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*Czechoslovak Mathematical Journal*, Vol. 53 (2003), No. 3, 641–653

**Czechoslovak Mathematical Journal, 53 (128) (2003), 641–653**

# HIGHER DEGREES OF DISTRIBUTIVITY IN $MV$-ALGEBRAS

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(Received October 6, 2000)

*Abstract.* In this paper we deal with the $(\alpha,\beta)$-distributivity of an $MV$-algebra $\mathscr{A}$, where $\alpha$ and $\beta$ are nonzero cardinals. It is proved that if $\mathscr{A}$ is singular and $(\alpha,2)$-distributive, then it is $(\alpha,\alpha)$-distributive. We show that if $\mathscr{A}$ is complete then it can be represented as a direct product of $MV$-algebras which are homogeneous with respect to higher degrees of distributivity.

*Keywords*: $MV$-algebra, archimedean $MV$-algebra, completeness, singular $MV$-algebra, higher degrees of distributivity

*MSC 2000*: 06D35, 06F20, 06D10

## Introduction

For $MV$-algebras, several equivalent systems of axioms have been applied in the literature. We use the axioms as in the forthcoming monograph [2]. The definition is recalled in Section 1 below.

If $A$ is an $MV$-algebra, then by means of the basic operations defined in $A$ we can introduce a partial order $\leqslant$ on $A$; it turns out that $(A,\leqslant)$ is a bounded distributive lattice. We denote it by $\ell(A)$.

Let $\alpha$ and $\beta$ be nonzero cardinals. In the present paper we deal with the condition of $(\alpha,\beta)$-distributivity in the lattice $\ell(A)$. (We often speak about $(\alpha,\beta)$-distributivity of $A$ meaning the corresponding condition for the lattice $\ell(A)$.)

The condition of $(\alpha,\beta)$-distributivity has been studied in several papers in Boolean algebras (for references, cf. [14]) and in lattice ordered groups (cf. [1], [3], [6], [7], [12], [16], [17]).

Supported by Grant VEGA 2/6087/99.

Each $MV$-algebra $A$ can be constructed by means of an appropriate abelian lattice ordered group $G$ with a strong unit $u$; we put $A = \Gamma(G, u)$ (cf. [2]); in [8], [9], [10] the notation $A_0(G, u)$ was used instead of $\Gamma(G, u)$.

We prove that if $A = \Gamma(G, u)$, then the lattice $\ell(A)$ is $(\alpha, \beta)$-distributive if and only if the lattice ordered group $G$ is $(\alpha, \beta)$-distributive.

Assume that $A$ is an archimedean $MV$-algebra. If $A$ is $(\alpha, \beta)$-distributive and each nontrivial interval of $\ell(A)$ has a nontrivial subinterval whose cardinality is less than or equal to $\beta$, then the Dedekind completion of $A$ is $(\alpha, \beta)$-distributive. This yields, in particular, that $A$ is completely distributive if and only if the Dedekind completion of $A$ is completely distributive.

An $MV$-algebra $A$ is called singular if for each $0 < a \in A$ there exists $0 < a_1 \leqslant a$ such that the interval $[0, a_1]$ of $\ell(A)$ is complemented. If $A$ is singular and $(\alpha, 2)$-distributive, then it is $(\alpha, \alpha)$-distributive.

Let $A$ be an $MV$-algebra and $a \in A$. There exists a convex sublattice $L(a, \alpha, \beta)$ of $\ell(A)$ such that $a \in L(a, \alpha, \beta)$ and $L(a, \alpha, \beta)$ is maximal with respect to the property of being $(\alpha, \beta)$-distributive. If $A$ is a complete $MV$-algebra, then we can introduce in a natural way the $MV$-structure on the lattice $L(0, \alpha, \beta)$; we prove that the corresponding $MV$-algebra is a direct factor of the $MV$-algebra $A$.

We show that each complete $MV$-algebra can be represented as a direct product of $MV$-algebras which are homogeneous with respect to the higher degrees of distributivity.

## 1. Preliminaries

We recall that a lattice $L$ is called infinitely distributive if it satisfies the following condition ($a_1$) and the condition ($a_2$) which is dual to ($a_1$).

($a_1$) Whenever $(x_i)_{i \in I}$ is an indexed system of elements of $L$ such that $\bigvee_{i \in I} x_i$ exists in $L$ and $y \in L$, then $\bigvee_{i \in I} (y \wedge x_i)$ exists in $L$ and

$$y \wedge \Big(\bigvee_{i \in I} x_i\Big) = \bigvee_{i \in I} (y \wedge x_i).$$

Let $\alpha$ and $\beta$ be nonzero cardinals. Consider the following condition for the lattice $L$:

($b_1$) Whenever $T$, $S$ are nonempty sets of indices with $\operatorname{card} T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$ and $(x_{t,s})_{t \in T, s \in S}$ is an indexed system of elements of $L$ such that

$$\bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} \quad \text{and} \quad \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)}$$

exist in $L$, then

$$\bigwedge_{t\in T}\bigvee_{s\in S} x_{t,s} = \bigvee_{\varphi\in S^T}\bigwedge_{t\in T} x_{t,\varphi(t)}.$$

We denote by $(\mathrm{b}_2)$ the condition which is dual to $(\mathrm{b}_1)$.

The lattice $L$ is called $(\alpha,\beta)$-distributive if it satisfies both conditions $(\mathrm{b}_1)$ and $(\mathrm{b}_2)$.

$L$ is said to be $\alpha$-distributive if it is $(\alpha,\alpha)$-distributive. $L$ is called completely distributive if it is $\alpha$-distributive for every nonzero cardinal $\alpha$.

It is obvious that any interval of an $(\alpha,\beta)$-distributive lattice is $(\alpha,\beta)$-distributive as well.

An interval of $L$ is called nontrivial if it has more than one element.

For lattice ordered groups we use the notation as in [4]. In particular, the group operation in a lattice ordered group $G$ is denoted by $+$. The underlying lattice of $G$ is denoted by $\ell(G)$, but often we say lattice $G$ rather than lattice $\ell(G)$.

It is well-known that the lattice $\ell(G)$ is infinitely distributive. Further, the mapping $\varphi(x)=-x$ is a dual automorphism of $\ell(G)$; hence $\ell(G)$ is $(\alpha,\beta)$-distributive if and only if it satisfies the condition $(\mathrm{b}_1)$.

**1.1. Lemma.** *Let $G$ be a lattice ordered group, $0<u\in G$. The following conditions are equivalent:*

(i) *The interval $[0,u]$ is $(\alpha,\beta)$-distributive.*

(ii) *The interval $[0,u]$ satisfies the condition $(\mathrm{b}_1)$.*

Proof. Clearly (i) $\Rightarrow$ (ii). Assume that (ii) is valid. For $x\in[0,u]$ put $\varphi(x)=-x$. Then $\varphi$ is a dual isomorphism of $[0,u]$ onto the interval $[-u,0]$. Hence $[-u,0]$ satisfies the condition $(\mathrm{b}_2)$.

Further, for each $y\in[-u,0]$ we set $\psi(y)=y+u$. Then $\psi$ is an isomorphism of $[-u,0]$ onto $[0,u]$. Thus the condition $(\mathrm{b}_2)$ is satisfied in $[0,u]$. □

An $MV$-algebra $\mathscr{A}$ is defined to be an algebraic structure $(A;\oplus,\neg,0)$, where $A$ is a nonempty set, $\oplus$ is a binary operation, $\neg$ a unary operation and $0$ a constant in $A$ such that the following identities are satisfied:

MV 1) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$;

MV 2) $x\oplus y=y\oplus x$;

MV 3) $x\oplus 0=x$;

MV 4) $\neg\neg x=x$;

MV 5) $x\oplus\neg 0=\neg 0$;

MV 6) $\neg(\neg x\oplus y)=\neg(\neg y\oplus x)\oplus x$.

(Cf. [2].)

We often use the symbol $A$ in place of $\mathscr{A}$.

Let $A$ be an $MV$-algebra. For any $x, y \in A$ we write $x \leqslant y$ if $\neg x \oplus y = \neg 0$. Then $(A; \leqslant)$ is a distributive lattice with the least element 0 and the greatest element $\neg 0$. (Cf. [2, Chapter 1].) Hence without loss of generality we can consider also the lattice operations $\wedge$ and $\vee$ as belonging to the basic operations on $A$.

Let $G$ be an abelian lattice ordered group and $0 < u \in G$. For each $x, y \in [0, u]$ we put

$$x \oplus y = u \wedge (x + y), \quad \neg x = u - x.$$

The structure $([0, u]; \oplus, \neg, 0)$ will be denoted by $\Gamma(G, u)$. Then (cf. [2, 2.1.2]), $\Gamma(G, u)$ is an $MV$-algebra with $u = \neg 0$. Conversely, for every $MV$-algebra $\mathscr{A}$ there exists an abelian lattice ordered group $G$ with a strong unit $u$ such that $\mathscr{A} = \Gamma(G, u)$ (cf. [2, Section 7.1]).

From these results and from 1.1 we conclude:

**1.2. Lemma.** *Let $\mathscr{A}$ be an $MV$-algebra. The following conditions are equivalent:*

(i) *The lattice $\ell(\mathscr{A})$ is $(\alpha, \beta)$-distributive.*

(ii) *The lattice $\ell(\mathscr{A})$ satisfies the condition* $(\mathrm{b}_1)$.

Hence when considering the condition of $(\alpha, \beta)$-distributivity in lattice ordered groups or in $MV$-algebras it suffices to take the condition $(\mathrm{b}_1)$ into account.

## 2. Infinite distributivity and $(\alpha, \beta)$-distributivity

**2.1. Lemma** (Cf. [6, 1.3].)**.** *Let $L$ be a lattice. The following conditions are equivalent:*

(i) *$L$ does not satisfy* $(\mathrm{b}_1)$.

(ii) *There exist nonempty sets $T$, $S$ with* $\operatorname{card} T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$, *elements $u, v \in L$ and an indexed system $(x_{t,s})_{t \in T, s \in S}$ of elements of $L$ such that the relations*

$$v = \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s}, \quad u = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)}, \quad u < v$$

*are valid.*

**2.2. Lemma.** *Let $L$ be an infinitely distributive lattice. The following conditions are equivalent:*

(i) *$L$ does not satisfy the condition* $(\mathrm{b}_1)$.

(ii) *There exists a nontrivial interval in $L$ which does not satisfy the condition* $(\mathrm{b}_1)$.

P r o o f. The implication (ii) $\Rightarrow$ (i) is obviously valid. Suppose that (i) holds. Then there exists an indexed system $(x_{t,s})_{t\in T, s\in S}$ with the properties as in the condition (ii) of 2.1. Denote

$$x'_{t,s} = (x_{t,s} \wedge v) \vee u \quad \text{for each} \quad t \in T, \ \ s \in S.$$

Then we have

$$v = \bigwedge_{t\in T} \bigvee_{s\in S} x'_{t,s}, \quad u = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x'_{t,\varphi(t)}, \quad u < v \tag{1}$$

and $x'_{t,s} \in [u, v]$ for each $t \in T$ and each $s \in S$. Hence in view of 2.1, $[u, v]$ does not satisfy $(\mathrm{b}_1)$. □

**2.3. Lemma.** *Let $L$ be an infinitely distributive lattice. Suppose that the condition* (ii) *of* 2.1 *is valid. Let (under the notation as in* 2.1*)* $u_1, v_1 \in L$, $u \leqslant u_1 < v_1 \leqslant v$. *Then the interval* $[u_1, v_1]$ *of $L$ does not satisfy* $(\mathrm{b}_1)$.

P r o o f. For each $t \in T$ and $s \in S$ we put

$$x''_{t,s} = (x_{t,s} \wedge v_1) \vee u_1.$$

Then, by using infinite distributivity, we obtain

$$v_1 = \bigwedge_{t\in T} \bigvee_{s\in S} x''_{t,s}, \quad u_1 = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x''_{t,\varphi(t)}.$$

Since $x''_{t,s} \in [u_1, v_1]$ for each $t \in T$ and each $s \in S$, the interval $[u_1, v_1]$ does not satisfy $(\mathrm{b}_1)$. □

**2.4. Lemma.** *Let $G$ be a lattice ordered group with the strong unit $u_0$. The following conditions are equivalent:*

(i) *$G$ is $(\alpha, \beta)$-distributive.*

(ii) *The interval* $[0, u_0]$ *of $G$ is $(\alpha, \beta)$-distributive.*

P r o o f. The implication (i) $\Rightarrow$ (ii) obviously holds. Assume that (ii) is valid. By way of contradiction, suppose that $G$ is not $(\alpha, \beta)$-distributive. Then the condition (ii) from 2.1 is valid. Thus according to 2.3, each subinterval of $[u, v]$ having more than one element fails to be $(\alpha, \beta)$-distributive. Denote $b_1 = v - u$. The intervals $[u, v]$ and $[0, b_1]$ are isomorphic, hence for each $b_2 \in G$ with $0 < b_2 \leqslant b_1$, the interval $[0, b_2]$ is not $(\alpha, \beta)$-distributive. Consider the interval $[0, b_2]$ where $b_2 = b_1 \wedge u_0$. We have $b_1 > 0$, hence $0 < b_2 \leqslant u_0$. According to (ii), the interval $[0, b_2]$ must be $(\alpha, \beta)$-distributive. We have arrived at a contradiction. □

Now let $\mathscr{A}$ be an $MV$-algebra. In view of Section 1 there exists an abelian lattice ordered group $G$ with a strong unit $u$ such that $\mathscr{A} = \Gamma(G,u)$.

From 2.4 and from the relation between $\mathscr{A}$ and $G$ we immediately obtain

**2.5. Proposition.** *Let $\mathscr{A}$ be an $MV$-algebra, $\mathscr{A} = \Gamma(G,u)$. The following conditions are equivalent:*

(i) *$G$ is $(\alpha,\beta)$-distributive.*

(ii) *$\ell(\mathscr{A})$ is $(\alpha,\beta)$-distributive.*

**2.6. Lemma.** *Let $\mathscr{A}$ be an $MV$-algebra, $\mathscr{A} = \Gamma(G,u)$. Let $\beta$ be a cardinal. The following conditions are equivalent:*

(i) *Each nontrivial interval in $G$ has a nontrivial subinterval whose cardinality is less than or equal to $\beta$.*

(ii) *The same condition as in* (i) *with $G$ replaced by $\ell(\mathscr{A})$.*

Proof. The relation (i) $\Rightarrow$ (ii) is obvious. The converse implication is a consequence of the fact that $u$ is a strong unit in $G$. □

The Dedekind completion of an archimedean lattice ordered group $G$ will be denoted by $D(G)$; cf., e.g., Fuchs [5]. The definition of $D(G)$ yields:

**2.7. Lemma.** *Let $0 < x \in D(G)$. Then there exists a subset $X$ of $G^+$ such that*

(i) *$X$ is upper bounded in $G$;*

(ii) *the relation $\sup X = x$ is valid in $D(G)$.*

*Moreover, $G$ is a regular $\ell$-subgroup of $D(G)$ (in the sense that if $x_1 \in G$ is a supremum in $G$ of a subset $X_1$ of $G$, then this remains valid in $D(G)$, and dually).*

**2.8. Proposition** (Cf. [7]). *Let $G$ be an archimedean $(\alpha,\beta)$-distributive lattice ordered group, $\beta \geqslant \aleph_0$. Assume that for each $0 < a$ there is $b \in G$ such that $0 < b \leqslant a$ and $\operatorname{card}[0,b] \leqslant \beta$. Then the lattice ordered group $D(G)$ is $(\alpha,\beta)$-distributive.*

For the notion of archimedean $MV$-algebra cf., e.g., [10] (in [2], the term 'semisimple' was used).

Let $\mathscr{A}$ be an archimedean $MV$-algebra. Assume that $\mathscr{B}$ is a complete $MV$-algebra such that

(i) $\mathscr{A}$ is a subalgebra of $\mathscr{B}$,

(ii) the lattice $\ell(\mathscr{B})$ is the Dedekind completion of the lattice $\mathscr{A}$.

Then $\mathscr{B}$ is called the Dedekind completion of $\mathscr{A}$.

Let $\mathscr{A}_1$ be any archimedean $MV$-algebra. There exists an abelian lattice ordered group $G_1$ with a strong unit $u_1$ such that $\mathscr{A}_1 = \Gamma(G_1,u_1)$. Then $G_1$ is archimedean (see [1]). In [9] it was proved that $\mathscr{A}_1$ is complete if and only if $G_1$ is a complete lattice ordered group.

From this we immediately conclude that if $\mathscr{A}$ is as above and $\mathscr{A}=\Gamma(G,u)$, then the $MV$-algebra $\mathscr{B}=\Gamma(D(G),u)$ is the Dedekind completion of $\mathscr{A}$. We denote $\mathscr{B}=D(\mathscr{A})$.

It is well-known that there exists a canonical embedding of the lattice $L=\ell(\mathscr{A})$ into its Dedekind completion $D(L)$ and that $D(L)$ is defined uniquely up to isomorphisms leaving all elements of $L$ fixed. By applying 2.7 we can verify that the same is valid for $D(\mathscr{A})$.

(We remark that in [11] the notion of the maximal completion $M(\mathscr{A})$ of an $MV$-algebra $\mathscr{A}$ was investigated without assuming the archimedean property; if $\mathscr{A}$ is archimedean, then $M(\mathscr{A})=D(\mathscr{A})$.)

Thus from 2.5 and 2.8 we obtain

**2.9. Proposition.** *Let $\mathscr{A}$ be an archimedean $(\alpha,\beta)$-distributive $MV$-algebra, $\beta\geqslant\aleph_0$. Assume that for each $0<a\in A$ there is $b\in A$ with $0<b\leqslant a$ such that* $\operatorname{card}[0,b]\leqslant\beta$*. Then the $MV$-algebra $D(\mathscr{A})$ is $(\alpha,\beta)$-distributive.*

Now let $G$ be an archimedean lattice ordered group. Put $D(G)=H$. If $\{h_i\}_{i\in I}$ is an upper-bounded subset of $H$, then there exists $\sup\{h_i\}_{i\in I}$ in $H$; we denote this element by $\bigvee\limits_{i\in I}^{H} h_i$. For a lower-bounded subset $\{h'_j\}_{j\in J}$, the meaning of $\bigwedge\limits_{j\in J}^{H} h'_j$ is analogous.

**2.10. Proposition.** *Let $G$ be an archimedean lattice ordered group, $H=D(G)$. The following conditions are equivalent:*

(i) *$G$ is $(\alpha,\beta)$-distributive.*

(ii) *Whenever $T$, $S$ are nonempty sets of indices with $\operatorname{card}T\leqslant\alpha$, $\operatorname{card}S\leqslant\beta$, and $\{x_{t,s}\}_{t\in T,s\in S}$ is an indexed system of elements of $G$ which is bounded in $G$ then*

$$\bigwedge_{t\in T}^{H}\bigvee_{s\in S}^{H} x_{t,s}=\bigvee_{\varphi\in S^T}^{H}\bigwedge_{t\in T}^{H} x_{t,\varphi(t)}.$$

P r o o f. Assume that (i) fails to hold. Then in view of 1.1 and 2.1 the condition (ii) from 2.1 is satisfied. Since $G$ is a regular $\ell$-subgroup of $H$ we obtain

$$v=\bigwedge_{t\in T}^{H}\bigvee_{s\in S}^{H} x_{t,s}>u=\bigvee_{\varphi\in S^T}^{H}\bigwedge_{t\in T}^{H} x_{t,\varphi(t)}, \tag{2}$$

whence (ii) is not valid.

Further suppose that (i) holds. By way of contradiction, assume that (ii) is not satisfied. Hence there are $u, v \in H$, $\{x_{t,s}\}_{t\in T, s\in S} \subseteq G$, $\operatorname{card} T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$ such that (2) is valid in $H$.

Each element of $H^+$ is a join (in $H$) of some elements of $G^+$. Since $u < v$, there is $a \in G$ such that $a \leqslant v$, $0 < a \nleqslant u$. For each $t \in T$ and $s \in S$ we put

$$x'_{t,s} = x_{t,s} \wedge a.$$

In view of the infinite distributivity of $H$ we obtain

$$a = v \wedge a = \bigwedge_{t\in T}^{H} \bigvee_{s\in S}^{H} x'_{t,s}, \tag{3}$$

$$u \wedge a = \bigvee_{\varphi\in S^T}^{H} \bigwedge_{t\in T}^{H} x'_{t,\varphi(t)} \tag{4}$$

and $u \wedge a < a$. Since $x'_{t,s}, a$ and $u \wedge a$ are elements of $G$, from (3), (4) we infer that

$$a = \bigwedge_{t\in T} \bigvee_{s\in S} x'_{t,s},$$

$$u \wedge a = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x'_{t,\varphi(t)}.$$

Thus $G$ is not $(\alpha, \beta)$-distributive, which is a contradiction. □

Let $\mathscr{A}$ be an archimedean $MV$-algebra and let $G$ be an abelian lattice ordered group with a strong unit $u$ such that $\mathscr{A} = \Gamma(G, u)$.

Put $\mathscr{B} = D(\mathscr{A})$. The lattice operations in $\mathscr{B}$ will be denoted by $\wedge^{\mathscr{B}}$ and $\vee^{\mathscr{B}}$.

From 2.10 and from the relation between $D(G)$ and $D(\mathscr{A})$ we conclude

**2.11. Proposition.** *Let $\mathscr{A}$ be an archimedean $MV$-algebra, $\mathscr{B} = D(\mathscr{A})$. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is $(\alpha, \beta)$-distributive.*

(ii) *Whenever $T$, $S$ are nonempty sets of indices with card $T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$ and $\{x_{t,s}\}$ is an indexed system of elements of $A$, then*

$$\bigwedge_{t\in T}^{\mathscr{B}} \bigvee_{s\in S}^{\mathscr{B}} x_{t,s} = \bigvee_{\varphi\in S^T}^{\mathscr{B}} \bigwedge_{t\in T}^{\mathscr{B}} x_{t,\varphi(t)}.$$

Since each Boolean algebra $B$ can be viewed as an archimedean $MV$-algebra (i.e., there exists an archimedean $MV$-algebra $\mathscr{A}$ such that $\ell(\mathscr{A}) = B$), we obtain as a corollary that 2.11 holds in the case when $\mathscr{A}$ is a Boolean algebra (cf. also [14, 35.4]).

## 3. Singular $MV$-algebras

The notion of singular $MV$-algebra was defined in the Introduction above. By the same condition we define also the notion of a singular lattice ordered group (cf. [4]). Let $\mathscr{A} = \Gamma(G, u)$; it is obvious that $\mathscr{A}$ is singular if and only if $G$ is singular.

**3.1. Lemma.** *Let $G$ be a singular lattice ordered group. The following conditions are equivalent:*
(i) *$G$ is $(\alpha, \beta)$-distributive.*
(ii) *If $0 < a \in G$ and if the interval $[0, a]$ of $G$ is complemented, then it is $(\alpha, \beta)$-distributive.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is obvious. Assume that (ii) is valid. By way of contradiction, suppose that (i) does not hold. Hence the condition (ii) from 2.1 is satisfied. Put $b = v - u$. The intervals $[u, v]$ and $[0, b]$ are isomorphic; thus in view of 2.3, no nontrivial interval of $[0, b]$ is $(\alpha, \beta)$-distributive. Since $G$ is singular, we have arrived at a contradiction. □

**3.2. Proposition.** *Let $G$ be a singular lattice ordered group. Then the following conditions are equivalent:*
(i) *$G$ is $\alpha$-distributive.*
(ii) *$G$ is $(\alpha, 2)$-distributive.*

P r o o f. It is obvious that (i) $\Rightarrow$ (ii). Suppose that (ii) is satisfied. Let $0 < a \in G$ such that the interval $[0, a]$ is complemented. Then $[0, a]$ is a Boolean algebra. Thus in view of [13] (cf. also [14, 19.1]), the interval $[0, a]$ is $\alpha$-distributive. Now 3.1 yields that $G$ is $\alpha$-distributive. □

From 2.5 and 3.2 we conclude

**3.3. Proposition.** *Let $\mathscr{A}$ be a singular $MV$-algebra. Then $\mathscr{A}$ is $\alpha$-distributive if and only if it is $(\alpha, 2)$-distributive.*

Another result of this type is

**3.4. Proposition.** *Let $\mathscr{A}$ be a complete $MV$-algebra. The following conditions are equivalent:*
(i) *$\mathscr{A}$ is $\alpha$-distributive.*
(ii) *$\mathscr{A}$ is $(\alpha, 2)$-distributive.*

P r o o f. Clearly (i) $\Rightarrow$ (ii). Let (ii) be valid. As usual, let $\mathscr{A} = \Gamma(G,u)$. According to 2.5, $G$ is $(\alpha,2)$-distributive. In view of [9], $G$ is complete. Thus [7, Theorem 3.9] yields that $G$ is $\alpha$-distributive. By applying 2.5 again we conclude that $\mathscr{A}$ is $\alpha$-distributive. □

## 4. The sublattices $L^{\mathscr{A}}(a,\alpha,\beta)$

Let $\alpha$ and $\beta$ be as above.

First suppose that $G$ is a lattice ordered group and $a \in G$. We denote by $\mathscr{L}^G(a,\alpha,\beta)$ the system of all convex sublattices $X$ of $\ell(G)$ such that the element $a$ belongs to $X$ and $X$ is $(\alpha,\beta)$-distributive. Let the system $\mathscr{L}^G(a,\alpha,\beta)$ be partially ordered by the set-theoretical inclusion.

The assertions of the following proposition have been proved in [6].

**4.1. Proposition.** *Let $G$ be a lattice ordered group and $a \in G$.*

(i) *The system $\mathscr{L}^G(a,\alpha,\beta)$ has a greatest element which will be denoted by $L^G(a,\alpha,\beta)$.*
(ii) *$L^G(a,\alpha,\beta)$ is a closed sublattice of $\ell(G)$.*
(iii) *If $b \in G$, then either $L^G(a,\alpha,\beta) = L^G(b,\alpha,\beta)$ or $L^G(a,\alpha,\beta) \cap L^G(b,\alpha,\beta) = \emptyset$.*
(iv) *$L^G(0,\alpha,\beta)$ is an $\ell$-ideal of $G$ and*

$$L^G(a,\alpha,\beta) = L^G(0,\alpha,\beta) + a.$$

Now let $\mathscr{A}$ be an $MV$-algebra. Suppose that $G$ is an abelian lattice ordered group with a strong unit $u$ such that $\mathscr{A} = \Gamma(G,u)$.

For an element $a$ of $A$ we define $\mathscr{L}^{\mathscr{A}}(a,\alpha,\beta)$ analogously as we defined $\mathscr{L}^G(a,\alpha,\beta)$ in the case when $a$ was an element of $G$.

**4.2. Proposition.** *Let $\mathscr{A}$ be an $MV$-algebra and $a \in A$.*

(i) *The system $L^{\mathscr{A}}(a,\alpha,\beta)$ has a greatest element which will be denoted by $L^{\mathscr{A}}(a,\alpha,\beta)$.*
(ii) *$L^{\mathscr{A}}(a,\alpha,\beta)$ is a closed sublattice of $\ell(\mathscr{A})$.*
(iii) *If $b \in A$, then either $L^{\mathscr{A}}(a,\alpha,\beta) = L^{\mathscr{A}}(b,\alpha,\beta)$ or*

$$L^{\mathscr{A}}(a,\alpha,\beta) \cap L^{\mathscr{A}}(b,\alpha,\beta) = \emptyset.$$

(iv) *$L^{\mathscr{A}}(0,\alpha,\beta)$ is closed with respect to the operation $\oplus$.*

P r o o f. The assertions (i), (ii) and (iii) were proved (for any infinitely distributive lattice) in [6, Section 2].

From the relations between $\mathscr{A}$ and $G$ we conclude that

$$L^{\mathscr{A}}(a,\alpha,\beta)=L^G(a,\alpha,\beta)\cap A$$

is valid for each $a\in A$.

Since $x\oplus y=(x+y)\wedge a$ holds for each $x,y\in A$, from 4.1 (iv) we obtain that $L^{\mathscr{A}}(0,\alpha,\beta)$ is closed with respect to the operation $\oplus$. □

Suppose that the $MV$-algebra $\mathscr{A}$ is complete (i.e., $\ell(\mathscr{A})$ is a complete lattice). Then in view of 4.2 (ii), $L^{\mathscr{A}}(0,\alpha,\beta)$ has a greatest element; let us denote it by $u_1$. Hence $L^{\mathscr{A}}(0,\alpha,\beta)$ is the interval $[0,u_1]$ of the lattice $\ell(\mathscr{A})$. We also have

$$[0,u_1]=L^G(0,\alpha,\beta)\cap A. \tag{5}$$

Further, in view of [9], $G$ is a complete lattice ordered group. Thus [6, Theorem 7.7] yields that $L^G(0,\alpha,\beta)$ is a direct factor of the lattice ordered group $G$.

We denote by $G_1$ the convex $\ell$-subgroup of $G$ generated by the element $u_1$. Then $u_1$ is a strong unit in $G_1$. Put $\mathscr{A}_1=\Gamma(G_1,u_1)$. Then $L^{\mathscr{A}}(0,\alpha,\beta)$ is the underlying lattice of $\mathscr{A}_1$.

From the fact that $L^G(0,\alpha,\beta)$ is a direct factor of $G$, from (5) and from [8, Lemma 3.2] we conclude

**4.3. Proposition.** *Let $\mathscr{A}$ be a complete $MV$-algebra and let $\mathscr{A}_1$ be as above. Then $\mathscr{A}_1$ is a direct factor of the $MV$-algebra $\mathscr{A}$.*

## 5. $d$-homogeneous $MV$-algebras

Let $L$ be a distributive lattice having more than one element. If $L$ is completely distributive, then we put $d(L)=\infty$.

Suppose that $L$ fails to be completely distributive. Then there exists an infinite cardinal $\alpha$ such that

(i) $L$ is not $\alpha$-distributive;

(ii) if $\beta$ is a nonzero cardinal with $\beta<\alpha$, then $L$ is $\beta$-distributive.

We put $d(L)=\alpha$.

A lattice $L$ will be called homogeneous with respect to the higher degrees of distributivity if either $\operatorname{card} L=1$, or $\operatorname{card} L>1$ and for each nontrivial interval $[a,b]$ in $L$ the relation

$$d([a,b])=d(L)$$

is valid.

We denote by $C$ the class of all infinite cardinals; put $C_1 = C \cup \{\infty\}$. Let $G$ be a lattice ordered group. For $i \in C_1$ let $H_i(0)$ be the set of all elements $x \in G$ such that either $x = 0$, or $x \neq 0$ and

$$d([x \wedge 0, x \vee 0]) = i.$$

Then (cf. [6, 7.2]) $H_i(0)$ is a convex $\ell$-subgroup of $G$. The results of [6, Sections 3 and 4] yield if $H_i(0) \neq \{0\}$, then $d(H_i(0)) = i$ and that $H_i(0)$ is $d$-homogeneous.

The notion of completely subdirect decomposition of a lattice ordered group has been introduced in [15]; cf. also [6].

Assume that $G$ is a complete lattice ordered group, $G \neq \{0\}$. Put

$$C_1^0 = \{i \in C_1 \colon H_i(0) \neq \{0\}\}.$$

**5.1. Proposition.** *Each complete $MV$-algebra $\mathscr{A}$ with $A \neq \{0\}$ can be represented as a direct product of $d$-homogeneous $MV$-algebras.*

Proof. Let $\mathscr{A}$ be a complete $MV$-algebra, $A \neq \{0\}$, $\mathscr{A} = \Gamma(G, u)$. Then $G$ is a complete lattice ordered group and $G \neq \{0\}$. Thus from [6, Theorem 7.9] we infer that $G$ can be represented as a completely subdirect product of its $\ell$-subgroups $H_i(0)$ ($i \in C_1^0$). All these $H_i(0)$ are $d$-homogeneous.

For each $i \in C_1^0$ let $u_i$ be the component of the element $u$ in $H_i(0)$. Put $\mathscr{A}_i = \Gamma(G, u_i)$. Clearly $\mathscr{A}_i = \Gamma(H_i(0), u_i)$. Then the lattice $\ell(\mathscr{A}_i)$ has more than one element and is $d$-homogeneous.

According to Theorem 4.2 in [8], the $MV$-algebra $\mathscr{A}$ is a direct product of the $MV$-algebras $\mathscr{A}_i$ ($i \in C_1^0$). □

Let $\alpha$ be an infinite cardinal. For the notion of weak $\alpha$-distributivity in a Boolean algebra and also for the detailed references to authors and papers dealing with this notion cf. [14, § 30] (sample names: Banach, Kuratowski, Horn, Tarski, Sikorski).

The above-mentioned definition given in [14] can be applied also for complete $MV$-algebras. Similarly as in the case of homogeneity with respect to higher degrees of distributivity we can introduce the notion of homogeneity with respect to higher degrees of weak distributivity.

By the same reasoning as above we can prove the result analogous to Proposition 5.1 where distributivity is replaced by weak distributivity.

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