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Darko Žubrinić

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GENERATING SINGULARITIES OF SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS USING WOLFF'S POTENTIAL

Darko Žubrinić, Zagreb

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Abstract. We consider a quasilinear elliptic problem whose left-hand side is a Leray-Lions operator of p-Laplacian type. If $p < \gamma < N$ and the right-hand side is a Radon measure with singularity of order γ at $x_0 \in \Omega$, then any supersolution in $W^{1,p}_{\mathrm{loc}}(\Omega)$ has singularity of order at least $(\gamma - p)/(p-1)$ at x_0. In the proof we exploit a pointwise estimate of $\mathscr{A}$-superharmonic solutions, due to Kilpeläinen and Malý, which involves Wolff's potential of Radon's measure.

Keywords: quasilinear elliptic, singularity, Sobolev function

MSC 2000: 31B05, 35B05

1. Introduction

Quasilinear elliptic problems having singular solutions have aroused a considerable interest in recent years. See for example Díaz [1], Kilpeläinen [5], Mou [10], Grillot [3], Simon [11], Korkut, Pašić, Žubrinić [7], [8], Žubrinić [12], and the references therein. The aim of this note is to extend the oscillation estimate stated in [7, Theorem 7] (see also [8, Theorem 3]). This will enable us to derive that if the right-hand side of a quasilinear elliptic equation possesses a singularity of sufficiently high order at a given point, then any weak solution is singular at the same point, see Corollary 1.

To make suitable comparisons, we consider the following quasilinear elliptic problem of Leray-Lions type:

$$-\operatorname{div} a(x, u, \nabla u) = f(x) \quad \text{in } \mathscr{D}'(\Omega), \tag{1}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geqslant 1$, $f \in L^1_{\mathrm{loc}}(\Omega)$ is given, and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution. Here $1 < p < \infty$ and $a(x, \eta, \xi)$ is a Carathéodory function with values in $\mathbb{R}^N$ satisfying conditions of Leray-Lions type, see [9]:

$$\exists\alpha > 0,\ \ a(x,\eta,\xi)\cdot\xi \geqslant \alpha|\xi|^p \quad \text{for a.e. } x\in\Omega,\ \ \forall\eta\in\mathbb{R},\ \ \forall\xi\in\mathbb{R}^N, \tag{2}$$

$$\begin{cases} \exists a_1 \geqslant 0,\ \ \exists a_2 > 0,\ \ \exists g\in L^{p'}(\Omega),\ \ \forall\eta\in\mathbb{R},\ \ \forall\xi\in\mathbb{R}^N, \\ |a(x,\eta,\xi)| \leqslant g(x) + a_1|\eta|^{p-1} + a_2|\xi|^{p-1}\ \text{ a.e. in }\ \Omega. \end{cases} \tag{3}$$

In [7, Theorem 7] we have obtained the following result: if $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a supersolution of (1), then for any ball $B_{2r}(x_0)\subset\Omega$ such that $f(x)\geqslant 0$ a.e. on $B_{2r}(x_0)$ we have the a priori estimate

$$\operatorname*{ess\,sup}_{B_{2r}(x_0)} u \geqslant \operatorname*{ess\,inf}_{B_{2r}(x_0)} u + br^{p'} \operatorname*{ess\,inf}_{B_r(x_0)} f(x)^{p'-1}. \tag{4}$$

Here b is an explicit constant given by

$$b = \frac{\alpha}{(a_2 p)^{p'}(2^N-1)^{p'-1}}.$$

Using the above estimate it is possible to show that if $f(x)$ has a singularity of order γ at x_0, that is,

$$f(x) \geqslant \frac{C}{|x-x_0|^\gamma} \tag{5}$$

in a neighbourhood of x_0, and $p<\gamma<N$, then any supersolution u is singular at x_0. More precisely, $\operatorname{osc}_{x_0} u = \infty$, where the oscillation of u at the point x_0 is defined by

$$\operatorname*{osc}_{x_0} u = \lim_{r\to 0} \operatorname*{osc}_{B_r(x_0)} u, \qquad \operatorname*{osc}_{B_r(x_0)} u = \operatorname*{ess\,sup}_{B_r(x_0)} u - \operatorname*{ess\,inf}_{B_r(x_0)} u. \tag{6}$$

In [12, Theorem 3] we have improved this result in the case when instead of a Leray-Lions type operator $-\operatorname{div} a(x,u,\nabla u)$ on the left-hand side of (1) we have a p-Laplace operator $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, that is, $a(x,\eta,\xi) = |\xi|^{p-2}\xi$. In this case, if $p<N$ and (5) is fulfilled with $\gamma\in(p, 1+\frac{N}{p'})$, then any supersolution $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (1) such that $u\geqslant 0$ in a ball $B_r(x_0)$ for some $r>0$, has a singularity of order at least

$$\frac{\gamma-p}{p-1}. \tag{7}$$

Moreover, we have obtained an explicit and sharp lower bound of $u(x)$ in the ball, see [12]:

$$u(x) \geqslant \frac{p-1}{\gamma-p}\Big(\frac{C}{N-\gamma}\Big)^{p'-1}\big[|x-x_0|^{-\frac{\gamma-p}{p-1}} - r^{-\frac{\gamma-p}{p-1}}\big]. \tag{8}$$

In proving this we have used some explicit computations together with Tolksdorf's comparison principle.

Here we obtain an analogous result for a class of quasilinear elliptic differential operators which is narrower than the Leray-Lions class, but still includes the case of p-Laplace operators. Furthermore, we allow Radon measures on the right-hand side:

$$-\operatorname{div} \mathscr{A}(x, \nabla u) = \mu. \tag{9}$$

More precisely, we consider a Carathéodory mapping $\mathscr{A}\colon\ \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\mathscr{A}(x,\xi)\cdot\xi \geqslant \alpha|\xi|^p, \quad |A(x,\xi)| \leqslant a_2|\xi|^{p-1}, \tag{10}$$

$$(\mathscr{A}(x,\xi) - \mathscr{A}(x,\zeta))\cdot(\xi-\zeta) > 0 \tag{11}$$

for a.e. $x \in \Omega$ and all $\xi, \zeta \in \mathbb{R}^N$, where a_2 is a positive constant, and for all $\lambda \in \mathbb{R}$,

$$\mathscr{A}(x, \lambda\xi)\cdot\xi = \lambda|\lambda|^{p-2}\mathscr{A}(x,\xi). \tag{12}$$

The corresponding quasilinear elliptic operator $-\operatorname{div}\mathscr{A}(x,\nabla u)$ is said to be of p-Laplacian type. Such operators are discussed in detail in Heinonen, Kilpeläinen and Martio [4].

In order to state our main result, we recall some terminology introduced in [4]. First, any solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the distribution equation $-\operatorname{div}\mathscr{A}(x,u) = 0$ always has a continuous representative, which is called an $\mathscr{A}$-harmonic function.

Second, a lower semicontinuous function $u\colon\ \Omega \to \mathbb{R}\cup\{+\infty\}$ is said to be $\mathscr{A}$-superharmonic in Ω, if it is not identically equal to $+\infty$ on any component of connectedness of Ω, and if it satisfies the following comparison principle: for all open $D \subset\subset \Omega$ and any $h \in C(\overline{D})$ which is $\mathscr{A}$-harmonic in D, the condition $h \leqslant u$ on ∂D implies that $h \leqslant u$ in D.

Now we list some basic properties of $\mathscr{A}$-superharmonic functions.

Proposition 1 (see [4, 7.25]).

(i) *If $u \in W^{1,p}_{loc}(\Omega)$ is such that*

$$-\operatorname{div}\mathscr{A}(x,\nabla u) \geqslant 0 \tag{13}$$

in the weak sense, then there exists an $\mathscr{A}$-superharmonic function $\hat{u}\colon\ \Omega \to \mathbb{R}\cup\{+\infty\}$ such that $\hat{u} = u$ a.e. Furthermore, for all $x \in \Omega$ we have

$$\hat{u}(x) = \lim_{r\to 0} \operatorname*{ess\,inf}_{B_r(x)} u. \tag{14}$$

(ii) *If u is $\mathscr{A}$-superharmonic, then $u(x) = \lim_{r\to 0} \operatorname{ess\,inf}_{B_r(x)} u$ holds for all $x \in \Omega$. Furthermore, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then it satisfies* (13).

The main result of this paper is the following.

Theorem 1. *Assume that $\mathscr{A}(x,\xi)$ satisfies conditions* (10)–(12). *Let μ be a Radon measure on Ω and let $f \in L^1_{\mathrm{loc}}(\Omega)$, $f(x) \geqslant 0$ a.e. in Ω, be such that $\mu \geqslant f(x)$ in Ω in the weak sense. Then there exists a constant $b_1 = b_1(p,N) > 0$ such that for any $\mathscr{A}$-supersolution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of* (9) *and any $x \in \Omega$, $r > 0$, such that $B_{2r}(x) \subset \Omega$, we have*

$$u(x) \geqslant \operatorname*{ess\,inf}_{B_{2r}(x)} u + b_1 \cdot r^{p'} \operatorname*{ess\,inf}_{y\in B_r(x)} f(y)^{p'-1}. \tag{15}$$

Under the conditions of Theorem 1 we conclude that

$$u(x) \geqslant \operatorname*{ess\,inf}_{\Omega} u + b_1 \cdot \Big(\frac{d(x,\partial\Omega)}{2}\Big)^{p'} \operatorname*{ess\,inf}_{y\in\Omega} f(y)^{p'-1} \tag{16}$$

for all $x \in \Omega$, where $d(x,\partial\Omega)$ is the distance from x_0 to the boundary of Ω.

As we see, when we deal with Leray-Lions operators of p-Laplacian type, then Theorem 1 gives us a better estimate than (4) in the sense that the estimate (15) is pointwise for $u(x)$. In this sense it extends [8, Theorem 7]. Note, however, that the constant $b_1 = b_1(p,N) > 0$ appearing in (15) is not known, while we have a precise value for the analogous constant b appearing in (4). As a consequence of Theorem 1 we obtain the following result about generating singularities of $\mathscr{A}$-superharmonic solutions, which extends [8, Corollary 15] from the setting of p-Laplace operators to general Leray-Lions operators of p-Laplacian type.

Corollary 1. *Assume that $\mathscr{A}(x,\xi)$ is as in the preceding theorem, and let $p < \gamma < N$. Let μ be a Radon measure on Ω such that*

$$\mu \geqslant \frac{C}{|x-x_0|^\gamma}$$

in the weak sense, where $C > 0$ and $x_0 \in \Omega$. Then any $\mathscr{A}$-superharmonic solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (9) *such that $u \geqslant 0$ on $B_R(x_0)$ for some $R > 0$, has a singularity of order at least $\frac{\gamma-p}{p-1}$. More precisely, there exists a constant $D = D(p,N) > 0$ such that for all x, $|x-x_0| < \frac{1}{2}R$, we have*

$$u(x) \geqslant \operatorname*{ess\,inf}_{B_R(x_0)} u + \frac{D}{|x-x_0|^{\frac{\gamma-p}{p-1}}}. \tag{17}$$

2. Proofs

Let μ be a nonnegative Radon measure on Ω. The Wolff potential of μ in a ball $B_r(x)$ is defined by

$$\mathbf{W}^{\mu}_{1,p}(x;r)=\int_0^r [t^{p-N}\mu(B_t(x))]^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}. \tag{18}$$

The following result due to Kilpeläinen and Malý [6] is crucial in proving Theorem 1. We state it here in a slightly different form.

Theorem 2. *Let μ be a nonnegative Radon measure, $B_{2r}(x)\subseteq\Omega$, and let u be a supersolution to problem* (9). *Then there exists a constant $C=C(p,N)>0$ such that*

$$u(x)\geqslant \operatorname*{ess\,inf}_{B_{2r}(x)} u + c_1\cdot \mathbf{W}^{\mu}_{1,p}(x;r). \tag{19}$$

P r o o f of Theorem 1. Let us denote

$$K=\operatorname*{ess\,inf}_{y\in B_r(x)} f(y).$$

Then by the assumption we have $\mu\geqslant K$, and therefore for any $t\in(0,r)$,

$$\mu(B_t(x))\geqslant K|B_t(x)|=KC_Nt^N,$$

where $|B_t(x)|$ is the Lebesgue measure of $B_t(x)$ and C_N is the volume of the unit ball in $\mathbb{R}^N$. Hence,

$$\begin{aligned}\mathbf{W}^{\mu}_{1,p}(x;r)&\geqslant\int_0^r [t^{p-N}KC_Nt^N]^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}\\ &=(KC_N)^{\frac{1}{p-1}}\int_0^r t^{p'-1}\,\mathrm{d}t=(KC_N)^{\frac{1}{p-1}}\frac{r^{p'}}{p'}.\end{aligned} \tag{20}$$

Now using Theorem 2 we obtain

$$u(x)\geqslant \operatorname*{ess\,inf}_{B_{2r}(x)} u + c_1\mathbf{W}^{\mu}_{1,p}(x;r)\geqslant \operatorname*{ess\,inf}_{B_{2r}(x)} u+\frac{c_1C_N^{p'-1}}{p'}\cdot r^{p'}\cdot K^{p'-1}.$$

The claim follows with $b_1=c_1C_N^{p'-1}/p'$. $\square$

P r o o f of Corollary 1. Let us fix $x \in B_{R/2}(x_0)$ and define $r = |x - x_0|$. Then clearly, $B_{2r}(x) \subset B_R(x_0)$. Using estimate (15) and $|y - x_0| \leqslant |y - x| + |x - x_0| \leqslant 2r$, we obtain that

$$\begin{aligned} u(x) &\geqslant \operatorname*{ess\,inf}_{B_{2r}(x)} u + b_1 \cdot r^{p'} \operatorname*{ess\,inf}_{y \in B_r(x)} \Big(\frac{C}{|y - x_0|^\gamma} \Big)^{p'-1} \\ &\geqslant \operatorname*{ess\,inf}_{B_R(x_0)} u + b_1 \cdot r^{p'} \Big(\frac{C}{(2r)^\gamma} \Big)^{p'-1} \\ &= \operatorname*{ess\,inf}_{B_R(x_0)} u + D r^{-\frac{\gamma-1}{p-1}}, \end{aligned}$$

where $D = c_1/p'(C \cdot C_N/2^\gamma)^{p'-1}$. $\square$

As we see, in order to have precise values of constants b_1 and D appearing in (15) and (17), respectively, it would be necessary to know the precise value of $c_1 = c_1(p, N)$ in Theorem 2.

In [8, Corollary 15] it has been shown that if x_0 is a boundary point of Ω having a weak cone property and $f(x) \geqslant C/|x - x_0|^\gamma$ with $\gamma = p$, then any supersolution of (1) has a singularity in x_0 with at least finite, positive oscillation. We say that a point $x_0 \in \partial\Omega$ has a weak cone property if there exists $d \in (0, 1)$ and a sequence of balls $B_{r_k}(x_k) \subset \Omega$ such that $x_k \to x_0$, $r_k \to 0$ as $k \to \infty$, and $r_k > d|x_k - x_0|$ for all k. It is easy to see that if a boundary point x_0 has the cone property, then it has the weak cone property. The converse is not true. Cusps do not have weak cone property.

Here we provide an example showing that a finite and positive oscillation of supersolution $u \in W^{1,p}(\Omega)$ at a boundary point x_0 of the domain can indeed be achieved, provided Ω has the weak cone property at x_0. We consider a distribution equation

$$-\Delta u = (N - 1)\frac{x_1}{|x|^3} \quad \text{in } \mathscr{D}'(\Omega), \tag{21}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, $x = (x_1, \ldots, x_N)$. In order to define the domain Ω, we introduce polar coordinates $(r, \theta_1, \ldots, \theta_{N-1})$ in $\mathbb{R}^N$, where $r \geqslant 0$, $\theta_1 \in (0, 2\pi)$, $\theta_i \in (0, \pi)$ for $i = 2, \ldots, N - 1$, and

$$\begin{aligned} x_1 &= r \cdot \sin\theta_{N-1} \sin\theta_{N-2} \ldots \sin\theta_2 \sin\theta_1, \\ x_2 &= r \cdot \sin\theta_{N-1} \sin\theta_{N-2} \ldots \sin\theta_2 \cos\theta_1, \\ &\;\;\vdots \\ x_N &= r \cdot \cos\theta_{N-1}. \end{aligned}$$

Let us define Ω in polar coordinates as the set of all $(r, \theta_1, \ldots, \theta_{N-1}) \in \mathbb{R}^N$ satisfying the following inequalities:

$$r \in (0, R), \quad \theta_i \in \Big(\frac{\pi}{4}, \frac{3\pi}{4}\Big), \quad i = 1, \ldots, N-1.$$

It is easy to see that $0 \in \partial\Omega$ and Ω has the cone property at $x_0 = 0$ (and also at any $x_0 \in \partial\Omega$). If we denote the right-hand side of (21) by $f(x)$, then

$$f(x) \geqslant \frac{C}{|x|^2}$$

where $C = (N-1)R \cdot 2^{-(N-1)/2}$, since $\sin\theta_i \geqslant 2^{-1/2}$. Here we have $\gamma = p = 2$. It is not difficult to check that the function $u(x) = \frac{x_1}{|x|}$ is indeed a weak solution of (21) in $H^1(\Omega)$, and u has a finite positive oscillation at $x_0 = 0$, precisely, $\operatorname{osc}_{x_0=0} u = 2$. This example stems from a well known diagonal elliptic system $-\Delta w = w|\nabla w|^2$, $w = (w_1, \ldots, w_N)$, which possesses a weak solution $w = x \cdot |x|^{-1} \in H^1(B_R(0), \mathbb{R}^N)$, see Giaquinta [2, p. 62].

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Author's address: Department of Applied Mathematics, Faculty of Electrical Engineering, Unska 3, 10000 Zagreb, Croatia, e-mail: darko.zubrinic@fer.hr.