F. Karabacak; Adnan Tercan

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MATRIX RINGS WITH SUMMAND INTERSECTION PROPERTY

F. Karabacak, Eskisehir, and A. Tercan, Ankara

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Abstract. A ring R has right SIP (SSP) if the intersection (sum) of two direct summands of R is also a direct summand. We show that the right SIP (SSP) is the Morita invariant property. We also prove that the trivial extension of R by M has SIP if and only if R has SIP and $(1-e)Me = 0$ for every idempotent e in R. Moreover, we give necessary and sufficient conditions for the generalized upper triangular matrix rings to have SIP.

Keywords: modules, Summand Intersection Property, Morita invariant

MSC 2000: 16D10, 16D15

1. Introduction

Throughout this paper R denotes an associative ring with identity and all R-modules are unital right R-modules.

A right R-module M has *SIP* if the intersection of every pair of direct summands of M is a direct summand of M. The ring R has *right SIP* provided that the right R-module R has SIP, i.e., the intersection of every pair of right ideals of R which are generated by idempotents is also generated by an idempotent. A module M has SSP if the sum of every pair of direct summands of M is a direct summand of M.

In [6], I. Kaplansky showed that a free module over a principal ideal domain has the Summand Intersection Property. Since then these modules have been investigated by several authors (see for example [3], [5] and [7]).

In this note we deal with matrix rings which have SIP. To this end we prove necessary and sufficient conditions for full matrix rings and trivial extensions to have SIP.

For any unexplained terminology please see [1] and [4].

The following lemma is well known but its proof is given for completeness.

Lemma 1. *If M_R has* SIP (SSP) *then so does every direct summand of M.*

Proof. Let M have SIP and let X be a direct summand of M. Let K and L be direct summands of X. Thus both K and L are direct summands of M. There exists a submodule F of M such that $M = (K \cap L) \oplus F$. Hence

$$X = X \cap M = X \cap \big((K \cap L) \oplus F\big) = (K \cap L) \oplus (X \cap F).$$

It follows that $K \cap L$ is a direct summand of‘ X. Hence X has SIP. SSP case is similar. $\square$

Lemma 2 [3, Lemma 3.1]. *Let R be a product of rings, $R = \prod\limits_I R_i$. Then R has* SSP *(*SIP *on the left) if and only if each R_i has* SSP *(*SIP *on the left).*

Proof. Straightforward (see for example [3]). $\square$

Next we provide an example which illustrates that a direct sum of modules which have SIP need not have SIP in general (see [7]).

Example. Let p be a prime integer. Let M be the Z-module $Z \oplus (Z/Zp)$. Then M does not have SIP.

Proof. Note that Z-modules Z and Z/Zp both have SIP. Let us consider submodules $A = (1, \overline{p})Z$ and $B = (1, \overline{1})Z$; it is clear that A and B are direct summands of M. However, $A \cap B$ is not a direct summand of M. $\square$

2. Matrix rings

We shall give an example which does not have SIP. The following example is taken from [2, Example 1.5].

Example 4. Let F be a field and

$$T = \left\{ \begin{bmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{bmatrix} : a, b, x, y \in F \right\}.$$

Let

$$e = e^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad c = c^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $eT \cap cT$ is nilpotent. Hence $eT \cap cT$ is not a direct summand of T. It follows that T does not have SIP.

Naturally a question arises, namely, when the full matrix ring over a ring has SIP. For, let R be any ring with identity, e an idempotent in R such that $R = ReR$ and S the subring eRe. Let M be a right-module. Then Me is a right S-module.

Lemma 5. *Let K be a submodule of M_R. Then K is a direct summand of M_R if and only if Ke is a direct summand of $(Me)_S$.*

P r o o f. Suppose K is a direct summand of M_R. Then $M = K \oplus N$ for some submodule N of M_R. Thus $Me = Ke + Ne$. However, $Ke \cap Ne \leqslant K \cap N = 0$. Therefore $Me = Ke \oplus Ne$. Conversely, suppose that $Me = Ke \oplus L$ for some submodule of $(Me)_S$. It is easy to see that $K \cap LR = 0$ and

$$M = MeR = (Ke + L)R = KeR + LR = K + LR.$$

Thus $M_R = K \oplus LR$. □

By using Lemma 5 we obtain the following result.

Theorem 6. *With the above notation, let M be a right R-module. The right R-module M has* SIP *(*SSP*) if and only if the right S-module Me has* SIP *(*SSP*).*

Corollary 7. *The ring R has right* SIP *(*right SSP*) if and only if the right eRe-module Re has* SIP *(*SSP*). In this case, S has right* SIP *(*right SSP*).*

P r o o f. Immediate by Theorem 6. □

Now we let S be a ring with identity 1, n a positive integer and R the ring $M_n(S)$ of all $n \times n$ matrices with entries in S. Let e_{11} denote the matrix in R with $(1,1)$ entry 1 and all other entries 0. It is well known that e_{11} is idempotent and $S \cong e_{11}Re_{11}$ and $R = Re_{11}R$.

Thus Theorem 6 gives without further proof the following result which was pointed out above.

Theorem 8. *With the above notation, $R = M_n(S)$ has right* SIP *(right* SSP*) if and only if the free right S-module S^n has* SIP *(*SSP*).*

Corollary 9. *If S has* SIP *(*SSP*) then $R = M_n(S)$ has right* SIP *(*SSP*).*

P r o o f. Since S has SIP then so does S^n by Lemma 2. Then the result follows by Theorem 8. The SSP case is similar. □

Let $\wp$ be a ring theoretic property. Then $\wp$ is said to be Morita invariant if and only if the following holds: whenever a ring R has $\wp$ then so do $M_n(R)$ for all $n \geqslant 2$ and eRe for all $e^2 = e \in R$ such that $R = ReR$.

Combining Corollary 7 with Corollary 9 we arrive at the following fact.

Corollary 10. *The right* SIP *(*SSP*) is Morita invariant.*

Now we prove a result on trivial extensions.

Theorem 11. *Let R be a ring, M a R-R bimodule and A the corresponding trivial extension. Then A has* SIP *if and only if the following two conditions hold:*

(i) *R has* SIP,

(ii) *for every idempotent e of R we have $(1-e)Me = 0$.*

Proof. Suppose that A has SIP and let us prove that (i) and (ii) hold.

(i) Let e, c be idempotent elements of R. Then (eR, eM) $(= (e,0)A)$ and (cR, cM) are direct summands of A and therefore $(eR \cap cR, eM \cap cM)$ is also a summand of A, whence $eR \cap cR$ is a direct summand of R and so R has SIP.

(ii) Suppose that $(1-e)xe \neq 0$ for some idempotent of R and $x \in M$. Then $y = (1-e)xe$ verifies $ey = 0$, $ye = y$; hence $(e,y)^2 = (e,y)$ and $I = (e,y)A$ is a direct summand of A. Since $J = (1-e,0)A$ also is a direct summand of A, then $I \cap J$ is, by hypothesis, a direct summand of A and it is easy to see that $I \cap J = \big(0, yR \cap (1-e)M\big)$. Therefore, there exists an idempotent element (f,z) of A such that $(f,z)A = I \cap J$. Thus, $fR = 0$ and so $f = 0$. Since $fz + zf = z$ hence $z = 0$. It follows that $yR \cap (1-e)M = 0$. So $y = 0$ as required.

Assume that (i) and (ii) hold and let (e,x) be an idempotent of A. Then $ex + xe = x$, so that $(1-e)xe = xe = 0$ by hypothesis, and $x = ex \in eM$. Now we have that every direct summand of A is of the form (eR, eM) for some idempotent element e of R. Since

$$eM \cap cM = eRM \cap cRM = (eR \cap cR)M$$

and R satisfies SIP, we have that

$$(eR, eM) \cap (cR, cM) = \big(eR \cap cR, (eR \cap cR)M\big) = (fR, fM).$$

It follows that A has SIP. $\square$

In the rest of this paper let A be the ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ where R and S are rings with identities and M is left R, right S-bimodule.

Lemma 12. *If* $\operatorname{Soc} A$ *is a direct summand of A then $M = 0$.*

Proof. By hypothesis, there exists $e^2 = e \in A$ such that $\operatorname{Soc} A = eA$. Hence $(1-e)A = 0$. Thus

$$A = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \cong \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix}$$

(see [4]). Since $\operatorname{Soc} A_A = \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \leqslant \operatorname{Soc} A$, there exists $f^2 = f \in \operatorname{Soc} A$ such that $f(\operatorname{Soc} A) = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$. Now $f = \begin{bmatrix} r & m \\ 0 & 0 \end{bmatrix}$ and $f^2 = \begin{bmatrix} r^2 & rm \\ 0 & 0 \end{bmatrix}$. Hence

$$\begin{bmatrix} r & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} rR & rM \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}.$$

But $r = 0$. It follows that $M = 0$. □

Theorem 13. *Let* $\operatorname{Soc} A$ *be a direct summand of* A. *Then* A *has right* SIP *if and only if both* R *and* S *have right* SIP.

Proof. Immediate by Lemma 12 and Lemma 2. □

As comparison to Theorem 13 we state the following example.

Example 14. Let F be a field and R the upper triangular matrix ring, i.e.

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F \right\}.$$

Then $\operatorname{Soc} R = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is not a direct summand of R_R. It is easy to check that R has right SIP.

Theorem 15. *A has right* SIP *if and only if the following two conditions hold:*

(i) *both* R *and* S *have right* SIP,

(ii) $eMf = 0$ *for every* $e^2 = e \in R$, $f^2 = f \in S$ *with at least one of* e *and* f *not being identity.*

Proof. Assume that A has right SIP. Then

$$A = \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}.$$

By Lemma 1, both $\begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$ have SIP and hence S has right SIP. Now, let I and J be direct summands of R. Thus $I = aR$ and $J = bR$ for some $a, b \in R$ such that $a^2 = a$ and $b^2 = b$. Since

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} A \cap \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} A = \begin{bmatrix} aR \cap bR & aM \cap bM \\ 0 & 0 \end{bmatrix}$$

is a direct summand of $\begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix}$ hence $aR \cap bR$ is clearly a direct summand of R.

Suppose $eMf \neq 0$ for some idempotents in R with at least one of e and f not being identity. Hence there exists $0 \neq y = exf \in eMf$ for some $x \in M$. Note that $ey = y$. Let $\alpha = \begin{bmatrix} e & 0 \\ 0 & 1-f \end{bmatrix}$, $\beta = \begin{bmatrix} 1-e & y \\ 0 & f \end{bmatrix}$. It is straightforward to check that $\alpha^2 = \alpha$ and $\beta^2 = \beta$. Hence αA and βA are direct summands of A. So,

$$\begin{aligned} \alpha A \cap \beta A &= \begin{bmatrix} eR & eM \\ 0 & (1-f)S \end{bmatrix} \cap \begin{bmatrix} (1-e)R & (1-e)M + yS \\ 0 & fS \end{bmatrix} \\ &= \begin{bmatrix} 0 & eM \cap \big((1-e)M + yS\big) \\ 0 & 0 \end{bmatrix} \end{aligned}$$

is a direct summand of A. It follows that $eM \cap \big((1-e)M + yS\big) = 0$. Thus $y \in (1-e)M$, i.e. $y = (1-e)m$ for some $m \in M$. But $ey = y = 0$ which is contradiction.

For the converse, let $\begin{bmatrix} e & x \\ 0 & f \end{bmatrix}$ be any idempotent of A. Note that $ex + xf = x$ and $exf = 0$, $(1-e)xf = 0$. Therefore $xf = 0$ and hence $x \in eM$. It follows that every direct summand of A is of the form $\begin{bmatrix} eR & eM \\ 0 & fS \end{bmatrix}$ for some $e^2 = e \in R$ and $f^2 = f \in S$. Since $eM \cap cM = eRM \cap cRM = (eR \cap cR)M$, we have

$$\begin{bmatrix} eR & eM \\ 0 & fS \end{bmatrix} \cap \begin{bmatrix} cR & cM \\ 0 & gS \end{bmatrix} = \begin{bmatrix} eR \cap cR & eM \cap cM \\ 0 & fS \cap gS \end{bmatrix} = \begin{bmatrix} \alpha R & \alpha M \\ 0 & \beta S \end{bmatrix}$$

where $\alpha^2 = \alpha$ and $\beta^2 = \beta$. It follows that A has right SIP. $\square$

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Authors' addresses: F. Karabacak, Anadolu University Education Faculty, Department of Mathematics, 26470 Eskisehir, Turkey, e-mail: fkarabac@anadolu.edu.tr; A. Tercan, Hacettepe University Department of Mathematics, Beytepe Campus, 06532 Ankara, Turkey, e-mail: tercan@hacettepe.edu.tr.