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# CONNECTED RESOLVABILITY OF GRAPHS

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*Abstract.* For an ordered set $W = \{w_1, w_2, \dots, w_k\}$ of vertices and a vertex $v$ in a connected graph $G$, the representation of $v$ with respect to $W$ is the $k$-vector $r(v|W) = (d(v, w_1),\ d(v, w_2), \dots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices $x$ and $y$. The set $W$ is a resolving set for $G$ if distinct vertices of $G$ have distinct representations with respect to $W$. A resolving set for $G$ containing a minimum number of vertices is a basis for $G$. The dimension $\dim(G)$ is the number of vertices in a basis for $G$. A resolving set $W$ of $G$ is connected if the subgraph $\langle W\rangle$ induced by $W$ is a nontrivial connected subgraph of $G$. The minimum cardinality of a connected resolving set in a graph $G$ is its connected resolving number $\operatorname{cr}(G)$. Thus $1 \leqslant \dim(G) \leqslant \operatorname{cr}(G) \leqslant n-1$ for every connected graph $G$ of order $n \geqslant 3$. The connected resolving numbers of some well-known graphs are determined. It is shown that if $G$ is a connected graph of order $n \geqslant 3$, then $\operatorname{cr}(G) = n-1$ if and only if $G = K_n$ or $G = K_{1,n-1}$. It is also shown that for positive integers $a$, $b$ with $a \leqslant b$, there exists a connected graph $G$ with $\dim(G) = a$ and $\operatorname{cr}(G) = b$ if and only if $(a, b) \notin \{(1, k)\colon\ k = 1 \text{ or } k \geqslant 3\}$. Several other realization results are present. The connected resolving numbers of the Cartesian products $G \times K_2$ for connected graphs $G$ are studied.

*Keywords*: resolving set, basis, dimension, connected resolving set, connected resolving number

*MSC 2000*: 05C12, 05C25, 05C35

## 1. Introduction

The *distance* $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$–$v$ path in $G$. For an ordered set $W = \{w_1, w_2, \dots, w_k\} \subset V(G)$ and a vertex $v$ of $G$, we refer to the $k$-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$$

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as the (*metric*) *representation of* $v$ *with respect to* $W$. The set $W$ is called a *resolving set* for $G$ if distinct vertices have distinct representations with respect to $W$. A resolving set for $G$ containing a minimum number of vertices is a *minimum resolving set* or a *basis* for $G$. The (*metric*) *dimension* $\dim(G)$ is the number of vertices in a basis for $G$.

The concepts of resolving set and minimum resolving set have previously appeared in the literature. In [9] and later in [10], Slater introduced these ideas and used *locating set* for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph $G$ as its *location number*. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [7] discovered these concepts independently as well but used the term *metric dimension* rather than location number, the terminology that we have adopted. These concepts were rediscovered by Johnson [8] of the Pharmecia Company while attempting to develop a capability of large datasets of chemical graphs. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1], [2], [4]. It was noted in [6, p. 204] that determining the dimension of a graph is an NP-complete problem. The dimension of directed graphs has been studied in [5]. We refer to the book [3] for graph theory notation and terminology not described here.

In this paper, we study the resolving sets of a graph with some additional property. For a nontrivial connected graph $G$, its vertex set $V(G)$ is always a resolving set. Moreover, $\langle V(G)\rangle = G$ is a nontrivial connected graph. A resolving set $W$ of $G$ is *connected* if the subgraph $\langle W\rangle$ induced by $W$ is a nontrivial connected subgraph of $G$. The minimum cardinality of a connected resolving set $W$ in a graph $G$ is the *connected resolving number* $\operatorname{cr}(G)$. A connected resolving set of cardinality $\operatorname{cr}(G)$ is called a $\operatorname{cr}$-*set* of $G$. Since every connected resolving set is a resolving set, $\dim(G) \leqslant \operatorname{cr}(G)$ for all connected graphs $G$. To illustrate this concept, consider the graph $G$ of Figure 1.

$G$:

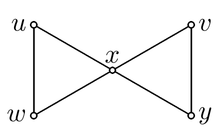


Figure 1. A graph $G$ with $\dim(G) = 2$ and $\operatorname{cr}(G) = 3$

The set $W = \{u, v\}$ is a basis for $G$ and so $\dim(G) = 2$. The representations for the vertices of $G$ with respect to $W$ are

$$
\begin{aligned}
&r(u|W) = (0,2) \qquad r(v|W) = (2,0) \qquad r(w|W) = (1,2)\\
&r(x|W) = (1,1) \qquad r(y|W) = (2,1).
\end{aligned}
$$

Since $\langle\{u, v\}\rangle$ is disconnected, $W$ is not a connected resolving set. On the other hand, the set $W' = \{u, v, x\}$ is a connected resolving set. The representations for the vertices of $G$ with respect to $W'$ are

$$
\begin{aligned}
&r(u|W') = (0,2,1) \qquad r(v|W') = (2,0,1) \qquad r(w|W') = (1,2,1)\\
&r(x|W') = (1,1,0) \qquad r(y|W') = (2,1,1).
\end{aligned}
$$

Since $G$ contains no 2-element connected resolving set, that is, a resolving set consisting of two adjacent vertices, $\mathrm{cr}(G) = 3$.

The example just presented also illustrates an important point. hen determining whether a given set $W$ of vertices of a graph $G$ is a resolving set for $G$, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of $G$ whose distance from $w$ is 0. We make a few other observations that will be of use on several occasions.

**Observation 1.1.** Let $W$ be a set of vertices of a graph $G$. If $W$ contains a resolving set of $G$ as its subset, then $W$ is also a resolving set of $G$.

**Observation 1.2.** If a connected graph $G$ contains a set $S$ of vertices of $G$ of cardinality $p \geqslant 2$ such that $d(u, x) = d(v, x)$ for all $u, v \in S$ and $x \in V(G) - \{u, v\}$, then every resolving set must contain at least $p - 1$ vertices of $S$.

Two vertices $u$ and $v$ of a connected graph $G$ is defined to be *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. Certainly, distance similarity in a graph $G$ is an equivalence relation in $V(G)$. Let $V_1, V_2, \ldots, V_k$ be the $k$ ($k \geqslant 1$) distinct distance-similar equivalence classes of $V(G)$. By Observation 1.2, if $W$ be a resolving set of $G$, then $W$ contains at least $|V_i| - 1$ vertices from each equivalence class $V_i$ for all $i$ with $1 \leqslant i \leqslant k$. Thus we have the following

**Observation 1.3.** Let $G$ be a nontrivial connected graph of order $n$. If $G$ has $k$ distance-similar equivalence classes, then $\dim(G) \geqslant n - k$ and so $\mathrm{cr}(G) \geqslant n - k$

**Observation 1.4.** Let $G$ be a connected graph. Then $\dim(G) = \mathrm{cr}(G)$ if and only if $G$ contains a connected basis.

By Observation 1.2, every basis $W$ of the graph $G$ in Figure 1 contains exactly one vertex from each of the sets $\{u, w\}$ and $\{v, y\}$ and so $W$ is not a connected resolving set. Thus $G$ contains no connected basis and so $\mathrm{cr}(G) > \dim(G)$ by Observation 1.4. On the other hand, by adding the vertex $x$ to a basis of $G$, we obtain a connected resolving set by Observation 1.1. In fact, every cr-set $S$ of $G$ is obtained from a basis of $G$ by adding the vertex $x$ so that $\langle S\rangle$ is a connected. Thus every cr-set of $G$ contains a basis of $G$. However, it is not true in general. For example, consider the graph $H$ of Figure 2. The set $\{u_2, v_1\}$ is a basis of $H$ and so $\dim(H) = 2$. Next we show that $\mathrm{cr}(H) = 3$. Since $\{u_1, u_2, v_1\}$ is a connected resolving set, $\mathrm{cr}(H) \leqslant 3$. Assume, to the contrary, that $\mathrm{cr}(H) = 2$. Let $S = \{x, y\}$ be a cr-set of $H$. By Observation 1.2, $S$ contains at least one of $v_1$ and $v_2$, say $x = v_1$. Since $\langle S\rangle$ is connected, it follows that $y = u_1$ or $y = u_4$. However, neither $\{u_2, v_1\}$ nor $\{u_4, v_1\}$ is a resolving set, which is a contradiction. Thus $\mathrm{cr}(H) = 3$. Notice that the cr-set $\{u_1, u_2, v_1\}$ of $H$ contains a basis $\{u_2, v_1\}$ of $H$, while the cr-set $\{u_1, v_1, v_2\}$ of $H$ contains no basis of $H$.

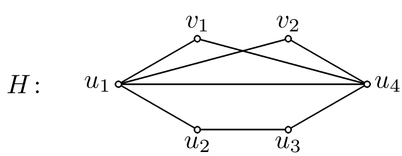

Figure 2. A graph $H$

In fact, for each integer $k \geqslant 4$, the structure of the graph $H$ of Figure 2 can be extended to produce a new graph $G$ by adding the $k-3$ new vertices $v_3, v_4, \ldots, v_{k-1}$ and joining each vertex $v_i$ $(3 \leqslant i \leqslant k-1)$ with the vertices $u_1$ and $u_4$ of $H$. Then the resulting graph $G$ has two cr-sets $S_1 = \{u_1, u_2, v_1, v_2, \ldots, v_{k-2}\}$ and $S_2 = \{u_1, v_1, v_2, \ldots, v_{k-1}\}$ of cardinality $k$, where $S_1$ contains a basis $B$ of $G$, namely $B = \{u_2, v_1, v_2, \ldots, v_{k-2}\}$, and $S_2$ contains no basis of $G$. These observations yield the following

**Proposition 1.5.** *For each integer $k \geqslant 3$, there is a connected graph $G$ with two* cr-*sets $S_1$ and $S_2$ of cardinality $k$ such that $S_1$ contains a basis of $G$ and $S_2$ contains no basis of $G$.*

## 2. Connected resolving numbers of some graphs

The dimensions of some well-known classes of graphs have been determined in [2], [7], [9], [10]. We state these results in the next two theorems.

**Theorem A.** *Let $G$ be a connected graph of order $n \geqslant 2$.*

(a) *Then* $\dim(G) = 1$ *if and only if* $G = P_n$*, the path of order* $n$.

(b) *Then* $\dim(G) = n - 1$ *if and only if* $G = K_n$*, the complete graph of order* $n$.

(c) *For* $n \geqslant 3$, $\dim(C_n) = 2$*, where* $C_n$ *is the cycle of order* $n$.

A vertex of degree at least 3 in a tree $T$ is called a *major vertex*. An end-vertex $u$ of $T$ is said to be a *terminal vertex of a major vertex* $v$ of $T$ if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $T$. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ of $T$ is an *exterior major vertex* of $T$ if it has positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of $T$ and let $\mathrm{ex}(T)$ denote the number of exterior major vertices of $T$. For example, the tree $T$ of Figure 3 has four major vertices, namely, $v_1$, $v_2$, $v_3$, $v_4$. The terminal vertices of $v_1$ are $u_1$ and $u_2$, the terminal vertices of $v_3$ are $u_3$, $u_4$, and $u_5$, and the terminal vertices of $v_4$ are $u_6$ and $u_7$. The major vertex $v_2$ has no terminal vertex and so $v_2$ is not an exterior major vertex of $T$. Therefore, $\sigma(T) = 7$ and $\mathrm{ex}(T) = 3$. We can now state a known formula for the dimension of a tree (see [2], [9]).

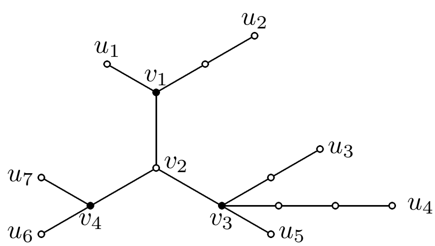


Figure 3. A tree with its major vertices

**Theorem B.** *If $T$ is a tree that is not a path, then*

$$\dim(T) = \sigma(T) - \mathrm{ex}(T).$$

If $G$ is a connected graph of order $n$, then every set of $n - 1$ vertices of $G$ is a resolving set of $G$. Moreover, every nontrivial connected graph $G$ contains a vertex $v$ that is not a cut-vertex and so $V(G) - \{v\}$ is a connected resolving set for $G$. Thus

$$2 \leqslant \mathrm{cr}(G) \leqslant n - 1 \tag{1}$$

for all connected graphs $G$ of order $n \geqslant 3$. The lower and upper bounds in (1) are both sharp. For example, if $G = P_n, C_n$, where $n \geqslant 2$, then any two adjacent vertices of $G$ form a cr-set of $G$ and so we have the following.

**Proposition 2.1.** *Let $n \geqslant 2$. If $G = P_n$, or $G = C_n$ for $n \geqslant 3$, then* $\operatorname{cr}(G) = 2$.

Next, we show that the complete graph $K_n$ and the star $K_{1,n-1}$ are the only connected graphs of order $n \geqslant 3$ having connected resolving number $n-1$.

**Theorem 2.2.** *Let $G$ be a connected graph of order $n \geqslant 3$. Then $\operatorname{cr}(G) = n-1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.*

Proof. Since $\dim(K_n) = n-1$ and every induced subgraph of $K_n$ is connected, $\operatorname{cr}(K_n) = \dim(K_n) = n-1$ by Observation 1.4. For $G = K_{1,n-1}$, let $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$, where $v$ is the central vertex of $G$. By Observation 1.2, every cr-set of $G$ contains at least $n-2$ end-vertices and so $\operatorname{cr}(G) \geqslant n-2$. On the other hand, the subgraph induced by any $n-2$ end-vertices of $G$ is the graph $\overline{K}_{n-2}$, which is not connected, and so $\operatorname{cr}(G) \geqslant n-1$. Hence $\operatorname{cr}(G) = n-1$.

For the converse, if $G$ is a connected graph of order 3, then $G = K_3$ or $G = K_{1,2}$ and the result holds for $n = 3$. Next we show that if $G$ is a connected graph of order $n \geqslant 4$ that is neither a complete graph nor a star, then $\operatorname{cr}(G) \leqslant n-2$. To do this, it suffices to show the following stronger statement: if $G$ is connected graph of order $n \geqslant 4$ that is neither a complete graph nor a star, then $G$ contains distinct vertices $u$, $v$, $w_1$, $w_2$ such that $V(G) - \{u, v\}$ is a connected resolving set and (1) $u$ is adjacent to $w_1$ and (2) $v$ is adjacent to $w_2$ but not to $w_1$. We proceed by induction on the order $n$ of $G$. For $n = 4$, the graphs $G_i$ $(1 \leqslant i \leqslant 4)$ of Figure 4 are only connected graphs order 4 that are different from $K_4$ or $K_{1,3}$. For each $i$ $(1 \leqslant i \leqslant 4)$, the vertices $u$, $v$, $w_1$, $w_2$ are shown in Figure 4 and $W = V(G_i) - \{u, v\} = \{w_1, w_2\}$ is a connected resolving set in $G_i$. Moreover, $u$ is adjacent to $w_1$ and $v$ is adjacent to $w_2$ but not to $w_1$. Thus the statement is true for $n = 4$. Assume that the statement is true for $n - 1 \geqslant 4$.

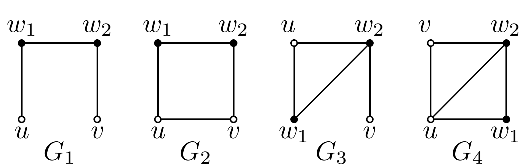


Figure 4. Graphs $G_i$ $(1 \leqslant i \leqslant 4)$

Let $G$ be a connected graph of order $n \geqslant 5$ that is not $K_n$ or $K_{1,n-1}$ and let $x$ be vertex of $G$ such that $G' = G - x$ is connected and $G' \neq K_{n-1}, K_{1,n-2}$. By the induction hypothesis, $G'$ contains distinct vertices $u$, $v$, $w_1$, $w_2$ such that

$W' = V(G') - \{u, v\}$ is a connected resolving set and $u$ is adjacent to $w_1$ and $v$ is is adjacent to $w_2$ but not to $w_1$. Since $G$ is connected, $x$ is adjacent to some vertex in $G$. We consider two cases.

*Case* 1. $x$ is adjacent to at least one vertex in $W'$. Let $W = W' \cup \{x\} = V(G) - \{u, v\}$. Then $\langle W \rangle$ is connected. Since $d(u, w_1) = 1$ and $d(v, w_1) \geqslant 2$, it follows that $r(v|W) \neq r(u|W)$ and so $W$ is a connected resolving set of $G$. Moreover, $u$ is adjacent to $w_1$ and $v$ is adjacent to $w_2$ but not to $w_1$.

*Case* 2. $x$ is adjacent to no vertex in $W'$. There are three subcases.

*Subcase* 2.1. $x$ is adjacent to both $u$ and $v$. Let $W = W' \cup \{v\} = V(G) - \{u, x\}$. Then $\langle W \rangle$ is connected. Since $d(u, w_1) = 1$ and $d(x, w_1) = 2$, it follows that $r(u|W) \neq r(x|W)$ and so $W$ is connected resolving set of $G$. Moreover, $u$ is adjacent to $w_1$ and $x$ is adjacent to $v$ but not to $w_1$.

*Subcase* 2.2. $x$ is adjacent to $u$ but not to $v$. Let $W = W' \cup \{u\} = V(G) - \{v, x\}$. Then $\langle W \rangle$ is connected. Since $d(v, w_2) = 1$ and $d(x, w_2) \geqslant 3$, it follows that $r(v|W) \neq r(x|W)$ and so $W$ is connected resolving set of $G$. Moreover, $v$ is adjacent to $w_2$ and $x$ is adjacent to $u$ but not to $w_2$.

*Subcase* 2.3. $x$ is adjacent to $v$ but not to $u$. Let $W = W' \cup \{v\} = V(G) - \{u, x\}$. Then $\langle W \rangle$ is connected. Since $d(u, w_1) = 1$ and $d(x, w_1) \geqslant 2$, it follows that $r(u|W) \neq r(x|W)$ and $W$ is connected resolving set of $G$. Moreover, $u$ is adjacent to $w_1$ and $x$ is is adjacent to $v$ but not to $w_1$.

Thus, in either case, $G$ contains a connected resolving set of cardinality $n - 2$. Therefore, $\operatorname{cr}(G) \leqslant n - 2$. □

We now determine the connected resolving numbers of complete $k$-partite ($k \geqslant 2$) graphs that are not stars.

**Proposition 2.3.** *For $k \geqslant 2$, let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete $k$-partite graph that is not a star. Let $n = n_1 + n_2 + \ldots + n_k$ and $l$ be the number of one's in $\{n_i \colon 1 \leqslant i \leqslant k\}$. Then*

$$\operatorname{cr}(G) = \begin{cases} n - k & \text{if } l = 0 \\ n - k + l - 1 & \text{if } l \geqslant 1. \end{cases}$$

Proof. Assume that $1 \leqslant n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$. For each $i$ with $1 \leqslant i \leqslant k$, let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ be a partite set of $G$. We consider two cases.

*Case* 1. $l = 0$. Then $n_i \geqslant 2$ for all $i$ with $1 \leqslant i \leqslant k$. Since $G$ has $k$ distinct distance-similar equivalence classes, namely $V_1, V_2, \ldots, V_k$, it follows from Observation 1.3 that $\operatorname{cr}(G) \geqslant n - k$. On the other hand, let $W = \bigcup\limits_{i=1}^{k} (V_i - \{v_{i,1}\})$. Since $W$ is a resolving set of $G$ and $\langle W \rangle = K_{n_1-1,n_2-1,\ldots,n_k-1}$ is connected, $W$ is a connected resolving set and so $\operatorname{cr}(G) \leqslant |W| = n-k$. Thus $\operatorname{cr}(G) = n-k$.

*Case* 2. $l \geqslant 1$. Then $n_i = 1$ for all $1 \leqslant i \leqslant l$ and $n_i \geqslant 2$ for all $l+1 \leqslant i \leqslant k$. Let $U_1 = \bigcup_{i=1}^{l} V_i = \{v_{11}, v_{21}, \ldots, v_{l1}\}$ and $U_j = V_{l+j-1}$ for all $j$ with $2 \leqslant j \leqslant k-l+1$. Then $U_1, U_2, \ldots, U_{k-l+1}$ are $k-l+1$ distinct distance-similar equivalence classes and so $\operatorname{cr}(G) \geqslant n-(k-l+1) = n-k+l-1$ by Observation 1.3. On the other hand, let

$$W = \{v_{21}, \ldots, v_{l1}\} \bigcup \Bigg( \bigcup_{i=l+1}^{k} (V_i - \{v_{i,1}\}) \Bigg).$$

Then $\langle W \rangle$ is connected. Since $d(v_{11}, w) = 1$ for all $w \in W$, $d(v_{i1}, w) = 1$ if $w \in W - (V_i - \{v_{i,1}\})$ and $d(v_{i1}, w) = 2$ if $w \in V_i - \{v_{i,1}\}$ for all $i$ with $l+1 \leqslant i \leqslant k$, it follows that $W$ is a resolving set. Thus $W$ is a connected resolving set and so $\operatorname{cr}(G) \leqslant |W| = n-k+l-1$. Therefore, $\operatorname{cr}(G) = |W| = n-k+l-1$. □

To determine the connected resolving numbers of trees that are not paths, we first state a lemma. We omit the proof of this lemma since it is straightforward.

**Lemma 2.4.** *Let $T$ be a nonpath tree of order $n \geqslant 4$ having $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of $v_i$, and let $P_{ij}$ be the $v_i - u_{ij}$ path ($1 \leqslant j \leqslant k_i$). Suppose that $W$ is a set of vertices of $T$. Then $W$ is a resolving set of $T$ if and only if $W$ contains at least one vertex from each of the paths $P_{ij} - v_i$ ($1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p$) with at most one exception for each $i$ with $1 \leqslant i \leqslant p$.*

Using Lemma 2.4, we are able to characterize the cr-sets in a tree $T$. Again, we omit the proof of the following result since it is routine.

**Theorem 2.5.** *Let $T$ be a nonpath tree of order $n \geqslant 4$ having $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of $v_i$, and let $P_{ij}$ be the $v_i - u_{ij}$ path ($1 \leqslant j \leqslant k_i$). Suppose that $W$ is a set of vertices of $T$. Then $W$ is a* cr-*set of $T$ if and only if*

(a) *$W$ contains exactly one vertex from each of the paths $P_{ij} - v_i$, $1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p$, with exactly one exception for each $i$ with $1 \leqslant i \leqslant p$,*
(b) *for each pair $i, j$ with $1 \leqslant j \leqslant k_i$ and $1 \leqslant i \leqslant p$, if $x_{ij} \in W$, then $x_{ij}$ is adjacent to $v_i$ in the path $P_{ij}$,*
(c) *$W$ contains all vertices in the paths between any two vertices described in* (b).

**Corollary 2.6.** *Let $T$ be a nonpath tree of order $n \geqslant 4$ having $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices* *of $v_i$ and let $P_{ij}$ be the $v_i - u_{ij}$ path of length $l_{ij}$ $(1 \leqslant j \leqslant k_i)$. Then*

$$\mathrm{cr}(T) = n + \dim(T) - \sum_{i,j} l_{ij}.$$

## 3. Graphs with prescribed connected resolving numbers and other parameters

We have seen that if $G$ is a connected graph of order $n$ with $\mathrm{cr}(G) = k$, then $2 \leqslant k \leqslant n-1$. In fact, every pair $k, n$ of integers with $2 \leqslant k \leqslant n-1$ is realizable as the connected resolving number and order of some graph as we show next.

**Theorem 3.1.** *For each pair $k, n$ with $2 \leqslant k \leqslant n-1$, there is a connected graph $G$ of order $n$ with connected resolving number $k$.*

Proof. For $k = 2$, let $G = P_n$, which has the desired property. For $k \geqslant 3$ let $G$ be that graph obtained from the path $P_{n-k+1}\colon u_1, u_2, \ldots, u_{n-k+1}$ by adding the $k-1$ new vertices $v_i$ $(1 \leqslant i \leqslant k-1)$ and joining each $v_i$ to $u_1$. Then the order of $G$ is $n$ and $\mathrm{cr}(G) = k$ by Corollary 2.6. $\square$

If $G$ is connected graph with $\dim(G) = a$ and $\mathrm{cr}(G) = b$, then $a \leqslant b$. Next we show that every pair $a, b$ of integers with $2 \leqslant a \leqslant b$ is realizable as the dimension and connected resolving number of some connected graph.

**Theorem 3.2.** *For every pairs $a, b$ of integers with $2 \leqslant a \leqslant b$, there exists a connected graph $G$ such that $\dim(G) = a$ and $\mathrm{cr}(G) = b$.*

Proof. For $b = a \geqslant 2$, let $G = K_{a+1}$; while for $b = a+1$, let $G = K_{1,a+1}$. Then the graph $G$ has the desired properties by Theorems A, B, and 2.2. So we may assume that $b \geqslant a+2$. Let $G$ be obtained from the path $P_{b-a+2}\colon u_1, u_2, \ldots, u_{b-a+2}$ of order $b-a+2$ by adding the $a$ new vertices $v_1, v_2, \ldots, v_a$ and the $a$ edges $v_iu_2$ $(1 \leqslant i \leqslant a-1)$ and $v_au_{b-a+1}$. The graph $G$ is shown in Figure 5. It then follows from Theorem B and Corollary 2.6 that $\dim(G) = a$ and $\mathrm{cr}(G) = b$, as desired. $\square$

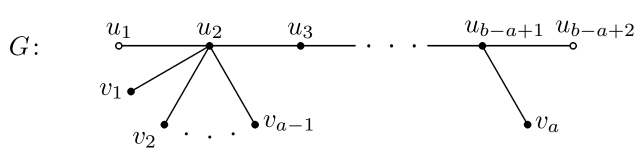


Figure 5. A graph $G$ with $\dim(G) = a$ and $\mathrm{cr}(G) = b$

By Theorem A, the path $P_n$ of order $n \geqslant 2$ is the only nontrivial connected graph with dimension 1. Since $\mathrm{cr}(P_n) = 2$, there is no connected graph $G$ with $\dim(G) = 1$ and $\mathrm{cr}(G) = k$ for some integer $k \neq 2$. Thus the following is a consequence of Theorem 3.2.

**Corollary 3.3.** *For positive integers $a$, $b$ with $a \leqslant b$, there exists a connected graph $G$ with $\dim(G) = a$ and $\mathrm{cr}(G) = b$ if and only if $(a, b) \notin \{(1, k)\colon k = 1$ or $k \geqslant 3\}$.*

By Theorem 2.2, if $G$ is a connected graph of order $n \geqslant 3$, then $\mathrm{cr}(G) = n-1$ if and only if $G = K_n, K_{1,n-1}$. Since $\dim(K_n) = n-1$ and $\dim(K_{1,n-1}) = n-2$, there is no connected graph $G$ of order $n \geqslant 5$ such that $\mathrm{cr}(G) = n-1$ and $1 \leqslant \dim(G) \leqslant n-3$. However, we have the following realization result.

**Theorem 3.4.** *Let $a$, $b$, $n$ be integers with $n \geqslant 5$. If $n-2 \leqslant a \leqslant b = n-1$ or $2 \leqslant a \leqslant b \leqslant n-2$, then there exists a connected graph $G$ of order $n$ such that $\dim(G) = a$ and $\mathrm{cr}(G) = b$.*

Proof. First, assume that $b = n-1$. For $a = n-1$, let $G = K_n$; while for $a = n-2$, let $G = K_{1,n-1}$. Since $\dim(K_n) = \mathrm{cr}(K_n) = n-1$, $\dim(K_{1,n-1}) = n-2$, and $\mathrm{cr}(K_{1,n-1}) = n-1$, the result holds for $b = n-1$. Now assume that $2 \leqslant a \leqslant b \leqslant n-2$. We consider two cases.

*Case* 1. $a = b$. Let $G$ be the graph obtained from the complete graph $K_{a+1}$, where $V(K_{a+1}) = \{u_1, u_2, \ldots, u_{a+1}\}$, and the path $P_{n-a-1}\colon v_1, v_2, \ldots, v_{n-a-1}$ by joining $u_{a+1}$ with $v_1$. Then the order of $G$ is $n$. By Observation 1.2, every resolving set contains at least $a$ vertices from $V(K_{a+1})$ and so $\mathrm{cr}(G) \geqslant \dim(G) \geqslant a$. On the other hand, let $W = \{u_1, u_2, \ldots, u_a\}$. Since $r(u_{a+1}|W) = (1, 1, \ldots, 1)$, $r(v_i|W) = (i+1, i+1, \ldots, i+1)$ for all $i$ with $1 \leqslant i \leqslant n-a-1$, and $\langle W \rangle = K_a$, it follows that $W$ is a connected resolving set. Thus $\dim(G) \leqslant |W| = a$ and $\mathrm{cr}(G) \leqslant |W| = a$. Therefore, $\dim(G) = \mathrm{cr}(G) = a$.

*Case* 2. $a < b$. If $b = a+1$, let $G$ be the graph obtained from the path $P\colon u_1, u_2, \ldots, u_{n-a}$ by adding the $a$ new vertices $v_1, v_2, \ldots, v_a$ and joining each $v_i$ ($1 \leqslant i \leqslant a$) to the end-vertex $u_1$ in $P$. Then the order of $G$ is $n$ and by Theorem B and Corollary 2.6, $\dim(G) = a$ and $\mathrm{cr}(G) = a+1 = b$. If $b \geqslant a+2$, let $G$ be the graph obtained from the path $P\colon u_1, u_2, \ldots, u_{n-a}$ by adding the $a$ new vertices $v_1, v_2, \ldots, v_a$ and joining each $v_i$ ($1 \leqslant i \leqslant a-1$) to $u_2$ and joining $v_a$ to $u_{b-a+1}$. Then the order of $G$ is $n$, $\dim(G) = a$, and $\mathrm{cr}(G) = b$ by Theorem B and Corollary 2.6. $\square$

In this section, we study the connected resolving numbers of the Cartesian products of a nontrivial connected graph $G$ and $K_2$. In the construction of $G \times K_2$, we have two copies $G_1$ and $G_2$ of $G$, where if $v_1 v_2$ is an edge of $G \times K_2$ and $v_i \in V(G_i)$, $i = 1, 2$, then $v_1$ and $v_2$ are said to *correspond* to each other. It was shown in [2] that $\dim(G) \leqslant \dim(G \times K_2) \leqslant \dim(G) + 1$ for every graph $G$. We show that it is also true for the connected resolving numbers. First, we state a lemma without proof.

**Lemma 4.1.** *Let $G$ be a nontrivial connected graph and let $G_1$ and $G_2$ be two copies of $G$ in $G \times K_2$. For a set $W$ of vertices in $G \times K_2$, let $W_1$ be the union of those vertices of $G_1$ belonging to $W$ and those vertices of $G_1$ corresponding to vertices of $G_2$ that belong to $W$. If $\langle W \rangle$ is connected in $G \times K_2$, then $\langle W_1 \rangle$ is connected in $G_1$.*

**Theorem 4.2.** *For a nontrivial connected graph $G$,*

$$\operatorname{cr}(G) \leqslant \operatorname{cr}(G \times K_2) \leqslant \operatorname{cr}(G) + 1.$$

Proof. Let $G \times K_2$ be formed from two copies $G_1$ and $G_2$ of $G$. We first show that $\operatorname{cr}(G) \leqslant \operatorname{cr}(G \times K_2)$. Let $W$ be a cr-set of $G \times K_2$ and $W_1$ be the union of those vertices of $G_1$ belonging to $W$ and those vertices of $G_1$ corresponding to vertices of $G_2$ that belong to $W$. Then $W_1 \subset V(G_1)$ and $|W_1| \leqslant |W|$. By Lemma 4.1, $\langle W_1 \rangle$ is connected in $G_1$. Thus it remains to show that $W_1$ is a resolving set of $G_1$. Observe that if $u \in V(G_1)$, then $d_{G_1}(u, w') = d_{G \times K_2}(u, w')$ for $w' \in W \cap V(G_1)$ and $d_{G_1}(u, w') = d_{G \times K_2}(u, w) - 1$ for $w' \notin W \cap V(G_1)$, where $w \in W \cap V(G_2)$ which corresponds to $w'$. This implies that $W_1$ is a resolving set of $G_1$.

Next, we show that $\operatorname{cr}(G \times K_2) \leqslant \operatorname{cr}(G) + 1$. Suppose that $W = \{w_1, w_2, \ldots, w_k\}$ is a cr-set of $G$. Let $W_1 = \{x_1, x_2, \ldots, x_k\}$ and $W_2 = \{y_1, y_2, \ldots, y_k\}$ be the corresponding cr-sets in $G_1$ and $G_2$, respectively. Let $W' = W_1 \cup \{y_1\}$. We show that $W'$ is a connected resolving set of $G \times K_2$, which implies that $\operatorname{cr}(G \times K_2) \leqslant \operatorname{cr}(G) + 1$. Since $\langle W_1 \rangle$ is a connected subgraph of $G_1$ and $\langle W' \rangle$ is obtained from $\langle W_1 \rangle$ by adding the pendant edge $x_1 y_1$, it follows that $\langle W' \rangle$ is a connected subgraph of $G \times K_2$. Thus it remains to verify that $W'$ is a resolving set of $G \times K_2$. Observe that if $u \in V(G_1)$, then $d_{G \times K_2}(u, x_i) = d_{G_1}(u, x_i)$ $(1 \leqslant i \leqslant k)$ and $d_{G \times K_2}(u, y_1) = d_{G_1}(u, x_1) + 1$; while if $u \in V(G_2)$, then $d_{G \times K_2}(u, x_i) = d_{G_1}(v, x_i) + 1$, where $v \in V(G_1)$ corresponds to $u$ and $d_{G \times K_2}(u, y_1) = d_{G_1}(v, x_1)$. Thus if $u \in V(G_1)$, then

$$r(u|W') = (d_{G_1}(u, x_1), d_{G_1}(u, x_2), \ldots, d_{G_1}(u, x_k), d_{G_1}(u, x_1) + 1).$$

If $u \in V(G_2)$, then

$$r(u|W') = (d_{G_1}(v, x_1) + 1, d_{G_1}(v, x_2) + 1, \dots, d_{G_1}(v, x_k) + 1, d_{G_1}(v, x_1))$$

where $v \in V(G_1)$ corresponds to $u$. Since $W_1$ is a resolving set of $G_1$, it follows that the representations $r(u|W')$ are distinct and so $W'$ is a resolving set of $G \times K_2$. $\square$

Both upper and lower bounds in Theorem 4.2 are sharp as we see in the next two results. We will prove only the second result since the proof of the first one is routine.

**Proposition 4.3.** *Let $k, n \geqslant 2$ be integers.*

(a) *If $G = K_n, P_n$, or $G = K_{n_1, n_2, \dots, n_k}$ that is not a star, then $\mathrm{cr}(G \times K_2) = \mathrm{cr}(G)$.*

(b) *If $n \geqslant 4$, then $\mathrm{cr}(C_n \times K_2) = \mathrm{cr}(C_n) + 1$.*

**Proposition 4.4.** *If $T$ is a nontrivial tree that is not a path, then*

$$\mathrm{cr}(T \times K_2) = \mathrm{cr}(T) + 1.$$

Proof. Let $T_1$ and $T_2$ be two copies of $T$ in $T \times K_2$, where $V(T_1) = \{u_1, u_2, \dots, u_n\}$, $V(T_2) = \{v_1, v_2, \dots, v_n\}$, and $u_i$ corresponds to $v_i$ for all $i$ with $1 \leqslant i \leqslant n$. By Theorem 4.2, it suffices to show that $\mathrm{cr}(T \times K_2) \neq \mathrm{cr}(T)$. Assume, to the contrary, that $\mathrm{cr}(T \times K_2) = \mathrm{cr}(T)$. Let $W$ be a cr-set of $T \times K_2$. We consider two cases.

*Case* 1. $W \subset V(T_1)$ or $W \subset V(T_2)$, say the former. Then $W$ is also cr-set of $T_1$. Let $u$ be an exterior major vertex of $T_1$ and $z_1, z_2, \dots, z_k$ ($k \geqslant 2$) be the terminal vertices of $u$ in $T$. For each $i$ with $1 \leqslant i \leqslant k$, let $P_i$ be the $u - z_i$ path in $T$ and $x_i$ is the vertex in $P_i$ that is adjacent to $v$. By Theorem 2.5, $u \in W$ and exactly one of $x_1, x_2, \dots, x_k$ does not belong to $W$, say $x_k \notin W$. Let $v$ be the vertex in $T_2$ that corresponds to $u$. Then $r(x_k|W) = r(v|W)$, which is a contradiction.

*Case* 2. $W \cap V(T_1) \neq \emptyset$ and $W \cap V(T_2) \neq \emptyset$. Let $W_1$ be the union of those vertices of $G_1$ belonging to $W$ and those vertices of $T_1$ corresponding to vertices of $T_2$ that belong to $W$. We claim that $|W_1| < |W|$. If $|W_1| = |W|$, then $W$ contains at most one of $u_i$ and $v_i$ for each pair $u_i, v_i$ ($1 \leqslant i \leqslant n$) of corresponding vertices in $T \times K_2$. This implies that $\langle W \rangle$ is disconnected, a contradiction. Thus $|W_1| < |W|$, as claimed. Since $|W| = \mathrm{cr}(T)$, it follows that $W_1$ is not a cr-set of $T_1$. However, by Lemma 4.1 $\langle W_1 \rangle$ is connected and so $W_1$ is not a resolving set of $T_1$. Let $x, y \in V(T_1)$ such that $r(x|W_1) = r(y|W_1)$. Assume that $W_1 = \{w_1, w_2, \dots, w_s, w_{s+1}, \dots, w_k\}$, where $w_i \in W$ for all $1 \leqslant i \leqslant s$ and $w_i \notin W$ for all $s + 1 \leqslant i \leqslant k$. Then $W = \{w_1, w_2, \dots, w_s, w'_{s+1}, \dots, w'_k\}$, where $w'_i$ corresponds to $w_i$ for all $s+1 \leqslant i \leqslant k$.

Then $d_{T\times K_2}(x,w_i)=d_{T_1}(x,w_i)$ for all $1\leqslant i\leqslant s$ and $d_{T\times K_2}(x,w_i')=d_{T_1}(x,w_i)+1$ for all $s+1\leqslant i\leqslant k$. Similarly, since $d_{T\times K_2}(y,w_i)=d_{T_1}(y,w_i)$ for all $1\leqslant i\leqslant s$ and $d_{T\times K_2}(y,w_i')=d_{T_1}(y,w_i)+1$ for $s+1\leqslant i\leqslant k$, it follows that $r(x|W)=r(y|W)$, which is a contradiction. □

Let $G$ be a nontrivial connected graph and $H$ a connected induced subgraph of $G$. We defined the *connected resolving ratio* of $G$ and $H$ by

$$r_{\mathrm{cr}}(H,G)=\frac{\mathrm{cr}(H)}{\mathrm{cr}(G)}.$$

By Theorem 2.2 that $\mathrm{cr}(K_{1,m})=m$ for all $m\geqslant 2$. Hence for $G=K_{1,m}$ and $H=K_2$, we can make the ratio $r_{\mathrm{cr}}(H,G)$ as small as we wish by choosing $m$ arbitrarily large. Although this may not be surprising, it may be unexpected that, in fact, we can make $r_{\mathrm{cr}}(H,G)$ as large as we wish. We now establish the truth of this statement.

For $n\geqslant 3$, we label the vertices of the star $K_{1,2^{n+1}}$ with $v_0,v_1,v_2,\ldots,v_{2^n}$, $v_1',v_2',\ldots,v_{2^n}'$, where $v_0$ is the central vertex. Then we add two new vertices $x$ and $x'$ and $2^{n+1}$ edges $xv_i$ and $x'v_i'$ for $1\leqslant i\leqslant 2^n$. Next, we add two sets $W=\{w_1,w_2,\ldots,w_n\}$ and $W'=\{w_1',w_2',\ldots,w_n'\}$ of vertices, together with the edges $w_ix$ and $w_i'x'$ for $1\leqslant i\leqslant n$. Finally, we add edges between $W$ and $\{v_0,v_1,v_2,\ldots,v_{2^n}\}$ so that each of the $2^n$ possible $n$-tuples of 1s and 2s appears exactly once such that the representations $(d(v_i,w_1),d(v_i,w_2),\ldots,d(v_i,w_n))$ are distinct for $1\leqslant i\leqslant 2^n$. Similarly, edges are added between $W'$ and $\{v_1',v_2',\ldots,v_{2^n}'\}$ so that $(d(v_i',w_1'),d(v_i',w_2'),\ldots,d(v_i',w_n'))$ are distinct for $1\leqslant i\leqslant 2^n$. Denote the resulting graph by $G$. The graph $G$ for $n=3$ is shown in Figure 6.

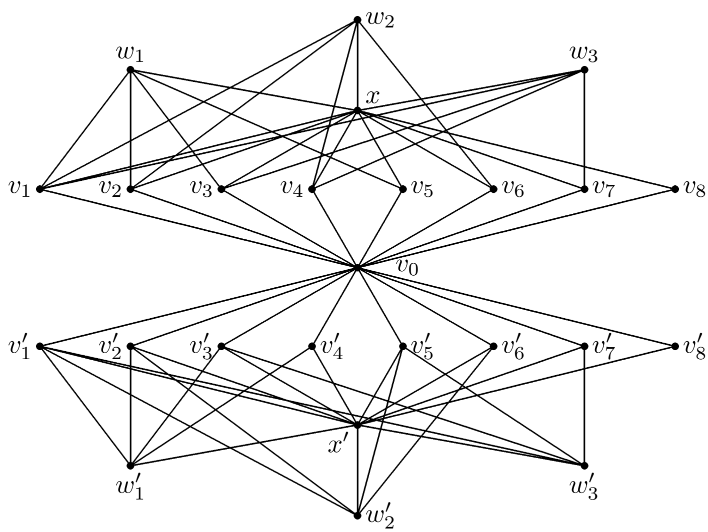


Figure 6. The graph $G$

Let $W^* = W \cup W' \cup \{v_0, v_1, v_1', x, x'\}$. We claim that $W^*$ is a connected resolving set of $G$. By construction, $\langle W^* \rangle$ is a connected subgraph of $G$. Moreover, $r(v_i|W^*) = r(v_j|W^*)$ implies that $i = j$ and $r(v_i'|W^*) = r(v_j'|W^*)$ implies that $i = j$. Observe that

$$r(v_i|W^*) = (*, *, \ldots, *, 3, 3, \ldots, 3, 3, 1, 2, 2, 3),\ \ 2 \leqslant i \leqslant 2^n$$
$$r(v_i'|W^*) = (3, 3, \ldots, 3, *, *, \ldots, *, 1, 2, 2, 2, 1),\ \ 2 \leqslant i \leqslant 2^n,$$

where $*$ represents an irrelevant coordinate. Thus $W^*$ is a connected resolving set of $G$ and so $\operatorname{cr}(G) \leqslant |W^*| = 2n+5$. Note that $G$ contains $H = K_{1,2^{n+1}}$ as an induced subgraph and

$$\frac{\operatorname{cr}(H)}{\operatorname{cr}(G)} \geqslant \frac{2^{n+1}}{2n+5}.$$

Since

$$\lim_{n\to\infty} \frac{2^{n+1}}{2n+5} = \infty$$

there exists a graph $G$ and an induced subgraph $H$ of $G$ such that $r_{\mathrm{cr}}(H, G) = \operatorname{cr}(H)/\operatorname{cr}(G)$ is arbitrary large.

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