\ldots<m_d\leqslant\operatorname{supp}(f_d)$. We define $B\subseteq\{m_1,\ldots,m_d\}$ as follows:

$$m_{i_s}\in B\leftrightarrow\begin{cases} i_s\in I',\\ i_s=\min\{i\notin I'\mid \operatorname{supp}(x'_j)\cap\operatorname{supp}(f_i)\neq\emptyset\}\ \text{ for some } j\in T'.\end{cases}$$

Let $m_{i_1}<\ldots<m_{i_r}$ be the elements of B. Observing that

$$|\{t\in\mathbb{N}\mid[m_{i_s},m_{i_{s+1}}]\cap\operatorname{supp}(v_t)\}|\geqslant 2,\ \ \forall\,1\leqslant s\leqslant r-1$$

and using property (c) of Lemma 1 we get that $r=|B|\leqslant n_k$. So $|I'|+|T'|\leqslant n_k$. $\square$

The proof of the next two corollaries easily follows from Theorem 1 from this paper and Corollaries 1 and 2 from [4].

Corollary 1. *Let* $T[(\mathscr{M}_k,\theta_k)_{k=1}^l]$, $1<p<\infty$, $n_k=i(\mathscr{M}_k)$ *and* $\theta_k\in(0,1)$. *The following conditions are equivalent:*

i) $T[(\mathscr{M}_k,\theta_k)_{k=1}^l]$ *contains a subspace isomorphic to* ℓ_p.

ii) $T[(\mathscr{M}_k,\theta_k)_{k=1}^l]$ *is* ℓ_p*-saturated.*

iii) $i(\mathscr{M}_k)$ *is finite,* $\theta_k>1/n_k$ *for some* $k=1,\ldots,l$ *and*

$$p=\min\Big\{\frac{1}{1-\log_{n_k}\frac{1}{\theta_k}}\mid n_k\cdot\theta_k>1\Big\}.$$

Corollary 2. *Let $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$, $\theta_k \in (0,1)$. The following conditions are equivalent:*

i) *$T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ contains a subspace isomorphic to c_0.*
ii) *$T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ is c_0-saturated.*
iii) *$i(\mathscr{M}_k)$ is finite and $\theta_k \leqslant 1/i(\mathscr{M}_k)$ for all $k = 1, \ldots, l$.*

In view of Proposition 1 and the previous corollaries we can include the case ℓ_1 in the discussion.

Corollary 3. *Let $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$, $2 \leqslant i(\mathscr{M}_k) \in \mathbb{N}$ and $\theta_k \in (0,1]$. The following conditions are equivalent:*

i) *$T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ contains a subspace isomorphic to ℓ_1.*
ii) *$T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ is ℓ_1-saturated.*
iii) *$\theta_k = 1$ for some $k = 1, \ldots, l$.*

So in particular we have proved the following criterion, which is useful to show when two Tsirelson type Banach spaces are totally incomparable.

Theorem 3. *Let $l, l' \in \mathbb{N}$, $\theta_k \in (0,1)$ and $i(\mathscr{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \ldots, l$ and $\theta'_k \in (0,1)$ and $i(\mathscr{M}'_k) = n'_k \in \mathbb{N}$ for all $k = 1, \ldots, l'$. Then $T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ and $T[(\mathscr{M}'_k, \theta'_k)_{k=1}^{l'}]$ are totally incomparable if and only if one of the following situations occurs:*

1. *$\theta_k \leqslant 1/n_k$ for all $k = 1, \ldots, l$ and $\theta'_k > 1/n'_k$ for some $k \in \{1, \ldots, l'\}$, or*
2. *$\theta'_k \leqslant 1/n'_k$ for all $k = 1, \ldots, l'$ and $\theta_k > 1/n_k$ for some $k \in \{1, \ldots, l\}$, or*
3. *$\theta_k > 1/n_k$ for some $k \in \{1, \ldots, l\}$ and $\theta'_k > 1/n'_k$ for some $k \in \{1, \ldots, l'\}$ and*

$$\min\Big\{\frac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \;\Big|\; n_k \cdot \theta_k > 1\Big\} \neq \min\Big\{\frac{1}{1 - \log_{n'_k} \frac{1}{\theta'_k}} \;\Big|\; n'_k \cdot \theta'_k > 1\Big\}.$$

Also we obtain a characterization of the reflexivity of this kind of spaces as in [1].

Proposition 5. *Let $l \in \mathbb{N}$. Let $\theta_k \in (0,1)$ and $i(\mathscr{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \ldots, l$. Then the following conditions are equivalent:*

1. *$T[(\mathscr{M}_k, \theta_k)_{k=1}^l]$ is reflexive.*
2. *$\theta_k > 1/i(\mathscr{M}_k)$ for some $k \in \{1, \ldots, l\}$.*

3. A criterion of total incomparability for spaces of the form $T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$

We will suppose throughout the section that $(\theta_k)_{k=1}^{\infty}\subset(0,1]$ is a non increasing null sequence since $T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$ is easily seen to be isometric to $T[(\mathscr{A}_k,\theta'_k)_{k=1}^{\infty}]$ where $\theta'_k=\sup\{\theta_j \mid j\geqslant k\}$ and $\inf\{\theta_k\}>0$ implies that $T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$ is isomorphic to ℓ_1.

The following properties of such spaces, stated as lemmas, are known.

Lemma 3. *Let $(u_i)_{i=1}^{n}$ be a normalized block sequence in $T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$. Then for all $a_1,\dots,a_n\in\mathbb{R}$,*

$$\Big\|\sum_{i=1}^{n}a_ie_i\Big\|\leqslant\Big\|\sum_{i=i}^{n}a_iu_i\Big\|.$$

P r o o f. It is easy to prove by induction on s that $\Big|\sum\limits_{i=1}^{n}a_ie_i\Big|_s\leqslant\Big\|\sum\limits_{i=i}^{n}a_iu_i\Big\|$. $\square$

The following lemma was proved in [11] with $\theta_k=(\log_2(1+k))^{-1}$, but the same proof works for any θ_k converging to zero.

Lemma 4 ([11])**.** *Let $T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$, let θ_k converge to 0. Let $(y_n)_{n=1}^{\infty}$ be a block sequence, let a strictly decreasing null sequence $(\varepsilon_n)_{n=1}^{\infty}\subset\mathbb{R}^+$ and a strictly increasing sequence $(k_n)_{n=1}^{\infty}\subset\mathbb{N}$ be such that for each n there is a normalized block sequence $(y(n,i))_{i=1}^{k_n}$, $(1+\varepsilon_n)$-equivalent to the $l_1^{k_n}$ basis and $y_n=\frac{1}{k_n}\sum\limits_{i=1}^{k_n}y(n,i)$. Then for all $l\in\mathbb{N}$,*

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\dots\lim_{n_l\to\infty}\Big\|\sum_{i=1}^{l}y_{n_i}\Big\|=\Big\|\sum_{i=1}^{l}e_i\Big\|.$$

We will consider spaces such that ℓ_1 is finitely block represented in every block subspace of the space but not containing ℓ_1. The role of ℓ_1 in this context, as well as that of c_0, can be easily described:

Proposition 6. *The following conditions are equivalent:*

i) *The identity is an isometric isomorphism from $T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$ onto c_0.*

ii) *$T[(\mathscr{A}_k,\theta_k)_{k=1}^{\infty}]$ contains a subspace isomorphic to c_0.*

iii) *For all $n\in\mathbb{N}$, $\Big\|\sum\limits_{i=1}^{n}e_i\Big\|=1$.*

iv) *$\theta_k\leqslant 1/k$ for all $k\in\mathbb{N}$.*

P r o o f. ii) $\Rightarrow$ iii): By the Bessaga-Pełczyński Principle and a theorem of R. C. James (see e.g. [8], p. 97), for every $\varepsilon > 0$ there exists a normalized block sequence $(u_i)_{i=1}^{\infty}$ such that for all $l \in \mathbb{N}$,

$$\max |a_i| \leqslant \left\| \sum_{i=1}^{l} a_i u_i \right\| \leqslant (1+\varepsilon) \max |a_i|, \quad a_1 \dots a_l \in \mathbb{R}$$

and so by Lemma 3, $\left\| \sum_{i=1}^{l} e_i \right\| \leqslant (1+\varepsilon)$ and iii) follows.

iii) $\Rightarrow$ iv): This is clear since $\theta \cdot l \leqslant \left\| \sum_{i=1}^{l} e_i \right\|$.

iv) $\Rightarrow$ i): By induction on $m \in \mathbb{N}$ it easily follows that $|\cdot|_m = |\cdot|_0$ on c_{00}. □

Proposition 7. *Let $T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$, let θ_k converge to 0. The following conditions are equivalent:*

i) *The identity is an isometric isomorphism from $T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$ onto ℓ_1.*

ii) *$T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$ contains a subspace isomorphic to ℓ_1.*

iii) *For all $n \in \mathbb{N}$, $\left\| \sum_{i=1}^{n} e_i \right\| = n$.*

iv) *$\theta_2 = 1$.*

P r o o f. ii) $\Rightarrow$ iii). Choose a strictly decreasing sequence $(\varepsilon_n))_{n=1}^{\infty} \subset \mathbb{R}_+$ converging to 0 and $k_n = n$. We will construct a block sequence $(y_n)_{n=1}^{\infty}$ as in Lemma 4 above.

By James' Theorem let $(u_i))_{i=1}^{\infty}$ be a normalized block sequence $(1+\varepsilon_1)$-equivalent to the unit vector basis of ℓ_1. Let $y_1 = u_1$. Again by James' theorem there exist a normalized block sequence $(u'_i))_{i=1}^{\infty}$ with $u'_i \in \mathrm{Span}\{u_i \mid i \in \mathbb{N}\}$ and $y_1 < u'_1$, $(1+\varepsilon_2)$-equivalent to the unit vector basis of ℓ_1. Let $y_2 = \frac{1}{2}(u'_1 + u'_2)$. We continue in the same way.

Let $l \in \mathbb{N}$. Since any block sequence $(y_{n_i})_{i=1}^{l}$ is $(1+\varepsilon_1)$-equivalent to the unit vector basis of ℓ_1^l, by Lemma 4 we have

$$(1-\varepsilon_1) l \leqslant \left\| \sum_{i=1}^{l} e_i \right\| \leqslant l$$

and the result follows.

iii) $\Rightarrow$ iv): Just notice that $2 = \|e_1 + e_2\| = 2\theta_2$.

iv) $\Rightarrow$ i): This follows by induction on $|\mathrm{supp}(x)|$. □

We now give sufficient conditions, in terms of the behaviour of $\lambda_n := \left\| \sum_{i=1}^{n} e_i \right\|$, guaranteeing that in a space of this kind ℓ_1 is finitely block represented in every block subspace.

Proposition 8 ([5]). *Let $n, l \in \mathbb{N}$, $0 < \varepsilon < 1$. Let $(X, \|\cdot\|)$ be a normed space with a normalized 1-unconditional normalized basis $(e_i)_{i=1}^{n^l}$ such that*

$$(n-\varepsilon)^l \leqslant \left\| \sum_{i=1}^{n^l} e_i \right\| \leqslant n^l.$$

Then there exists a normalized block sequence $(y_i)_{i=1}^{n}$ of $(e_i)_{i=1}^{n^l}$ such that

$$n-\varepsilon \leqslant \left\| \sum_{i=1}^{n} y_i \right\| \leqslant n.$$

Moreover, $(y_i)_{i=1}^{n}$ is $\frac{1}{1-\varepsilon}$-equivalent to the canonical basis of ℓ_1^n.

Proposition 9. *Let $T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$, let θ_k converge to 0. If there exists $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ unbounded and $(l_k)_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \left[n_k - \left(\lambda_{n_k^{l_k}}\right)^{1/l_k} \right] = 0,$$

then ℓ_1 is finitely block represented in every block subspace of $T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$.

Proof. Given $n \in \mathbb{N}$ and $0 < \varepsilon < 1$, take $k \in \mathbb{N}$ such that $n_k > n$ and $n_k - \left(\lambda_{n_k^{l_k}}\right)^{1/l_k} < \varepsilon$. Let $(u_i)_{i=1}^{\infty}$ be a normalized block sequence. Then

$$n_k^{l_k} \geqslant \left\| \sum_{i=1}^{n_k^{l_k}} u_i \right\| \geqslant \left\| \sum_{i=1}^{n_k^{l_k}} e_i \right\| = \lambda_{n_k^{l_k}} \geqslant (n_k - \varepsilon)^{l_k}$$

and, by Proposition 8, $l_1^{n_k}$ is finitely block represented in blocks of $(u_i)_{i=1}^{\infty}$. $\square$

Remark 5. By similar arguments it is easy to prove that the following condition is also sufficient:

1. There exits $m \geqslant 2$ such that $\lim\limits_{l\to\infty} (\lambda_{m^l})^{1/l} = m$.

We can also give sufficient conditions for the sequence $(\theta_k)_{k=1}^{\infty}$:

2. There exists $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ unbounded and $(l_k)_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} n_k \left[1 - \left(\theta_{n_k^{l_k}}\right)^{1/l_k} \right] = 0$ or
3. There exists $m \geqslant 2$ such that $\lim\limits_{l\to\infty} (\theta_{m^l})^{1/l} = 1$ or, equivalently, $\lim\limits_{l\to\infty} (\theta_{m^l})^{\frac{1}{l}} = 1$ for all $m \geqslant 2$.

Lemma 5. *Let $(X, \|\cdot\|)$ and $(X', \|\cdot\|')$ be Banach spaces not totally incomparable with Schauder bases $(e_i)_{i=1}^{\infty}$ and $(e'_i)_{i=1}^{\infty}$. If $(e_i)_{i=1}^{\infty}$ is shrinking, there exist block sequences $(u_i)_{i=1}^{\infty}$ and $(u'_i)_{i=1}^{\infty}$ of $(e_i)_{i=1}^{\infty}$ and $(e'_i)_{i=1}^{\infty}$ respectively such that the application $T\colon \overline{\operatorname{Span}}\{u_i \mid i \in \mathbb{N}\} \to \overline{\operatorname{Span}}\{u'_i \mid i \in \mathbb{N}\}$, given by $T(u_i) = u'_i$ for all $i \in \mathbb{N}$ is an isomorphism.*

Proof. There exist subspaces $Y \subseteq X$ and $Y' \subseteq X'$ and an isomorphism $S\colon Y \longrightarrow Y'$. We will see that for all $\varepsilon > 0$ we can find block sequences $(u_i)_{i=1}^{\infty}$ and $(u'_i)_{i=1}^{\infty}$ such that $(1-\varepsilon)\|S\|\,\|S^{-1}\| \leqslant \|T\|\,\|T^{-1}\| \leqslant (1+\varepsilon)\|S\|\,\|S^{-1}\|$.

Let $\varepsilon > 0$. There exists a normalized block sequence $(x_i)_{i=1}^{\infty}$ of $(e_i)_{i=1}^{\infty}$ and $\overline{\operatorname{Span}}\{y_i \mid i \in \mathbb{N}\} \subseteq Y$ such that the linear isomorphism defined by $U(x_i) = y_i$ verifies $\|U\|\,\|U^{-1}\| \leqslant 1+\varepsilon$. Let $y'_i := S(y_i)$ for all $i \in \mathbb{N}$.

Since $\inf\limits_{i\in\mathbb{N}} \|y'_i\| > 0$ and $(e_i)_{i=1}^{\infty}$ is a shrinking basis, y'_i tends to 0 weakly. So, by the Bessaga-Pełczyński principle, there is a subsequence $(y'_{i_k})_{k=1}^{\infty}$ and a block sequence $(u'_k)_{k=1}^{\infty}$ of $(e'_i)_{i=1}^{\infty}$ such that the isomorphism defined by $V(y'_{i_k}) = u'_k$ verifies $\|V\|\,\|V^{-1}\| \leqslant 1+\varepsilon$. Take $u_k = x_{i_k}$ and $T = V \circ S \circ U$. $\square$

Remark 6. Let $X = T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$, $\theta_k \in (0,1)$. Since its canonical basis $(e_i)_{i=1}^{\infty}$ is unconditional, hence being shrinking is equivalent to ℓ_1 not being isomorphic to any subspace of X and this is the case by Proposition 7.

Theorem 4. *Let $X = T[(\mathscr{A}_k, \theta_k)_{k=1}^{\infty}]$ and $X' = T[(\mathscr{A}_k, \theta'_k)_{k=1}^{\infty}]$ with $\theta_k, \theta'_k \in (0,1)$ be such that ℓ_1 is finitely block represented in every block subspace of X and X'. If X and X' are not totally incomparable, then there exists $C \geqslant 0$ such that for all $n \in \mathbb{N}$,*

$$(*) \qquad \frac{1}{C} \leqslant \frac{\lambda_l}{\lambda'_l} \leqslant C.$$

Proof. Denote by $\|\cdot\|$ and $\|\cdot\|'$ the norms of X and X', respectively. By Lemma 5, there exist block sequences $(u_i)_{i=1}^{\infty} \subseteq X$ and $(u'_i)_{i=1}^{\infty} \subseteq X'$ of their respective bases denoted by $(e_i)_{i=1}^{\infty}$ and $(e'_i)_{i=1}^{\infty}$, such that $T\colon \overline{\operatorname{Span}}\{u_i \mid i \in \mathbb{N}\} \longrightarrow \overline{\operatorname{Span}}\{u'_i \mid i \in \mathbb{N}\}$, given by $T(u_i) = u'_i$ for all $i \in \mathbb{N}$ is an isomorphism. Therefore, for all $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ and $n \in \mathbb{N}$,

$$\frac{1}{\|T\|}\Big\|\sum_{i=1}^{n} a_i u'_i\Big\|' \leqslant \Big\|\sum_{i=1}^{n} a_i u_i\Big\| \leqslant \|T^{-1}\|\,\Big\|\sum_{i=1}^{n} a_i u'_i\Big\|'.$$

By Lemma 4, given $\varepsilon > 0$ and $l \in \mathbb{N}$, there exists a normalized block sequence $y_1, \ldots, y_l$ of $(u_i)_{i=1}^{\infty}$, such that

$$\lambda_l - \varepsilon \leqslant \Big\|\sum_{i=1}^{l} y_i\Big\| \leqslant \lambda_l + \varepsilon.$$

Let $y_i' := T(y_i)$ for all $i = 1, \ldots, l$. Then we have

$$\begin{aligned}\lambda_l + \varepsilon &\geqslant \Big\|\sum_{i=1}^{l} y_i\Big\| \geqslant \frac{1}{\|T\|}\Big\|\sum_{i=1}^{l} y_i'\Big\|' \\ &= \frac{1}{\|T\|}\Big\|\sum_{i=1}^{l} \|y_i'\|' \frac{y_i'}{\|y_i'\|'}\Big\|' \geqslant \frac{1}{\|T\|}\min_{1\leqslant i\leqslant l}\|y_i'\|'\Big\|\sum_{i=1}^{l} \frac{y_i'}{\|y_i'\|'}\Big\|' \\ &\geqslant \frac{1}{\|T\|\,\|T^{-1}\|}\Big\|\sum_{i=1}^{l} e_i'\Big\|' = \frac{1}{\|T\|\,\|T^{-1}\|}\lambda_l'\end{aligned}$$

(note that in the last inequality we use Lemma 3). Since the inequality is true for all $\varepsilon > 0$, we have proved that $\lambda_l \geqslant (\|T\|\,\|T^{-1}\|)^{-1}\lambda_l'$.

Now we reverse the roles of X and X' to obtain $(\|T\|\,\|T^{-1}\|)^{-1}\lambda_l' \leqslant \lambda_l \leqslant \|T\|\,\|T^{-1}\|\lambda_l'$. □

Remark 7. If X and X' contain isometric subspaces Y and Y', then $\lambda_l = \lambda_l'$ for all $l \in \mathbb{N}$. Actually, the same equality holds if for every $\varepsilon > 0$, X and X' contain $(1+\varepsilon)$-isomorphic subspaces.

Remark 8. There are special cases when the calculus of λ_l is easy. For instance when (θ_k), (θ_k') belong to the so called class $\mathscr{F}$ defined in [11] we have $\lambda_l = l\cdot\theta_l$ and the condition $(*)$ of Theorem 4 yields $1/C \leqslant \theta_l/\theta_l' \leqslant C$ for all l or $\theta_l = \theta_l'$ if we can find isometric subspaces or $(1+\varepsilon)$-isomorphic subspaces for all $\varepsilon > 0$.

Example 3. Let $f_r(x) = \log_2^r(1+x)$ with $0 < r < 3\log 2 - 1$. Then $(f_r^{-1}(k)) \in \mathscr{F}$ and if $0 < r < s < 3\log 2 - 1$, the spaces $T\big[\big(\mathscr{A}_k, 1/f_r(k)\big)_{k=1}^{\infty}\big]$ and $T\Big[\big(\mathscr{A}_k, 1/f_s(k)\big)_{k=1}^{\infty}\Big]$ are, by Theorem 4, totally incomparable. Moreover, it is easy to check that these spaces are also totally incomparable to ℓ_p, $1 \leqslant p < \infty$ or c_0.

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