Ülo Lumiste

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SEMIPARALLEL ISOMETRIC IMMERSIONS OF 3-DIMENSIONAL SEMISYMMETRIC RIEMANNIAN MANIFOLDS

Ülo Lumiste, Tartu

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Abstract. A Riemannian manifold is said to be semisymmetric if $R(X,Y)\cdot R=0$. A submanifold of Euclidean space which satisfies $\overline{R}(X,Y)\cdot h=0$ is called semiparallel. It is known that semiparallel submanifolds are intrinsically semisymmetric. But can every semisymmetric manifold be immersed isometrically as a semiparallel submanifold? This problem has been solved up to now only for the dimension 2, when the answer is affirmative for the positive Gaussian curvature. Among semisymmetric manifolds a special role is played by the foliated ones, which in the dimension 3 are divided by Kowalski into four classes: elliptic, hyperbolic, parabolic and planar. It is shown now that only the planar ones can be immersed isometrically into Euclidean spaces as 3-dimensional semiparallel submanifolds. This result is obtained by a complete classification of such submanifolds.

Keywords: semisymmetric Riemannian manifolds, semiparallel submanifolds, isometric immersions, planar foliated manifolds

MSC 2000: 53C25, 53C42, 53B25

Introduction

1. A Riemannian manifold M^m is said to be *locally symmetric* if at its arbitrary point the geodesic reflection, defined in general only locally, is local isometry. By the famous result of É. Cartan the analytic condition expressing local symmetry is that the Riemannian curvature tensor R be parallel with respect to the Levi-Cività connection ∇, i.e. $\nabla R=0$.

In the extrinsic theory of submanifolds M^m in Euclidean spaces $\mathbb{E}^n$ the analogous concept of locally symmetric (extrinsically) submanifold was introduced by D. Ferus [5] and W. Strübing [24] using the normal reflection as follows.

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For a submanifold M^m in $\mathbb{E}^n$ at every its point $x \in M^m$ the tangent vector subspace T_xM^m is complemented by the normal vector subspace $T_x^{\perp}M^m$ so that $T_xM^m \oplus T_x^{\perp}M^m$ is the orthogonal decomposition of the vector space $\mathbb{E}^n$. Here $T_x^{\perp}M^m$ together with x determines the normal $(n-m)$-plane of M^m at x in $\mathbb{E}^n$. The reflection σ_x of $\mathbb{E}^n$ with respect to this $(n-m)$-plane, which maps a point $y \in \mathbb{E}^n$ into a point $\sigma_x(y) \in \mathbb{E}^n$, symmetric to y with respect to this plane, is called the *normal reflection* σ_x for M^m at x.

A submanifold M^m in $\mathbb{E}^n$ is said to be *locally symmetric* (*extrinsically*) if for all points $x \in M^m$ the normal reflection σ_x for M^m at x induces a local isometry of M^m.

Ferus [5] and Strübing [24] showed that a submanifold M^m in $\mathbb{E}^n$ is a locally symmetric (extrinsically) submanifold if and only if its second fundamental form h is *parallel*, i.e. $\tilde{\nabla}h = 0$, with respect to the van der Waerden-Bortolotti connection $\tilde{\nabla} = \nabla \oplus \nabla^{\perp}$, where ∇ is the Levi-Cività connection and $\nabla^{\perp}$ is the normal connection of M^m.

This result gave rise to calling the submanifolds with parallel second fundamental form (previously studied by J. Vilms [28]), which coincide with the locally symmetric (extrinsically) submanifolds, the *parallel* submanifolds by M. Takeuchi [27]. Now this name has become the most popular one.

The well-known Gauss equation, which expresses the curvature tensor R of a submanifold M^m by means of its second fundamental form h, shows that every parallel submanifold M^m in $\mathbb{E}^n$ is intrinsically a Riemannian locally symmetric manifold. By a result due to Ferus [5], it is not an arbitrary one, but in the irreducible case is locally a symmetric R-space, and the immersion $M^m \to \mathbb{E}^n$ is locally a standard imbedding of this symmetric R-space.

2. The condition $\nabla R = 0$ of local symmetry of a Riemannian manifold M^m is a differential system. The integrability condition for this system is the point-wise condition $R(X,Y) \cdot R = 0$, where $R(X,Y)$, for arbitrary tangent vector fields X and Y, is a linear operator acting as a derivation on the curvature tensor R. This last condition was introduced by É. Cartan [3] (independently in 1943 also by P. A. Shirokov; see [21], p. 389). Its importance for geodesic maps was shown by N. S. Sinjukov in [22], where the Riemannian manifolds satisfying this condition were called *semisymmetric* (see also [23]). K. Nomizu conjectered in [19] that all complete, irreducible semisymmetric Riemannian manifolds M^m, $m \geqslant 3$, are locally symmetric, but soon this conjecture was refuted in [26] (for $m = 3$) and [20]. For $m = 2$ the situation is trivial: every Riemannian M^2 is semisymmetric.

The extrinsic analogue of this is the integrability condition $\overline{R}(X,Y) \cdot h = 0$ for the differential system $\tilde{\nabla}h = 0$, which characterizes the parallel submanifolds. Here $\overline{R}(X,Y)$ is the curvature operator of the van der Waerden-Bortolotti connection. The Riemannian submanifolds M^m in $\mathbb{E}^n$ satisfying this condition are called *semisym-*

metric (*extrinsically*) [8]–[11], or more often *semiparallel* [4], [14]. Intrinsically every semiparallel submanifold is a semisymmetric Riemannian manifold; this follows again from the Gauss equation and the expressions for the curvature tensors of $\nabla^{\perp}$ (see [4], [15]).

Geometrically, semiparallel submanifolds are characterized as the second order envelopes of the parallel ones (see [10]). For the corresponding theorem some generalizations have been given recently in [17], [18]. It has inspired also a purely intrinsic consideration: it is established in [7] that the metric of each semisymmetric Riemannian manifold is a second order envelope of a family of locally symmetric metrics (see also [6]).

This analogy gives rise to the question what is correspondence between semisymmetric Riemannian manifolds, on the one hand, and semiparallel submanifolds on the other hand. More explicitly, can every semisymmetric manifold be immersed isometrically as a semiparallel submanifold of an Euclidean space?

Recall that for the particular case of symmetric manifolds and parallel submanifolds the answer is negative: due to Ferus' result only symmetric R-spaces can be immersed isometrically as parallel submanifolds. But what will be the answer like in our more general situation?

Up to now only the two-dimensional case has been investigated in [14]. The result can be summarized in the following way.

It is known that every two-dimensional Riemannian manifold M^2 is semisymmetric.

In [4] all semiparallel surfaces M^2 in $\mathbb{E}^n$ were divided into three classes: (i) totally geodesic or totally umbilical surfaces (i.e. planes or spheres or their open parts), (ii) surfaces with zero Gaussian curvature and flat normal connection, (iii) the second order envelopes of Veronese surfaces. (Here the description of the class (iii) is modified using the result of [10]; see also [15].)

It is shown in [9] that if $n = 5$ then the only semisymmetric surfaces of the class (iii) in $\mathbb{E}^5$ are the parallel ones, namely the single Veronese surfaces, every one of which has constant positive Gaussian curvature K. In [14] there is added that into pseudo-Euclidean space ${}_3\mathbb{E}^5$ (with 3 minus signs in the canonical form of $\mathrm{d}s^2$) also M^2 of negative constant K can be immersed isometrically as a semiparallel surface of the class (iii), again as a parallel one.

The main result in [14] is that into ${}_s\mathbb{E}^6$ with $s \in \{0, 3, 4\}$ also a Riemannian M^2 of non-constant Gaussian curvature K can be immersed isometrically as a non-parallel semiparallel surface of the class (iii), but in ${}_s\mathbb{E}^7$ with $s \in \{0, 3, 4, 5\}$ every Riemannian M^2 can be isometrically realized in the class (iii).

So it turns out that for $m = 2$ the situation depends essentially on the dimension n and on the number s of the negative coefficients in the canonical form of the metric quadratic form of the ambient space ${}_s\mathbb{E}^n$.

In the present paper the same problem is investigated for the dimension $m = 3$ and only for the Euclidean ambient space $\mathbb{E}^n$: can every semisymmetric Riemannian M^3 be immersed isometrically into some $\mathbb{E}^n$ as a semiparallel submanifold?

An irreducible semisymmetric M^3 is, by a result, due to Szabó [25], either locally symmetric or foliated by Euclidean leaves of codimension two. The latter were further divided by Kowalski [6] into four classes, formed by the elliptic, hyperbolic, parabolic and planar spaces, respectively.

The aim of the present paper is to prove the following

Main Theorem. *Let (M^3, g) be a Riemannian 3-manifold which can be immersed isometrically into an Euclidean space $\mathbb{E}^n$ as a semiparallel submanifold. Then either (M^3, g) is a space of constant curvature, locally, or it is a foliated semisymmetric space of planar type.*

Note that a general Riemannian M^3 is not semisymmetric any more. Therefore first the classification of semisymmetric Riemannian M^m is needed, especially for $m = 3$. This is given, according to [25] and [6], in Section 1, where especially the planar foliated M^3 are characterized.

The classification of three-dimensional semiparallel submanifolds in Euclidean spaces $\mathbb{E}^n$ is given separately for submanifolds whose principal normal subspace has dimension $m_1 \leqslant 3$ (Section 3), and for those with $3 < m_1 \leqslant 6$ (Section 4). Here our earlier publications [16] and [11] could be used with some additions concerning the inner metric. A special attention is given to submanifolds which are intrinsically planar foliated manifolds.

The proof of the Main Theorem in Section 5 reduces then to a comparison of these two classifications. The submanifolds realizing the foliated semisymmetric 3-manifolds are described geometrically in Section 6. Finally, some concluding remarks are formulated in Section 7.

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1. Classification of semisymmetric Riemannian M^3

The general local classification of the semisymmetric Riemannian M^m was made by Z. I. Szabó in [25]. First he proved by means of the infinitesimal or the local holonomy group that for every semisymmetric Riemannian manifold M^m there exists a dense open subset U such that around the points of U the manifold M^3 is locally isometric to a direct product of semisymmetric manifolds $M_0 \times M_1 \times \ldots \times M_r$, where M_0 is an open part of a Euclidean space and the manifolds M_i, $i > 0$, are infinitesimally irreducible simple semisymmetric leaves. Here a semisymmetric M is called a *simple leaf* if at every its point x the primitive holonomy group determines a simple decomposition $T_xM = V_x^{(0)} + V_x^{(1)}$, where this group acts trivially on $V_x^{(0)}$ and there is only one subspace $V_x^{(1)}$ which is invariant for this group. A simple leaf is said to be infinitesimally irreducible if at least at one point the infinitesimal holonomy group acts irreducibly on $V_x^{(1)}$.

The dimension $\nu(x) = \dim V_x^{(0)}$ is called the *index of nullity* at x and $u(x) = \dim M - \nu(x)$ the *index of non-nullity* at x.

The classification theorem by Szabó asserts the following (according to the formulation given in [1], [2]).

Let M be an infinitesimally irreducible simple semisymmetric leaf and x a point of M. Then one of the following cases occurs:

(a) $\nu(x) = 0$ and $u(x) > 2$: M is locally symmetric and hence locally isometric to a symmetric space;

(b) $\nu(x) = 1$ and $u(x) > 2$: M is locally isometric to an elliptic, a hyperbolic or a Euclidean cone;

(c) $\nu(x) = 2$ and $u(x) > 2$: M is locally isometric to a Kählerian cone;

(d) $\nu(x) = \dim M - 2$ and $u(x) = 2$: M is locally isometric to a space foliated by Euclidean leaves of codimension two (or to a two-dimensional manifold, this in the case whenen space $\dim M = 2$).

If here $\dim M = 3$, then $u(x) = 3 - \nu(x)$ and thus the cases (b) and (c) are impossible. Therefore they do not need more detailed description in the present paper. Hence the main attention must be turned to the case (d): to the three-dimensional semisymmetric Riemannian manifolds M^3 foliated by one-dimensional Euclidean leaves, which can be considered now as the geodesic lines (called in [6] the *principal geodesics* of the foliated M^3).

For such foliated M^3, O. Kowalski introduced in a preprint of 1991 the geometric concept of *asymptotic foliation* (see [6]), generalized afterwards by E. Boeckx [1] to a general foliated semisymmetric Riemannian manifolds.

A two-dimensional submanifold of a three-dimensional foliated Riemannian manifold M^3 is called an *asymptotic leaf* if it is generated by the principal geodesics of M^3

and if its tangent planes are parallel along each of these geodesics (with respect to the Levi-Cività connection ∇ of M^3).

A two-dimensional foliation of such an M^3 is called the *asymptotic foliation* if its integral manifolds are asymptotic leaves.

What follows here in the present section gives a treatment of the asymptotic foliations according to Kowalski [6] (and Boeckx [1]).

Let the bundle of orthonormal frames (e_1, e_2, e_3) be adapted to the considered M^3 so that at each point $x \in M^3$ the unit vector e_3 is tangent to the principal geodesic. For the bundle of dual coframes $(\omega^1, \omega^2, \omega^3)$ the following structure equations hold:

$$\mathrm{d}\omega^i = \omega^j \wedge \omega^i_j, \quad \mathrm{d}\omega^i_j = \omega^k_j \wedge \omega^i_k + \Omega^i_j,$$

where ω^i_j and Ω^i_j are the connection 1-forms and the curvature 2-forms, correspondingly, of ∇. Here orthonormality yields $\omega^i_j + \omega^j_i = 0$, $\Omega^i_j + \Omega^j_i = 0$.

A vector field $X = e_i X^i$ is parallel along the one-dimensional leaf (principal geodesic) with respect to ∇ if and only if $dX^i + X^j\omega^i_j$ is zero as a consequence of $\omega^1 = \omega^2 = 0$. For the adapted frame bundle above and for a principal geodesic it means that this condition must be satisfied for $X^i = \delta^i_3$, when $dX^i + X^j\omega^i_j = \omega^i_3$, i.e.

$$\omega^1_3 = a\omega^1 + b\omega^2, \quad \omega^2_3 = c\omega^1 + e\omega^2. \tag{1.1}$$

Let $X = e_1 \cos\varphi + e_2 \sin\varphi$ be a unit vector in the tangent plane $\operatorname{span}\{X, e_3\}$ of the asymptotic leaf. Then $\nabla_{e_3} X = \nabla_X e_3 + [e_3, X]$ must belong to this tangent plane. Since the tangent distribution of this leaf is a foliation the same can be said about $[e_3, X]$. Thus $\nabla_X e_3 = \nabla_{e_1} e_3 \cos\varphi + \nabla_{e_2} e_3 \sin\varphi = (\omega^k_3(e_1)e_k)\cos\varphi + (\omega^k_3(e_2)e_k)\sin\varphi = (ae_1 + ce_2)\cos\varphi + (be_1 + ee_2)\sin\varphi$ must belong to $\operatorname{span}\{X, e_3\}$ and therefore must be a multiple of $X = e_1\cos\varphi + e_2\sin\varphi$. This last condition is equivalent to

$$b\sin^2\varphi + (a-e)\cos\varphi\sin\varphi - c\cos^2\varphi = 0.$$

But along the asymptotic leaf $\omega^1 \sin\varphi = \omega^2\cos\varphi$, so that this condition reduces to

$$c(\omega^1)^2 + (e-a)\omega^1\omega^2 - b(\omega^2)^2 = 0. \tag{1.2}$$

According to [6], [2] a foliated M^3 is said to be *planar* if it admits infinitely many asymptotic foliations. If it admits just two (or one, or none, respectively) asymptotic foliations, it is said to be *hyperbolic* (or *parabolic*, or *elliptic*, respectively).

From (1.2) it is seen that each planar foliated M^3 is characterized by $a - e = b = c = 0$, i.e. by the fact that (1.1) reduces to

$$\omega^1_3 = a\omega^1, \quad \omega^2_3 = a\omega^2. \tag{1.3}$$

2. Three-dimensional semiparallel submanifolds

For our problem also the classification of semiparallel submanifolds M^3 in the Euclidean space $\mathbb{E}^n$ is important. This classification is made in [16] and [11] (some generalizations for M^3 in Riemannian space forms were made later in [15]).

A general information about semiparallel submanifolds is given e.g. in [4], [14], [15].

Let M^m be a Riemannian submanifold in $\mathbb{E}^n$. Let a point $x \in M^m$ and its radius vector with respect to the origin $O \in \mathbb{E}^n$ be identified. Let the bundle of orthogonal frames $\{x; e_1, \dots, e_n\}$ be reduced to the subbundle adapted to M^m, so that $e_i \in T_x M^m$, $e_\alpha \in T_x^\perp M^m$, where the indices $i, j, \dots$ run over the set $\{1, \dots, m\}$ and $\alpha, \beta, \dots$ run over the set $\{m+1, \dots, n\}$. There hold then the derivation formulae

$$\mathrm{d}x = e_I \omega^I, \quad \mathrm{d}e_I = e_J \omega_I^J \tag{2.1}$$

(independent of O), where

$$\omega_J^I + \omega_I^J = 0, \tag{2.2}$$

and the structure equations

$$\mathrm{d}\omega^I = \omega^J \wedge \omega_J^I, \quad \mathrm{d}\omega_I^J = \omega_I^K \wedge \omega_K^J. \tag{2.3}$$

For subbundle adapted to M^3, $\omega^\alpha = 0$. Now (2.2) yields

$$\omega_j^i + \omega_i^j = 0, \quad \omega_\alpha^i = -\omega_i^\alpha, \quad \omega_\beta^\alpha + \omega_\alpha^\beta = 0, \tag{2.4}$$

but (2.3) give $\omega^i \wedge \omega_i^\alpha = 0$, thus, due to Cartan's lemma,

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.5}$$

For $X, Y \in T_x M^m$ with $X = e_i X^i$, $Y = e_j Y^j$ the map $h\colon (X, Y) \longmapsto e_\alpha h_{ij}^\alpha X^i Y^j$ is the second fundamental form of the submanifold M^m in $\mathbb{E}^n$. The differential prolongation used for (2.5) showes that $\tilde{\nabla} h_{ij}^\alpha = h_{ijk}^\alpha \omega^k$, $h_{ijk}^\alpha = h_{ikj}^\alpha$, where

$$\tilde{\nabla} h_{ij}^\alpha := dh_{ij}^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k + h_{ij}^\beta \omega_\beta^\alpha \tag{2.6}$$

are the components of $\tilde{\nabla} h$.

Further, by exterior differentiation,

$$\tilde{\nabla} h^{\alpha}_{ijk} \wedge \omega^k = -h^{\alpha}_{kj}\Omega^k_i - h^{\alpha}_{ik}\Omega^k_j + h^{\beta}_{ij}\Omega^{\alpha}_{\beta}, \tag{2.7}$$

where

$$\Omega^j_i := d\omega^j_i - \omega^k_i \wedge \omega^j_k = -\sum_{\alpha} \omega^{\alpha}_i \wedge \omega^{\alpha}_j, \tag{2.8}$$

$$\Omega^{\beta}_{\alpha} := d\omega^{\beta}_{\alpha} - \omega^{\gamma}_{\alpha} \wedge \omega^{\beta}_{\gamma} = -\sum_{i} \omega^{\alpha}_i \wedge \omega^{\beta}_i \tag{2.9}$$

are the curvature 2-forms of ∇ and $\nabla^{\perp}$, respectively. Due to (2.5) they are

$$\Omega^j_i = -R^j_{i,kl}\omega^k \wedge \omega^l, \quad \Omega^{\beta}_{\alpha} = -R^{\beta}_{\alpha,kl}\omega^k \wedge \omega^l,$$

where the coefficients, the components of the curvature tensors R and $R^{\perp}$ of ∇ and $\nabla^{\perp}$, respectively, can be expressed algebraically by the components of the second fundamental form:

$$R^j_{i,kl} = \sum_{\alpha} h^{\alpha}_{i[k}h^{\alpha}_{l]j}, \quad R^{\beta}_{\alpha,kl} = \sum_{i} h^{\alpha}_{i[k}h^{\beta}_{l]i}. \tag{2.10}$$

Here the first equation is the famous Gauss equation, which expresses the relationship between the intrinsic and extrinsic properties of a submanifold M^m in $\mathbb{E}^n$.

Submanifolds satisfying the differential system $\tilde{\nabla} h^{\alpha}_{ij} = 0$ (equivalently, $h^{\alpha}_{ijk} = 0$) are called *parallel*; they are intrinsically locally symmetric Riemannian manifolds. The submanifolds satisfying the integrability condition for this system, which by (2.7) is

$$h^{\alpha}_{kj}\Omega^k_i + h^{\alpha}_{ik}\Omega^k_j - h^{\beta}_{ij}\Omega^{\alpha}_{\beta} = 0, \tag{2.11}$$

are called *semiparallel*; they are intrinsically semisymmetric Riemannian manifolds. Both these assertions follow easily from the Gauss equation.

Summing in (2.11) over $i = j$ one obtains that the so called *mean curvature vector* $H = H^{\alpha}e_{\alpha}$ with $H^{\alpha} = \frac{1}{m}\sum_{i} h^{\alpha}_{ii}$, of a semiparallel submanifold satisfies

$$H^{\beta}\Omega^{\alpha}_{\beta} = 0. \tag{2.12}$$

This is due to the fact that h^{α}_{ij} and Ω^j_i are symmetric and antisymmetric, respectively, with respect to i, j and therefore their product annihilates after contraction.

Our problem is: can every semisymmetric Riemannian M^m be immersed into $\mathbb{E}^n$ as a semiparallel submanifold? This problem will be considered here for the dimension $m = 3$.

For this purpose the classification of all semiparallel submanifolds M^3 in $\mathbb{E}^n$ is needed. Recall that this is done in [16], [11] (see also [15], Sect. 20).

Here the preliminary classification goes by the dimension m_1 of the so called *first* (or *principal*) *normal subspace* $\mathrm{span}\{h^\alpha_{ij}e_\alpha\}$. This m_1 will be called further the *principal codimension.* If $m = 3$ then obviously $m_1 \leqslant 6$.

Note that the submanifolds M^3 with $m_1 = 0$ are totally geodesic and therefore are some open parts of three-dimensional Euclidean planes in $\mathbb{E}^n$; they are all trivially semiparallel.

Let now $m_1 > 0$. The orthonormal frame bundle can be adapted further so that $e_4, e_5, \ldots, e_{3+m_1}$ belong to $\mathrm{span}\{h^\alpha_{ij}e_\alpha\}$. Then $h^\varrho_{ij} = 0$ ($\varrho, \sigma, \ldots = m_1 + 4, \ldots, n$), thus

$$\Omega^\varrho_\alpha = -\Omega^\alpha_\varrho = 0. \tag{2.13}$$

If all $\Omega^\beta_\alpha = 0$ then M^3 is said to have *flat normal connection* $\nabla^\perp$. Due to (2.9) and (2.10) the matrices $\|h^\alpha_{ij}\|$ for every two different values of α then commute and therefore are simultanously diagonalizable by a suitable orthogonal transformation of $\{e_1, e_2, e_3\}$. After that $h^\alpha_{ij} = k^\alpha_i\delta_{ij}$, and thus $\Omega^j_i = -\langle k_i, k_j\rangle\omega^i \wedge \omega^j$, where $k_i = k^\alpha_i e_\alpha$ are the so called *principal curvature vectors*, but the semiparallelity condition (2.11) reduces to

$$(k_i - k_j)\langle k_i, k_j\rangle = 0; \quad i \neq j. \tag{2.14}$$

3. Classification of semiparallel M^3 of principal codimension $m_1 \leqslant 3$

This classification is given in [16] and will be reproduced here with some additions concerning the intrinsical metric.

3.1. The case of $m_1 = 1$.

Here $\nabla^\perp$ is obviously flat, in the adapted frame $k_i = \kappa_i e_4$ and thus $\langle k_i, k_j\rangle = \kappa_i\kappa_j$. Due to (2.14) there are the following possibilities (up to a permutation of 1, 2, 3).

Type (1): $\kappa_1\kappa_2\kappa_3 \neq 0$. Then $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$, where $\kappa \neq 0$ for $m_1 = 1$. In this subcase M^3 is totally umbilical, thus an open part of $S^3 \subset \mathbb{E}^4 \subset \mathbb{E}^n$, and intrinsically is a Riemannian manifold of constant curvature.

Type (2): $\kappa_3 = 0$, $\kappa_1 = \kappa_2 = \kappa \neq 0$. In the adapted orthonormal frame M^3 is then determined locally by the differential system

$$\omega^\alpha = 0, \quad \omega_1^4 = \kappa\omega^1, \quad \omega_2^4 = \kappa\omega^2, \quad \omega_3^4 = 0, \quad \omega_i^\varrho = 0 \tag{3.1}$$
$$(i = 1, 2, 3;\ \varrho = 5, \dots, n).$$

Differential prolongation by means of (2.3), (2.4), and Cartan's lemma leads to

$$\omega_3^1 = a\omega^1, \quad \omega_3^2 = a\omega^2, \quad \mathrm{d}\ln\kappa = -a\omega^3, \quad \omega_4^\varrho = 0 \tag{3.2}$$

and the next prolongation gives

$$\mathrm{d}a = -a^2\omega^3. \tag{3.3}$$

The system (3.1)–(3.2) is totally integrable and determines the considered M^3 up to arbitrary constants. From (2.1) now

$$\mathrm{d}x = e_i\omega^i, \quad \mathrm{d}e_1 = e_2\omega_1^2 + (-ae_3 + \kappa e_4)\omega^1, \quad \mathrm{d}e_2 = -e_1\omega_1^2 + (-ae_3 + \kappa e_4)\omega^2,$$
$$\mathrm{d}e_3 = a(e_1\omega^1 + e_2\omega^2), \quad \mathrm{d}e_4 = -\kappa(e_1\omega^1 + e_2\omega^2), \quad \mathrm{d}e_\varrho = e_\sigma\omega_\varrho^\sigma.$$

Hence this M^3 lies in the plane $\mathbb{E}^4 \subset \mathbb{E}^n$ spanned by x, e_1, e_2, e_3 and e_4. As is easily seen, the system $\omega^1 = \omega^2 = 0$ determines a foliation of M^3 whose leaves are the straight lines with direction vector e_3.

Let $a \neq 0$. Since $d(x - a^{-1}e_3) = 0$ along M^3, this M^3 is an open part of a cone with a point-vertex whose radius vector is $c = x - a^{-1}e_3$. The equation $\omega^3 = 0$ is totally integrable and determines a foliation of M^3 whose leaves are totally umbilical, with principal curvature vector $-ae_3 + \kappa e_4$, orthogonal to the straight generators of the cone. Hence M^3 is an open part of a round cone.

Intrinsically this M^3 is a foliated semisymmetric Riemannian manifold, which is due to (3.2) and (1.3) of planar type (in the sense of Kowalski, see Sect. 1).

If $a \equiv 0$ then the vertex has moved to infinity and M^3 is a right cylinder over a two-dimensional sphere, thus a product submanifold.

Type (3): $\kappa_3 = \kappa_2 = 0$. Here $\kappa_1 = \kappa \neq 0$ for $m_1 = 1$. In the adapted orthonormal frame

$$\omega^\alpha = 0, \quad \omega_1^4 = \kappa\omega^1, \quad \omega_2^4 = \omega_3^4 = 0, \quad \omega_i^\varrho = 0,$$

where i and ϱ run as in (3.1). Differential prolongation leads to

$$\omega_1^2 = \lambda\omega^1, \quad \omega_1^3 = \mu\omega^1, \quad \mathrm{d}\ln\kappa = \nu\omega^1 + \lambda\omega^2 + \mu\omega^3, \quad \omega_4^\varrho = \varphi^\varrho\omega^1.$$

A further investigation of this system shows that the considered M^3 exists with arbitrariness of $n-1$ real analytic functions of one real argument.

The equation $\omega^1 = 0$ is totally integrable, because $\mathrm{d}\omega^1 = 0$, and determines a foliation of M^3 whose leaves are, in view of $\mathrm{d}e_2 = e_3\omega_2^3 \pmod{\omega^1}$, $\mathrm{d}e_3 = -e_2\omega_2^3 \pmod{\omega^1}$, two-dimensional planes, along every of which the unit vectors e_1, e_4, e_ϱ are constant, thus the two-dimensional plane tangent to M^3 is invariant.

Here

$$\Omega_1^2 = \omega_1^4 \wedge \omega_4^2 + \omega_1^\varrho \wedge \omega_\varrho^2 = 0, \quad \Omega_4^\varrho = \omega_4^i \wedge \omega_i^\varrho = 0, \quad \Omega_\varrho^\sigma = \omega_\varrho^i \wedge \omega_i^\sigma = 0,$$

thus M^3 has flat $\tilde{\nabla}$. From the intrinsical point of view this means that M^3 is a locally Euclidean Riemannian manifold immersed into $\mathbb{E}^n$.

3.2. The case of $m_1 = 2$.

Then (2.12) reduces to $H^4\Omega_4^5 = H^5\Omega_4^5 = 0$. For the Riemannian semiparallel M^3 here $H^4 = H^5 = 0$ is impossible, because then $H = 0$ and M^3 is, due to Proposition 8.6 of [15], an open part of a three-dimensional plane $\mathbb{E}^3 \subset \mathbb{E}^n$, thus $m_1 = 0$; this contradicts to $m_1 = 2$. Therefore $\Omega_4^5 = 0$ and hence $\nabla^\perp$ is flat. Now (2.14) can be reduced, after a suitable permutation of 1, 2, 3, to the following two subcases.

Type (4): $0 \neq k_1 \perp k_2 \neq 0$, $k_3 = 0$. Here e_4 and e_5 can be taken so that they are collinear to k_1 and k_2, respectively, hence $k_1 = \kappa_1 e_4$ and $k_2 = \kappa_2 e_5$. Then

$$\omega_1^4 = \kappa_1\omega^1, \quad \omega_2^5 = \kappa_2\omega^2, \quad \omega_1^5 = \omega_2^4 = \omega_3^4 = \omega_3^5 = \omega_i^\varrho = 0,$$

where $i, j, \ldots = 1, 2, 3$ and $\varrho = 6, \ldots, n$. Now according to (2.8), (2.9) $\Omega_i^j = 0$ so that also ∇ is flat. This M^3 is intrinsically locally Euclidean.

Type (5): $0 \neq k_1 = k_2 \perp k_3 \neq 0$. Here $k_1 = k_2 = \kappa e_4$, $k_3 = \kappa_3 e_5$ by a suitable choice of e_4 and e_5. Hence

$$\omega_1^4 = \kappa\omega^1, \quad \omega_2^4 = \kappa\omega^2, \quad \omega_3^4 = 0,$$
$$\omega_1^5 = \omega_2^5 = 0, \quad \omega_3^5 = \kappa_3\omega^3, \quad \omega_i^\varrho = 0,$$

where $\mathrm{d}\omega_3^5 = 0$ and thus at least locally $\omega_3^5 = du^3$. The first two equations of the second row give by exterior differentiation

$$\omega_1^3 \wedge du^3 + \kappa\omega^1 \wedge \omega_4^5 = 0, \quad \omega_2^3 \wedge du^3 + \kappa\omega^2 \wedge \omega_4^5 = 0.$$

Hence by Cartan's lemma

$$\omega_3^1 = a\omega^1 + \lambda_1 du^3, \quad \omega_3^2 = a\omega^2 + \lambda_2 du^3, \quad \kappa\omega_4^5 = a du^3.$$

The third equation of the first row gives by exterior differentiation and substitution that $\lambda_1 = \lambda_2 = 0$, so that

$$\omega_3^1 = a\omega^1, \quad \omega_3^2 = a\omega^2. \tag{3.4}$$

Hence the next differential prolongation gives $\mathrm{d}a = -a^2\omega^3$.

Now $\omega^1 = \omega^2 = 0$ determines a foliation of M^3 whose leaves are the lines with $\mathrm{d}x = e_3\omega^3$, $\mathrm{d}e_3 = e_4\kappa_3\omega^3$, thus the geodesic lines of M^3. The equation $\omega^3 = 0$ is totally integrable because $\mathrm{d}\omega^3 = 0$. It determines a foliation of M^3 whose leaves are totally umbilical because for them $\mathrm{d}e_1 = e_2\omega_1^2 + (-ae_3 + \kappa e_4)\omega^1$, $\mathrm{d}e_2 = -e_1\omega_1^2 + (-ae_3 + \kappa e_4)\omega^2$. Hence these leaves are two-dimensional spheres. Along every such sphere the tangent lines of M^3, orthogonal to the sphere, intersect in a point with radius vector $x - a^{-1}e_3$, because $d(x - a^{-1}e_3) = 0 \pmod{\omega^3}$. (If $a \equiv 0$ along the sphere then these tangent lines are parallel.) Hence these sphere-leaves are the characteristics of a one-parametric family of three-dimensional spheres and M^3 is an envelope of this family. It follows that M^3 is a *canal submanifold*. For every of its geodesics, orthogonal to the characteristics, the principal curvature vector is $\kappa_3 e_4$, thus orthogonal to the three-dimensional plane containing the characteristic sphere and therefore spanned by e_1, e_2, $-ae_3 + \kappa e_4$. Such kind of canal M^3 can be called of *orthogonal type* (see [11]).

Intrinsically it is a foliated semisymmetric Riemannian three-dimensional manifold, which is in view of (3.4) and (1.3) of planar type (in the sense of Kowalski, see Sect. 1).

3.3. The case of $m_1 = 3$.

Here $\nabla^\perp$ can be but need not be flat. Let us start from the first possibility.

Type (6): Let $\nabla^\perp$ be flat. Now three principal curvature vectors k_1, k_2, k_3 must be different, due to $m_1 = 3$, and the semiparallelity condition (2.14) shows that they are mutually orthogonal vectors. Therefore the frame vectors e_{3+i} can be taken so that $k_i = \kappa_i e_{3+i}$. Then

$$\omega_i^{3+j} = \kappa_i\delta_i^j\omega^i, \quad \omega_i^\xi = 0 \quad (\xi, \eta, \ldots = 7, \ldots, n).$$

It follows that

$$\Omega_i^j = \omega_i^{3+k} \wedge \omega_{3+k}^j = \sum_k \kappa_i\delta_i^k\omega^i \wedge (-\kappa_j\delta_j^k\omega^j) = -\kappa_i\kappa_j\delta_{ij}\omega^i \wedge \omega^j = 0,$$

i.e. also ∇ is flat and M^3 is intrinsically locally Euclidean.

Further let $\nabla^\perp$ be nonflat. Here Proposition 8.6 of [15] yields $H \neq 0$, as for the case $m_1 = 2$. The frame vectors e_4, e_5, e_6 in the principal normal subspace can be taken so that e_4 is collinear to H. Then $H^4 \neq 0$ but the other components of H are zero. Now (2.12) gives that $\Omega_4^\alpha = 0$ in addition to (2.13). Since $\nabla^\perp$ is nonflat there must be $\Omega_5^6 \neq 0$. In particular, $\Omega_4^5 = 0$, but this yields that the matrices $\|h_{ij}^4\|$ and $\|h_{ij}^5\|$ commute and can be simultaneously diagonalized by a suitable choice of e_1, e_2, e_3. After that

$$h_{ij}^4 = \kappa_i \delta_{ij}, \quad h_{ij}^5 = \lambda_i \delta_{ij}.$$

Here $\lambda_1 + \lambda_2 + \lambda_3 = 0$ since $H^5 = 0$, but (2.9) gives

$$0 \neq \Omega_5^6 = (\lambda_2 - \lambda_1) h_{12}^6 \omega^1 \wedge \omega^2 + (\lambda_3 - \lambda_2) h_{23}^6 \omega^2 \wedge \omega^3 + (\lambda_1 - \lambda_3) h_{31}^6 \omega^3 \wedge \omega^1.$$

Hence at least one of the coefficients is non-zero. After a renumeration, if needed, one obtains $(\lambda_1 - \lambda_2) h_{12}^6 \neq 0$, $\lambda_1 \neq 0$.

Semiparallelity condition (2.11) for $\alpha = 5$ reduces to

$$(\lambda_j - \lambda_i) \Omega_i^j = h_{ij}^6 \Omega_6^5, \tag{3.5}$$

therefore

$$h_{ii}^6 = 0, \quad (\lambda_1 - \lambda_2) \Omega_1^2 = h_{12}^6 \Omega_5^6 \neq 0, \tag{3.6}$$

thus $\Omega_1^2 \neq 0$.

Further, $0 = \Omega_4^6 = \sum_{i<j} (\kappa_j - \kappa_i) h_{ij}^6 \omega^i \wedge \omega^j$ yields

$$\kappa_1 = \kappa_2 = \kappa, \quad (\kappa - \kappa_3) h_{13}^6 = (\kappa - \kappa_3) h_{23}^6 = 0. \tag{3.7}$$

There exist two possibilities.

Type (7): $\kappa_3 \neq \kappa$, $h_{13}^6 = h_{23}^6 = 0$. Here (2.11) for $\alpha = 6$ and $i = j = 1$, or $i = j = 2$ gives, respectively,

$$2h_{12}^6 \Omega_1^2 - \lambda_1 \Omega_5^6 = 0, \quad -2h_{12} \Omega_1^2 - \lambda_2 \Omega_5^6 = 0, \tag{3.8}$$

hence $(\lambda_1 + \lambda_2) \Omega_5^6 = 0$ and thus $\lambda_2 = -\lambda$, $\lambda_3 = 0$, where λ_1 is denoted by λ.

Now (3.6) and (3.8) reduce to

$$2\lambda \Omega_1^2 - h_{12}^6 \Omega_5^6 = 0, \quad 2h_{12}^6 \Omega_1^2 - \lambda \Omega_5^6 = 0$$

and yield $(h_{12}^6)^2 = \lambda^2$. Replacing e_6 by $-e_6$, if needed, one can take $h_{12}^6 = \lambda \neq 0$. Due to (2.8) and (2.9)

$$\Omega_1^2 = (2\lambda^2 - \kappa^2) \omega^1 \wedge \omega^2, \quad \Omega_2^3 = -\kappa\kappa_3 \omega^2 \wedge \omega^3, \quad \Omega_5^6 = -2\lambda^2 \omega^1 \wedge \omega^2.$$

Substitution into (3.5) gives $\kappa^2 = 3\lambda^2 \neq 0$, and replacing e_4 by $-e_4$, if needed, one gets $\kappa = \lambda\sqrt{3}$. Finally (2.11) for $\alpha = 6$ and $i = 1$, $j = 3$ yields $\lambda\kappa\kappa_3 = 0$, thus $\kappa_3 = 0$.

All this can be summarized with

$$\omega_1^4 = \lambda\sqrt{3}\omega^1, \quad \omega_2^4 = \lambda\sqrt{3}\omega^2, \quad \omega_3^4 = 0, \tag{3.9}$$

$$\omega_1^5 = \lambda\omega^1, \quad \omega_2^5 = -\lambda\omega^2, \quad \omega_3^5 = 0, \tag{3.10}$$

$$\omega_1^6 = \lambda\omega^2, \quad \omega_2^6 = \lambda\omega^1, \quad \omega_3^6 = 0, \tag{3.11}$$

$$\omega_1^\varrho = \omega_2^\varrho = \omega_3^\varrho = 0, \tag{3.12}$$

where $\varrho = 7, \ldots, n$. This differential system together with

$$\omega^4 = \omega^5 = \omega^6 = \omega^\varrho = 0$$

determines the considered M^3.

By exterior differentiation from $\omega_3^4 = \omega_3^5 = \omega_3^6 = 0$ it follows that

$$\omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 = \omega_1^3 \wedge \omega^1 - \omega_2^3 \wedge \omega^2 = \omega_1^3 \wedge \omega^2 + \omega_2^3 \wedge \omega^1 = 0,$$

therefore

$$\omega_1^3 = -a\omega^1, \quad \omega_2^3 = -a\omega^2. \tag{3.13}$$

Due to (3.13) the system $\omega^1 = \omega^2 = 0$ determines a foliation of M^3 whose leaves are the straight lines because for them $\mathrm{d}e_3 = 0$.

Thus intrinsically this M^3 is a foliated semisymmetric Riemannian three-dimensional manifold, which is due to (3.13) and (1.3) of planar type (in the sense of Kowalski).

Geometric characterization of this semiparallel submanifold will be given below in Section 6.

The other possibility in (3.7) is

Type (8): $\kappa_3 = \kappa$. Here (2.11) for $\alpha = 6$ and $i = 1$, $j = 3$, or $i = 2$, $j = 3$ gives, respectively,

$$h_{23}^6\Omega_1^2 + h_{12}^6\Omega_3^2 = 0, \quad h_{13}^6\Omega_2^1 + h_{21}^6\Omega_3^1 = 0, \tag{3.14}$$

but (3.5) leads to

$$(\lambda_3 - \lambda_1)\Omega_1^3 = h_{13}^6\Omega_6^5, \quad (\lambda_3 - \lambda_2)\Omega_2^3 = h_{23}^6\Omega_6^5.$$

After multiplying by $h^6_{12} \neq 0$, using (3.6) and $\lambda_1 + \lambda_2 + \lambda_3 = 0$ one obtains

$$h^6_{13}\lambda_1 = h^6_{23}\lambda_2 = 0.$$

Consequently, $h^6_{13} = 0$. The possibilty $h^6_{23} = 0$ leads to a contradiction, because, as in the subcase of type (7), then $\kappa_3 = 0$ and $\kappa^2 = 3\lambda^2 \neq 0$, but this contradicts to $\kappa_3 = \kappa$. Therefore $h^6_{23} \neq 0$, $\lambda_2 = 0$ and thus $\lambda_3 = -\lambda_1 = \lambda \neq 0$. From (3.14) now $\Omega^3_1 = 0$, $\Omega^3_2 \neq 0$.

The semiparallelity condition (2.11) for $\alpha = 6$ and $i = j = 1$, and for $\alpha = 5$ and $i = 1$, $j = 2$ yields, respectively,

$$2h^6_{12}\Omega^2_1 - \lambda\Omega^6_5 = 0, \quad \lambda\Omega^2_1 - h^6_{12}\Omega^6_5 = 0.$$

Hence $2(h^6_{12})^2 = \lambda^2$. But the same (2.11) for $\alpha = 6$ and $i = j = 2$, or $i = 1$, $j = 3$ gives, respectively,

$$h^6_{12}\Omega^2_1 - h^6_{23}\Omega^3_2 = 0, \quad h^6_{23}\Omega^2_1 - h^6_{12}\Omega^3_2 = 0,$$

hence $(h^6_{23})^2 = (h^6_{12})^2$. Replacing e_6 by $-e_6$ and e_2 by $-e_2$, if needed, one obtains $h^6_{23} = h^6_{12} = \frac{1}{\sqrt{2}}\lambda$.

Now

$$0 = \Omega^3_1 = (\lambda^2 - \kappa^2)\omega^2 \wedge \omega^3$$

and taking $-e_4$ instead of e_4, if needed, one gets finally $\kappa = \lambda$. It remains to rotate e_2 and e_3 in their plane on the angle $\pi/4$ to obtain the differential system determining the considered M^3 in the form:

$$\begin{gathered}
\omega^4 = \omega^5 = \omega^6 = \omega^\varrho = 0,\\
\omega^4_1 = \lambda\omega^1, \quad \omega^4_2 = \lambda\omega^2, \quad \omega^4_3 = \lambda\omega^3,\\
\omega^5_1 = \lambda\omega^2, \quad \omega^5_2 = \lambda\omega^1, \quad \omega^5_3 = 0,\\
\omega^6_1 = \lambda\omega^3, \quad \omega^6_2 = 0, \quad \omega^6_3 = \lambda\omega^1,\\
\omega^\varrho_1 = \omega^\varrho_2 = \omega^\varrho_3 = 0,
\end{gathered}$$

where $\varrho = 7, \dots, n$. The differential prolongation (exterior differentiation and Cartan's lemma) of this system gives

$$\begin{gathered}
\omega^2_1 = a\omega^2, \quad \omega^3_1 = a\omega^3,\\
\mathrm{d}\ln\lambda = -a\omega^1, \quad \omega^3_2 - \omega^6_5 = 0, \quad \omega^5_4 = a\omega^2, \quad \omega^6_4 = a\omega^3,\\
\omega^\varrho_4 = \varphi^\varrho\omega^1, \quad \omega^\varrho_5 = \varphi\omega^2, \quad \omega^\varrho_6 = \varphi^\varrho\omega^3
\end{gathered} \tag{3.15}$$

and from here in the same manner

$$\mathrm{d}a = -a^2\omega^1, \quad \omega_4^\varrho = \omega_5^\varrho = \omega_6^\varrho = 0.$$

This prolonged system is now totally integrable and determines the considered M^3 up to constants. This M^3 lies obviously in a $\mathbb{E}^6 \subset \mathbb{E}^n$.

Due to (3.15) the system $\omega^2 = \omega^3 = 0$ determines a foliation of M^3. For its leaves $\mathrm{d}e_1 = e_4\lambda\omega^1$, $\mathrm{d}e_4 = -e_1\lambda\omega^1$, therefore each of them is a plane curve, which is geodesic for M^3. Hence M^3 is intrinsically a foliated semisymmetric Riemannian three-dimensional manifold. If we exchange in (3.15) the roles of the indices 1 and 3 and compare the result with (1.3), it is seen that this M^3 is intrinsically of planar type (in the sense of Kowalski).

Geometric characterization of this semiparallel submanifold will be given below in Section 6.

4. Classification for the remaining cases

This classification is given in [11] and will be reproduced here with some additions concerning the intrinsical metric.

4.1. The case of $m_1 = 4$.

Here between six vectors $h_{ij} = e_\alpha h_{ij}^\alpha$ there must be two independent linear relations $h_{ij}\xi^{ij} = 0$ and $h_{ij}\eta^{ij} = 0$, which determine a one-parametric family $\varrho(h_{ij}\xi^{ij}) + \sigma(h_{ij}\eta^{ij}) = 0$ of such relations. In this family the singular case corresponds to a root of the cubic equation $\det|\varrho\xi^{ij} + \sigma\eta^{ij}| = 0$ with respect to ϱ/σ or σ/ϱ. There is at least one real root and thus one basic relation can be presented in the form $h_{ij}\xi_1^{(i}\xi_2^{j)} = 0$, where $\xi_1 = \xi_1^i e_i$ and $\xi_2 = \xi_2^j e_j$ determine some directions in T_xM^3.

A. Let these directions be distinct. After normalization of ξ_1 and ξ_2 the frame vector e_2 can be taken orthogonal to them and e_1 and e_3 collinear, respectively, to $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$. Then the special basic relation above is $h_{11}(\xi_1^1)^2 - h_{33}(\xi_1^3)^2 = 0$. Here $\xi_1^1\xi_1^3 \neq 0$; the roles of e_1 and e_3 can be interchanged taking $-\xi_2$ instead of ξ_2. Hence the following subcases occur.

(A$_1$) $(\eta^{12})^2 + (\eta^{23})^2 \neq 0$. Here it can be supposed that $\eta^{23} \neq 0$ and the basic relations are

$$h_{33} = \mu h_{11}, \quad h_{23} = \nu_1 h_{11} + \nu_2 h_{22} + \nu_3 h_{12} + \nu_4 h_{13}.$$

(A$_2$) $\eta^{12} = \eta^{23} = 0$, $\eta^{13} \neq 0$. Then

$$h_{33} = \mu h_{11}, \quad h_{13} = \nu_1 h_{11} + \nu h_{22}.$$

(A_3) $\eta^{12} = \eta^{23} = \eta^{13} = 0$. Here either

(A_3') $\eta^{22} \neq 0$ and $h_{33} = \mu h_{11}$, $h_{22} = \nu h_{11}$ or

(A_3'') $\eta^{22} = 0$, then $\eta^{11} \neq 0$ and $h_{33} = h_{11} = 0$.

B. Let ξ_1 and ξ_2 have the same direction, in which e_3 can be taken. Then the special basic relation above is $h_{33} = 0$ and the roles of e_1 and e_2 can be interchanged taking $-e_3$ instead of e_3. Here the following subcases occur.

(B_1) $(\eta^{13})^2 + (\eta^{23})^2 \neq 0$. It can be supposed that $\eta^{23} \neq 0$, but this leads to the limit case of (A_1), when $\mu = 0$.

(B_2) $\eta^{13} = \eta^{23} = 0$, $\eta^{12} \neq 0$. Then

$$h_{33} = 0, \quad h_{12} = \lambda_1 h_{11} + \lambda_2 h_{22}.$$

(B_3) $\eta^{13} = \eta^{23} = \eta^{12} = 0$. This leads either to the limit case of (A_3'), when $\mu = 0$, or to the case (A_3'').

So we must consider three subcases (A_2), (A_3''), (B_2), and two subcases (A_1), (A_3') with their limit cases when $\mu = 0$.

In each of them the semiparallelity condition (2.11) must be satisfied. By means of (2.5), (2.8) and (2.9) this condition can be represented as a purely algebraic system of cubic homogeneous equations

$$\sum_k \{h_{kj} H_{i[p,q]k} + h_{ik} H_{j[p,q]k} - H_{ij,k[p} h_{q]k}\} = 0, \tag{4.1}$$

where $H_{ij,kl} = \langle h_{ij}, h_{kl} \rangle = \sum_\alpha h^\alpha_{ij} h^\alpha_{kl}$. Every equation of this system is a linear dependence between the vectors h_{ij} with different pairs $\{ij\}$. In the considered case $m_1 = 4$ there must always be four linearly independent vectors among them, therefore the coefficient of every of the latter must be zero.

In the sequel the equation (4.1) will be referred to as $[ij, pq]$, and if the coefficient at a vector h_{rs} vanishes, then this condition will be referred to as $[ij, pq|rs]$.

Let us start with the subcase (B_2); here the vectors h_{11}, h_{22}, h_{13}, h_{23} are linearly independent. Now

$$[11, 13|13]\colon\ 2H_{13,13} + H_{11,11} = 0$$

gives a contradiction. Hence (B_2) is impossible for a semiparallel M^3 in $\mathbb{E}^n$. The same argument shows that (A_3'') is impossible, too.

Next let us consider the subcase (A_3'); here the vectors h_{11}, h_{12}, h_{13}, h_{23} are linearly independent. Then

$$[11, 12|12]\colon\ 3H_{11,22} - 2H_{12,12} - H_{11,11} = 0, \tag{4.2}$$

$$[22, 12|12]\colon\ 3H_{11,22} - 2H_{12,12} - H_{22,22} = 0, \tag{4.3}$$

thus $H_{22,22} = H_{11,11}$ and for (A'_3) $\nu^2 = 1$. The case $\nu = -1$ is impossible because (4.2) gives then a contradiction $2H_{11,11} + H_{12,12} = 0$. Hence $\nu = 1$ and $H_{11,22} = H_{12,12}$. Now

$$[13, 12|23]\colon\ H_{11,22} - H_{12,12} - H_{13,13} = 0$$

gives a contradiction $H_{13,13} = 0$ and so (A'_3) is impossible for a semiparallel M^3 in $\mathbb{E}^n$.

The same can be shown for (A_2), where the vectors h_{11}, h_{22}, h_{12}, h_{23} are linearly independent. Here as before (4.2), (4.3) hold and thus $H_{11,11} = H_{22,22}$. On the other hand, $[33, 12|12]$ gives $\mu(H_{11,22} - H_{11,11}) = 0$, hence $H_{11,11} = H_{22,22} = H_{11,22}$. This is impossible because h_{11}, h_{22} are linearly independent.

It follows that the semiparallelity condition (4.1) can be satisfied only in the subcase (A_1), when the vectors h_{11}, h_{22}, h_{12}, h_{13} are linearly independent and, recall,

$$h_{33} = \mu h_{11}, \quad h_{23} = \nu_1 h_{11} + \nu_2 h_{22} + \nu_3 h_{12} + \nu_4 h_{13}. \tag{4.4}$$

Type (9): Here the following conditions are to be used.

$$\begin{aligned}
&[11, 12|11]\colon && H_{11,12} - \nu_1 H_{11,13} = 0,\\
&[11, 12|22]\colon && -H_{11,12} - \nu_2 H_{11,13} = 0,\\
&[22, 12|11]\colon && H_{12,22} + \nu_1(2H_{12,23} - 3H_{22,13}) = 0,\\
&[22, 12|22]\colon && -H_{12,22} + \nu_2(2H_{12,23} - 3H_{22,13}) = 0,\\
&[12, 12|11]\colon && 2H_{12,12} - H_{11,22} + \nu_1(H_{11,23} - H_{12,13}) = 0,\\
&[12, 12|22]\colon && -2H_{12,12} + H_{11,22} + \nu_2(H_{11,23} - H_{12,13}) = 0,\\
&[33, 12|11]\colon && H_{33,12} + \nu_1(2H_{22,13} - 2H_{12,23} - H_{13,33}) = 0,\\
&[33, 12|22]\colon && -H_{33,12} + \nu_2(2H_{22,13} - 2H_{12,23} - H_{13,33}) = 0.
\end{aligned}$$

Suppose $\nu_1 + \nu_2 \neq 0$. From the first three pairs of these conditions

$$\begin{gathered}
H_{11,12} = H_{11,13} = 0,\\
H_{12,22} = 2H_{12,23} - 3H_{22,13} = 0,\\
2H_{12,12} - H_{11,22} = H_{11,23} - H_{12,13} = 0.
\end{gathered}$$

Therefore, due to (4.4), $H_{33,12} = \mu H_{11,12} = 0$, $H_{33,13} = \mu H_{11,13} = 0$. Now the third pair of these conditions reduces to $H_{22,13} = H_{12,23}$ and together with the relation above gives $H_{22,13} = H_{12,23} = 0$. Thus, due to (4.4), $\nu_3 H_{12,12} + \nu_4 H_{12,13} = 0$.

Further,

$$\begin{aligned}
&[13, 12|11]\colon\ \nu_1(H_{11,22} - H_{12,12} - H_{13,13}) + H_{12,13} = 0,\\
&[13, 12|22]\colon\ \nu_2(H_{11,22} - H_{12,12} - H_{13,13}) - H_{12,13} = 0,
\end{aligned}$$

hence $H_{12,13} = H_{11,22} - H_{12,12} - H_{13,13} = 0$, thus $H_{11,23} = 0$, $H_{12,12} = H_{13,13} = \chi^2 \neq 0$, $\nu_3 = 0$, $H_{11,22} = 2\chi^2$. Now from

$$\begin{aligned}
&[11,12|12]\colon\ 3H_{11,22} - 2H_{12,12} - H_{11,11} = 0,\\
&[22,12|12]\colon\ 3H_{11,22} - 2H_{12,12} - H_{22,22} = 0
\end{aligned}$$

one obtains $H_{11,11} = H_{22,22} = 4\chi^2$.

On the other hand $[13,12|13]\colon\ \nu_4(H_{11,22} - H_{12,12}) = 0$ gives $\nu_4\chi^2 = 0$, thus $\nu_4 = 0$ and now $[22,12|13]$ yields $H_{22,23} = 0$. But $H_{11,23} = 0$ and $H_{22,23} = 0$ give together a contradiction:

$$4\chi^2\nu_1 + 2\chi^2\nu_2 = 0, \quad 2\chi^2\nu_1 + 4\chi^2\nu_2 = 0, \quad \nu_1 + \nu_2 \neq 0.$$

Consequently $\nu_1 = -\nu_2 = \nu$ and

$$H_{11,12} = \nu H_{11,13}, \quad H_{22,12} = \nu(3H_{22,13} - 2H_{12,23}), \tag{4.5}$$

$$2H_{12,12} - H_{11,22} = \nu(H_{12,13} - H_{11,23}), \tag{4.6}$$

$$\mu H_{11,12} = \nu(2H_{12,23} - 2H_{22,13} + \mu H_{11,13}). \tag{4.7}$$

Next the following relations are to be used.

$$\begin{aligned}
&[11,13|11]\colon\ (1-\mu)H_{11,13} - \nu H_{11,12} = 0,\\
&[11,13|22]\colon\ \nu H_{11,12} = 0,\\
&[22,13|11]\colon\ (1-\mu)H_{22,13} + \nu(2\mu H_{11,12} - 2H_{22,13} - H_{22,12}) = 0,\\
&[22,13|22]\colon\ \nu(2\mu H_{11,12} - 2H_{22,13} - H_{22,12}) = 0,\\
&[12,13|11]\colon\ 2H_{12,13} - H_{11,23} - \mu H_{12,13} + \nu(\mu H_{11,11} - H_{12,12} - H_{13,13}) = 0,\\
&[12,13|22]\colon\ H_{11,23} - H_{12,13} - \nu(\mu H_{11,11} - H_{12,12} - H_{13,13}) = 0,\\
&[13,12|11]\colon\ \nu(H_{11,22} - H_{12,12} - H_{13,13}) + H_{12,13}\\
&\qquad\qquad + (1-\mu)(H_{12,13} - H_{11,23}) = 0,\\
&[13,12|22]\colon\ \nu(H_{11,22} - H_{12,12} - H_{13,13}) + H_{12,13} = 0,\\
&[23,12|11]\colon\ H_{12,23} + \mu(H_{12,23} - H_{13,22}) - \nu H_{13.23} = 0,\\
&[23,12|22]\colon\ H_{13,22} - 2H_{12,23} + \nu H_{13,23} = 0,\\
&[23,13|11]\colon\ H_{13,23} + \mu(\mu H_{11,12} - 2H_{13,23}) - \nu H_{12,23} = 0,\\
&[23,13|22]\colon\ H_{13,23} - \mu H_{11,12} + \nu H_{12,23} = 0.
\end{aligned}$$

Suppose that $\mu \neq 1$. Then from the first three pairs of relations $H_{11,13} = H_{22,13} = H_{12,13} = 0$ and (4.5) gives $H_{11,12} = 0$. Due to the next three pairs of relations

$H_{11,23} = H_{12,23} = H_{13,23} = 0$ and from (4.5)–(4.7) $H_{22,12} = 2H_{12,12} - H_{11,22} = 0$. Now $H_{12,23} = H_{13,23} = 0$ and (4.4) gives $\nu_3 H_{12,12} = 0$, $\nu_4 H_{13,13} = 0$, thus $\nu_3 = \nu_4 = 0$. Adding

$$\begin{aligned}
&[22,23|11]\colon\ -\mu H_{22,23} + \nu(3\mu H_{11,22} - 2H_{23,23} - H_{22,22}) = 0,\\
&[22,23|22]\colon\qquad H_{22,23} - \nu(3\mu H_{11,22} - 2H_{22,23} - H_{22,22}) = 0
\end{aligned}$$

one obtains $(1-\mu)H_{22,23} = 0$ and therefore $H_{22,23} = 0$. This together with $H_{11,23} = 0$ gives

$$\nu(H_{11,11} - H_{11,22}) = 0, \quad \nu(H_{11,22} - H_{22,22}) = 0.$$

Here $\nu \neq 0$ yields a contradiction: $H_{11,11} = H_{11,22} = H_{22,22}$ and h_{11}, h_{22} could not be linearly independent. But $\nu = 0$ leads to a contradiction, too. Indeed, then $h_{23} = 0$ and $[23,12|13]\colon H_{12,12} - H_{11,22} = 0$ contradicts to $2H_{12,12} - H_{11,22} = 0$ obtained above.

So, we must have $\mu = 1$, and thus

$$h_{33} = h_{11}, \quad h_{23} = \nu(h_{11} - h_{22}) + \nu_3 h_{12} + \nu h_{13}.$$

Suppose $\nu \neq 0$. Then $H_{11,12} = H_{11,13} = 0$ due to (4.5) and the above relation $[11,13|11]$. The relations $[33,13|22]$ and $[33,13|12]$ yield $H_{13,23} = H_{11,23} = 0$, but $[11,13|12]$ and $[11,13|13]$ give $H_{12,13} = H_{11,11} - H_{13,13} = 0$. Substitution into the above relation $[12,13|22]$ leads to a contradiction $\nu H_{12,12} = 0$.

Hence $\nu = 0$ and thus $h_{33} = h_{11}$, $h_{23} = \nu_3 h_{12} + \nu_4 h_{13}$. From (4.5) and other relations above it follows that

$$\begin{gathered}
H_{11,12} = H_{22,12} = H_{11,22} - 2H_{12,12} = H_{12,13} = H_{13,22} - 2H_{12,23} = 0,\\
H_{11,23} = H_{13,23} = 0.
\end{gathered}$$

Thus $\nu_4 H_{13,13} = 0$, so $\nu_4 = 0$ and $h_{23} = \nu_3 h_{12}$. Now

$$[12,23|13]\colon\ H_{11,22} - H_{12,12} - \nu_3^2 H_{12,12} = 0$$

gives $(1 - \nu_3^2)H_{12,12} = 0$, hence $\nu_3 = \pm 1$. Here the case $\nu_3 = 1$, when $h_{33} = h_{11}$, $h_{23} = h_{12}$, can be reduced to the case $\nu_3 = -1$ taking $-e_1$ instead of e_1.

Now the case $\nu_3 = -1$, when $h_{33} = h_{11}$, $h_{23} = -h_{12}$, turns after the transformation $e_3' = \frac{1}{\sqrt{2}}(e_1 + e_3)$, $e_2' = \frac{1}{\sqrt{2}}(-e_1 + e_3)$, $e_1' = e_2$ into

$$h_{13}' = h_{23}' = 0.$$

A straightforward check shows that all semiparallelity conditions (4.1) for $h_{13} = h_{23} = 0$ and linearly independent h_{11}, h_{22}, h_{12}, h_{33} reduce to

$$H_{11,12} = H_{11,33} = H_{22,12} = H_{22,33} = H_{12,33} = 0,$$
$$H_{11,11} = H_{22,22} = 2H_{11,22} = 4H_{12,12}.$$

Denoting now $H_{12,12} = \lambda^2$, $H_{33,33} = \kappa^2$ and taking e_4, e_5, e_6, e_7 collinear to $h_{11}-h_{22}$, $h_{11}+h_{22}$, h_{12} and h_{33}, respectively, one obtains for M^3 the differential system consisting of (3.9)–(3.12),

$$\omega_1^7 = 0, \quad \omega_2^7 = 0, \quad \omega_3^7 = \kappa\omega^3 \tag{4.8}$$

and

$$\omega^4 = \omega^5 = \omega^6 = \omega^7 = \omega^\varrho = 0, \tag{4.9}$$

where now $\varrho = 8, \dots, n$ and $\lambda > 0$, $\kappa > 0$.

By exterior differentiation the equations $\omega_3^4 = \omega_3^5 = \omega_3^6 = \omega_3^7 - \kappa\omega^3 = 0$ of this system lead to

$$\lambda\sqrt{3}\omega_3^1 \wedge \omega^1 + \lambda\sqrt{3}\omega_3^2 \wedge \omega^2 + \kappa\omega_4^7 \wedge \omega^3 = 0,$$
$$\lambda\omega_3^1 \wedge \omega^1 - \lambda\omega_3^2 \wedge \omega^2 + \kappa\omega_5^7 \wedge \omega^3 = 0,$$
$$\lambda\omega_3^2 \wedge \omega^1 + \lambda\omega_3^1 \wedge \omega^2 + \kappa\omega_6^7 \wedge \omega^3,$$
$$\omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 + \mathrm{d}\ln\kappa \wedge \omega^3 = 0.$$

Therefore

$$\lambda\sqrt{3}\omega_3^1 = A\omega^1 + F\omega^3, \quad \lambda\sqrt{3}\omega_3^2 = A\omega^2 + G\omega^3,$$
$$\kappa\omega_4^7 = F\omega^1 + H\omega^2 + I\omega^3,$$
$$\kappa\sqrt{3}\omega_5^7 = -F\omega^1 + H\omega^2 + J\omega^3, \quad \kappa\sqrt{3}\omega_6^7 = -H\omega^1 - F\omega^2 + K\omega^3,$$
$$\lambda\sqrt{3}\mathrm{d}\ln\kappa = F\omega^1 + H\omega^2 + L\omega^3.$$

The other equations (4.8) give

$$\kappa\omega_1^3 \wedge \omega^3 + \lambda\sqrt{3}\omega^1 \wedge \omega_4^7 + \lambda\omega^2 \wedge \omega_5^7 + \lambda\omega^2 \wedge \omega_6^7 = 0,$$
$$\kappa\omega_2^3 \wedge \omega^3 + \lambda\sqrt{3}\omega^2 \wedge \omega_4^7 - \lambda\omega^2 \wedge \omega_5^7 + \lambda\omega^1 \wedge \omega_6^7 = 0,$$

and thus $F = H = J = K = 0$, $3\lambda^2 I = \kappa^2 A$. Denoting $A/(\kappa\sqrt{3}) = a$, $L/(\kappa\sqrt{3}) = \varphi$ one obtains

$$\omega_3^1 = a\omega^1, \quad \omega_3^2 = a\omega^2 \tag{4.10}$$

$$\lambda\sqrt{3}\omega_4^7 = a\kappa\omega^3, \quad \omega_5^7 = \omega_6^7 = 0, \quad \mathrm{d}\ln\kappa = \varphi\omega^3. \tag{4.11}$$

Due to (4.10) the system $\omega^1 = \omega^2 = 0$ determines a foliation of M^3. Intrinsically this M^3 is a foliated semisymmetric Riemannian three-dimensional manifold, which is due to (4.10) and (1.3) of planar type (in the sense of Kowalski). Geometric characterization of this semiparallel submanifold will be given below in Section 6.

4.2. The case of $m_1 = 5$
is impossible for a semiparallel M^3 in $\mathbb{E}^n$. Indeed, there is then a linear dependence $h_{ij}\xi^{ij} = 0$ with $\sum(\xi^{ij})^2 \neq 0$. Here h_{ij} determine a vector valued symmetric tensor field, hence ξ^{ij} give a symmetric tensor field determined up to a multiplier. The tangent frame vectors e_1, e_2, e_3 can be taken at each point of M^3 so that this dependence is $h_{11}\xi^{11} + h_{22}\xi^{22} + h_{33}\xi^{33} = 0$, which after reordering, if needed, leads to

$$h_{33} = \mu_1 h_{11} + \mu_2 h_{22}. \tag{4.12}$$

The five vectors h_{11}, h_{22}, h_{12}, h_{13}, h_{23} are linearly independent.

The expression (4.12) is to be substituted into (4.1); the coefficients of the above five vectors in the resulting equation must be zero. So

$$[aa, ab|ab]\colon\ 3H_{aa,bb} - 2H_{ab,ab} - H_{aa,aa} = 0,$$

where a and b are 1 or 2, $a \neq b$. In particular this gives

$$H_{11,11} = H_{22,22} = \sigma^2 > 0. \tag{4.13}$$

Further,

$$[aa, a3|a3]\colon\ 3H_{aa,33} - 2H_{a3,a3} - H_{aa,aa} = 0. \tag{4.14}$$

This together with (4.12) and (4.13) leads to

$$(3\mu_a - 1)\sigma^2 + 3\mu_b H_{aa,bb} - 2H_{a3,a3} = 0. \tag{4.15}$$

Adding

$$[aa, b3|b3]\colon\ \mu_a\sigma^2 + (\mu_b - 1)H_{aa,bb} = 0 \tag{4.16}$$

one obtains a homogeneous linear system with non-trivial solution $(\sigma^2, H_{11,22})$, thus

$$(\mu_1 - \mu_2)(\mu_+\mu_2 - 1) = 0.$$

Here $\mu_1 + \mu_2 = 1$ is impossible: (4.16) gives $H_{11,22} = \sigma^2$, but this contradicts to (4.13), because h_{11} and h_{22} are linearly independent.

However, $\mu_1 = \mu_2$ leads to a contradiction, too. Due to (4.16) and (4.13) then $\mu_1 = \mu_2 = \mu \neq 1$, thus $H_{11,22} = \mu\sigma^2(1-\mu)^{-1}$. Now

$$[33, 3a|a3]\colon\ 3H_{aa,33} - 2H_{a3,a3} - H_{33,33} = 0$$

together with (4.13) and (4.14) gives $H_{33,33} = \sigma^2$. After substitution from (4.12) the result is $2\mu^2 + \mu - 1 = 0$. The root $\mu = \frac{1}{2}$ leads to a particular case of $\mu_1 + \mu_2 = 1$ and thus is impossible. For the root $\mu = -1$ there is $H_{aa,bb} = -\frac{1}{2}\sigma^2$, but this is impossible, too: (4.15) gives $-4\sigma^2 + \frac{3}{2}\sigma^2 - 2H_{a3,a3} = 0$ or $-\frac{5}{2}\sigma^2 = H_{a3,a3} > 0$.

4.3. The case of $m_1 = 6$.

Then all six vectors h_{ij}, where $i, j, \dots$ run over $\{1, 2, 3\}$, are linearly independent, therefore (4.1) leads to

$$\begin{gathered}
H_{12,12} = H_{23,23} = H_{13,13} = \lambda^2, \\
H_{11,22} = H_{22,33} = H_{11,33} = 2\lambda^2, \\
H_{11,11} = H_{22,22} = H_{33,33} = 4\lambda^2, \\
H_{aa,ab} = H_{aa,bc} = H_{ab,ac} = 0
\end{gathered}$$

for every three different values a, b, c from $\{1, 2, 3\}$. In short,

$$H_{ij,kl} = \lambda^2(2\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

(see [9], [11]). Now due to (2.8)

$$\Omega_i^j = -H_{i[k,l]j}\omega^k \wedge \omega^l = -\lambda^2\omega^i \wedge \omega^j$$

and according to the Schur theorem $\lambda = \text{const}$. This shows that intrinsically such an M^3 is a Riemannian space of constant curvature.

In [11] it is shown that this M^3 is a second order envelope of three-dimensional Veronese orbits $V^3(R)$ (see also [9]).

5. Proof of the Main Theorem

It is sufficient to look at the types (1), ...,(9), and 4.3 which were analyzed in the Sections 3 and 4. The types (1), (3), (4), (6), and 4.3 lead to Riemannian manifolds of constant curvature. The remaining types (2), (5), (7), (8), and (9) have been shown to produce foliated semisymmetric spaces M^3 of planar type. This concludes the proof.

6. Geometric descriptions of foliated semiparallel M^3

It remains to describe geometrically the foliated semiparallel M^3 of the cases (7)–(9) above.

Let start with (7), where M^3 in $\mathbb{E}^n$ is determined by the differential system (3.9)–(3.13). The next differential prolongation gives

$$\mathrm{d}a = -a^2\omega^3.$$

If here $a \neq 0$, then $d(x - a^{-1}e_3) = 0$ and thus the straight lines, by which this M^3 is foliated, are going through a fixed point. Hence this M^3 is then a cone. An orthogonal surface of the generators of this cone is determined by the differential system which is obtained after removing from (3.9)–(3.12) the equations of the last column. This system is investigated in [10], where κ is used instead of λ. The geometrical construction of such a surface depends essentially on the dimension of the ambient space.

If the M^3 under consideration lies in $\mathbb{E}^6$, then $a = 0$, as is shown in [12] (where λ is used instead of a), and M^3 is the cylinder $V^2(R) \times \mathbb{E}^1$ over a Veronese orbit $V^2(R) \in \mathbb{E}^5$, or an open part of such a cylinder.

If this M^3 lies essentially in $\mathbb{E}^n$, $n > 6$, then M^3 is a second order envelope of these Veronese cylinders (see [10], [12]). A more thorough investigation of such an envelope is rather complicated (see [12]).

Next let us consider the case (8). The differential system here shows that for the orthogonal surface of the family of principal geodesics, determined by $\omega^2 = \omega^3 = 0$, it holds that $\omega^1 = 0$,

$$\mathrm{d}e_2 = e_3\omega_2^3 + (-ae_1 + \lambda e_4)\omega^2,$$
$$\mathrm{d}e_3 = -e_2\omega_2^3 + (-ae_1 + \lambda e_4)\omega^3,$$

where a and λ are constants for this surface. It follows that this surface consists of totally umbilical points and therefore is a sphere (or its open part). The centre of this sphere has the radius vector

$$c = x + (a^2 + \lambda^2)^{-1}(-ae_1 + \lambda e_4)$$

and since $dc = 0$, as is easy to check (cf. [13]), all these spheres have a common centre, i.e. they are concentric.

The principal geodesics are plane curves, as is shown before, whose curvature is λ. Moreover, from (3.15) it follows that $\mathrm{d}\omega^1 = 0$, thus $\omega^1 = \mathrm{d}s$. Now $\mathrm{d}a = -a^2\mathrm{d}s$ gives for the case $a \neq 0$ that $a = s^{-1}$. Therefore $\ln\lambda = -a\omega^1$ yields $\lambda = ks^{-1}$ with $k = \text{const}$. Hence all these principal geodesics are congruent logarithmic spirals. It can be shown (see [13]) that the pole of every of these logarithmic spirals coincides with the common centre of the concentric spheres above. In the limit case when $a = 0$ these logarithmic spirals reduce to circles and then M^3 is a parallel orbit, namely the Segre orbit (see [13], [15]), or its open part. In the general case M^3 is a second order envelope of Segre orbits.

Finally, it remains to analyse the case (9), when M^3 in $\mathbb{E}^n$ is determined by the differential system (3.9)–(3.12), (4.7) together with (4.8), where $\varrho = 8, \ldots, n$.

Recall that partial differential prolongation of this system leads to the equations (4.9) and (4.10), but by the further prolongation the following consequences can be obtained (see [11]):

$$-\frac{1}{2}\mathrm{d}\ln\lambda = A\omega^1 + B\omega^2 - \frac{1}{2}\varphi\omega^3, \quad \frac{1}{5}(2\omega_1^2 - \omega_5^6) = -B\omega^1 + A\omega^2, \tag{6.1}$$

$$\frac{1}{\sqrt{3}}\omega_4^5 = A\omega^1 - B\omega^2, \quad \frac{1}{\sqrt{3}}\omega_4^6 = B\omega^1 + A\omega^2. \tag{6.2}$$

Let here $n = 7$. Then the equations in which ϱ appears as an upper index disappear. Now the two middle equations (4.10) give by exterior differentiation $\omega_5^4 \wedge \omega_4^7 = 0$, $\omega_6^4 \wedge \omega_4^7 = 0$, thus $Aa = Ba = 0$.

If here $a = 0$ then $\omega_3^1 = \omega_3^2 = 0$ and this together with (3.9)–(3.12) and (4.7)–(4.10) shows that M^3 is a product submanifold $M^2 \times M^1$. Here (6.1) and (6.2) give by differential prolongation (see [9], [11], [14])

$$\begin{aligned} dA &= B\omega_1^2 + \frac{1}{5}(14B^2 - 11A^2)\omega^1 - 5AB\omega^2, \\ dB &= -A\omega_1^2 - 5AB\omega^1 + \frac{1}{5}(14A^2 - 11B^2)\omega^2 \end{aligned}$$

and now the exterior differentiation leads to

$$A\Big[\lambda^2 + \frac{42}{25}(A^2 + B^2)\Big] = B\Big[\lambda^2 + \frac{42}{25}(A^2 + B^2)\Big] = 0,$$

thus $A = B = 0$ and $\lambda = \text{const}$. Hence M^2 is a Veronese orbit $V^2(R)$ (or its open part) in an $\mathbb{E}^5 \subset \mathbb{E}^n$. The principal geodesics of $M^3 = V^2(R) \times M^1$ are plane lines M^1 of curvature κ. Altogether this M^3 is a second order envelope of products $V^2(R) \times S^1(r)$ with $R = \text{const}$.

If $A = B = 0$ then $2\omega_1^2 = \omega_5^6$. Here an exterior differentiation leads to $a = 0$ and we return to the previous case.

Finally if $n > 7$ then the same system as above shows that M^3 is a general second order envelope of products $V^2(R) \times S^1(r)$.

7. Concluding remarks

In connection with the Main Theorem it is essential to remark that the foliated semisymmetric M^3 of the other types, i.e. hyperbolic, parabolic and elliptic, can not be immersed isometrically into $\mathbb{E}^n$ as semiparallel submanifolds, although they form much broader families as is seen from [6], where the exact forms of their metrics are given. Here the problem arises what happens if we replace $\mathbb{E}^n$ by another ambient space.

Also another question arises: can every foliated semisymmetric Riemannian manifold of planar type be immersed isometrically into $\mathbb{E}^n$ as a semiparallel submanifold?

If we try to answer this question the following result by Kowalski is important (see Theorem 7.10 in [6]).

On every planar foliated M^3 the local coordinates can be taken so that the orthonormal coframe is given by

$$\omega^1 = f(x^1, x^2)x^3 \mathrm{d}x^1, \quad \omega^2 = x^3 \mathrm{d}x^2, \quad \omega^3 = \mathrm{d}x^3.$$

The local isometry classes of such M^3 are parametrized by the function $f(x^1, x^2)$ modulo 2 arbitrary functions of 1 variable.

Let now check up what are the parametrizations of the foliated semiparallel submanifolds M^3 of the planar type in Euclidean spaces, i.e. in the cases (2), (5), (8), (7) and (9).

In the case (2) M^3 is an open part of a round cone in $\mathbb{E}^4$, which depends only on some constants and therefore can not be intrinsically a general planar foliated Riemannian M^3.

In the case (5) M^3 is determined by a one-parametric family of three-dimensional spheres, thus depends only on some real analytic functions of one real argument and therefore can not be intrinsically a general planar foliated Riemannian M^3 either.

The same can be said about the case (8), where M^3 is determined by a totally integrable differential system, as shown in Section 3, and thus depends on some constants.

Let now take the case (7), where M^3 is a second order envelope of the products $V^2(R) \times \mathbb{E}^1$ of Veronese surfaces and straight lines. Let consider first the subcase when the principal geodesics are the parallel straight lines. True enough, in this subcase M^3 is not an infinitesimally irreducible simple leaf, but a product $M^2 \times \mathbb{E}^1$, which nevertheless can be considered as a limit case of the general M^3 of the case (7), namely when a is tending to zero in (3.13). Here M^2 is a second order envelope of the Veronese surfaces $V^2(R)$.

In [14] it is shown that in $\mathbb{E}^n$, $n \geqslant 7$, the last envelope depends on one real analytic function of two real arguments. Of course at least the same arbitrariness must prevail also for the general M^3 of the case (7).

This makes plausible the following conjecture:

Into $\mathbb{E}^n$, $n \geqslant 8$, *every planar foliated Riemannian* M^3 *can be immersed isometrically as a semiparallel submanifold.*

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Author's address: Faculty of Mathematics, Tartu University, Vanemuise 46, 51014 Tartu, Estonia.