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PREDUALS OF SPACES OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS

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Abstract. For U a balanced open subset of a Fréchet space E and F a dual-Banach space we introduce the topology τ_γ on the space $\mathscr{H}(U,F)$ of holomorphic functions from U into F. This topology allows us to construct a predual for $(\mathscr{H}(U,F),\tau_\delta)$ which in turn allows us to investigate the topological structure of spaces of vector-valued holomorphic functions. In particular, we are able to give necessary and sufficient conditions for the equivalence and compatibility of various topologies on spaces of vector-valued holomorphic functions.

Keywords: holomorphic functions, Fréchet spaces, preduals

MSC 2000: 46G20, 46G25, 46A20, 46A32, 46A25

1. Introduction

For many years research workers in complex analysis on infinite dimensional spaces concentrated on scalar-valued holomorphic functions. Although the vector-valued holomorphic functions arose in certain contexts most attention was concentrated on the nature of the domain rather than the range. In recent times, however, it was discovered that consideration of the vector-valued case did lead to more than minor differences and it was necessary to look at this case if one wished to obtain necessary and sufficient conditions for certain problems such as the $\tau_o=\tau_\omega$ problem ([9], [17]).

Thus it is now appropriate that we begin a systematic study of the vector-valued situation. In this article we are mainly concerned with the representation of spaces of vector-valued holomorphic functions as dual spaces. We apply these representations to the problem of topologies and to reflexivity of spaces of holomorphic functions.

We will show that we can find U, a balanced open subset of a Fréchet Schwartz space, and F, a reflexive Banach space, such that $(\mathscr{H}(U,F),\tau_o)'=(\mathscr{H}(U,F),\tau_\omega)'$ but $(\mathscr{H}(U,F),\tau_o)\neq(\mathscr{H}(U,F),\tau_\omega)$.

Although a number of our results apply to spaces of holomorphic functions on arbitrary locally convex spaces with values in arbitrary Banach spaces the most fruitful investigations arise when we consider holomorphic functions on Fréchet spaces with values in a Banach space which is 1-complemented in its second dual.

2. Holomorphic functions with values in dual-Banach spaces

For U balanced open in a Fréchet space E much of what is known about the relationship between the three topologies τ_o, τ_ω and τ_δ on $\mathscr{H}(U)$ depends on the fact that $(\mathscr{H}(U), \tau_o)$ is semi-Montel. For F an infinite dimensional Banach space this is no longer true of $(\mathscr{H}(U,F), \tau_o)$. However, when F is a dual Banach space, we can introduce a new semi-Montel topology, τ_γ, on $\mathscr{H}(U,F)$. Many of the results for the scalar-valued case can be extended to the vector-valued case using τ_γ in place of τ_o. To do this, we first look at the Dixmier-Ng Theorem and what it tells us about the bounded subsets of dual-Banach spaces and spaces of holomorphic functions with values in a dual Banach space.

The Dixmier-Ng Theorem says that a Banach space F is the dual of a Banach space V if and only if there is a locally convex topology τ on F such that the unit ball of F is a τ-compact subset. An examination of the proof [24, p. 211] shows that if F is a dual Banach space, the following families of subsets of F coincide:

(a) the norm-bounded subsets of F,
(b) the τ-bounded subsets of F,
(c) the $\sigma(F,V)$-bounded subsets of F,
(d) the $\sigma(F,V)$-relatively compact subsets of F,
(e) the τ-relatively compact subsets of F.

Let F_τ denote the vector space F with the τ-topology. It follows from the above that if F is a dual-Banach space then F_τ is a semi-Montel space. For reflexive Banach spaces, the above result is true for $V = F'$ and $\tau = \sigma(F,F')$. If V is any predual of F we define F_{σ^*} by $F_{\sigma^*} = (F, \sigma(F,V))$. We now show that for U open in a Fréchet space E we have $\mathscr{H}(U,F) = \mathscr{H}(U,F_{\sigma^*})$. This has already been established by Nachbin for Banach spaces (see [30]).

Proposition 1. *Let U be an open subset of a Fréchet space E and let $F = V_b'$ be a dual-Banach space, then $\mathscr{H}(U,F) = \mathscr{H}(U,F_{\sigma^*})$.*

Proof. We clearly have $\mathscr{H}(U,F) \subseteq \mathscr{H}(U,F_{\sigma^*})$. Let $f \in \mathscr{H}(U,F_{\sigma^*})$ and let B_V denote the unit ball of V. We consider V as a subspace of F'. For K compact in U, $f(K)$ is norm bounded in F. It follows that $\mathscr{F} = (\varphi \circ f)_{\varphi \in B_V}$ is τ_o-bounded in $\mathscr{H}(U)$. Since E is a Fréchet space, $\mathscr{F}$ is locally bounded. This means that, for each $\zeta \in U$, there is a neighbourhood U_ζ of ζ such that $\sup_{\varphi \in B_V} \|\varphi \circ f\|_{U_\zeta} < \infty$. In particular, we have that $f(U_\zeta)$ is $\sigma(F, V)$-bounded and therefore, it is norm bounded in F. By [21, Lemma 2.8] and [23, Theorem 5.2] we have $f \in \mathscr{H}(U, F)$. $\square$

For U open in a Fréchet space E and F a dual-Banach space Proposition 1 allows us to define a new topology, τ_γ, on $\mathscr{H}(U, F)$ by

$$(\mathscr{H}(U, F), \tau_\gamma) = (\mathscr{H}(U, V'_\gamma), \tau_o),$$

where V'_γ is the space F endowed with the topology of uniform convergence on compact subsets of V. It should be observed that a Banach space need not have a unique predual and that the definition of τ_γ may well depend on our choice of V. However, what is important is the fact that τ_γ is a semi-Montel topology (see [18]).

For each integer n, $(P(^nE, F), \tau_\gamma)$ will denote $P(^nE, F)$ with the topology induced from $(\mathscr{H}(U, F), \tau_\gamma)$.

Theorem 2. *Let E be a Fréchet space and F be a Banach space which is complemented in its bidual. Then for each integer n, τ_ω is the barrelled topology associated with τ_o on $P(^nE, F)$.*

Proof. We first assume that F is a dual-Banach space. Let $\{U_n\}_n$ denote a fundamental system of neighbourhoods about 0 in E. Since $(P(^nE, F), \tau_\omega)$ is a countable inductive limit of Banach spaces it is a barrelled and bornological DF space. Hence $\tau_\omega \geqslant \tau_{\gamma,t}$. By [21, 3.93], $(P(^nE, F), \tau_\gamma)$ is semi-Montel and, since τ_γ is weaker than τ_o, we have that $B_n := \{P \in P(^nE, F)\colon \|P\|_{U_n} \leqslant 1\}$ is τ_γ-compact for each n. Applying Proposition 3.40 of [21] we conclude $\tau_\omega = \tau_{o,t}$.

If F is complemented in its bidual let π denote the projection from F'' into F. Then $I_{P(^nE)}\varepsilon\pi$ is a continuous projection from $(P(^nE, F''), \tau_o) = (P(^nE), \tau_o)\varepsilon F''$ into $(P(^nE, F), \tau_o) = (P(^nE), \tau_o)\varepsilon F$. Using [37, Theorem 1.2.1], we can show as in [3, Lemma 1] that the τ_ω topology on $P(^nE, F'')$ restricted to $P(^nE, F)$ is equal to $(P(^nE, F), \tau_\omega)$. By [3, Lemma 1] the barrelled topology associated with τ_o on $P(^nE, F)$ is equal to the restriction to $P(^nE, F)$ of the barrelled topology associated to τ_o on $P(^nE, F'')$. By the first paragraph this final topology is τ_ω. Therefore, $\tau_{o,t} = \tau_\omega$ on $P(^nE, F)$. $\square$

The class of Banach spaces which are 1-complemented in their bidual includes reflexive Banach spaces, dual Banach spaces and the space $L^1[0,1]$ which is neither reflexive nor dual. It does not include c_o.

By [13, Lemma 6.5] it follows that if U is balanced open in a Fréchet space E and F is 1-complemented in its bidual then τ_δ is the barrelled topology associated with τ_o on $\mathscr{H}(U, F)$.

Let U be an open subset of a Fréchet space E and let F be a dual-Banach space. Denote by $G(U,F)$ the space of linear maps φ from $\mathscr{H}(U,F)$ into $\mathbb{C}$ such that the restriction of φ to each locally bounded set of $\mathscr{H}(U,F)$ is τ_γ-continuous.

By [29, Theorem 1.1], $G(U,F)$ is complete and $G(U,F)'_i = (\mathscr{H}(U,F),\tau_\delta)$. Since the $\sigma(F,V)$-bounded and norm-bounded subsets of F coincide the τ_γ-bounded subsets of $\mathscr{H}(U,F)$ are locally bounded. It follows from Grothendieck's Completeness Theorem, [25, Theorem 3.11.1], that $G(U,F)$ is the completion of $(\mathscr{H}(U,F),\tau_\gamma)'_b$. As $(\mathscr{H}(U,F),\,\tau_\gamma)$ is semi-Montel, $G(U,F)$ is barrelled.

Let K be a compact subset of a Fréchet space E, $F = V'_b$ a dual-Banach space and let $\{V_n\}_n$ be a neighbourhood basis about 0 in E. We define the τ_γ-topology on $\mathscr{H}(K,F)$ by

$$(\mathscr{H}(K,F),\tau_\gamma) = \operatorname*{ind}_{j} \mathscr{H}(K+V_j,F),\tau_\gamma).$$

By using arguments similar to the above, one can show that for each integer n (resp. each compact subset K of E) there is a complete locally convex space $Q(^nE,F)$ (resp. $G(K,F)$) such that $Q(^nE,F)'_i = (P(^nE,F),\tau_\omega)$ (resp. $G(K,F)'_i = (\mathscr{H}(K,F),\tau_\omega)$). As in the above, $Q(^nE,F)$ (resp. $G(K,F)$) is defined as the space of all linear maps from $P(^nE,F)$ (resp. $\mathscr{H}(K,F)$) to $\mathbb{C}$ whose restriction to each locally bounded set is τ_γ-continuous. It can be shown that $Q(^nE,F)$ (resp. $G(K,F)$) is the completion of $(P(^nE,F),\tau_\gamma)'_b$ (resp. $(\mathscr{H}(K,F),\tau_\gamma)'_b$). Both the spaces $Q(^nE,F)$ and $G(K,F)$ are Fréchet spaces. We write $Q(^nE)$ for $Q(^nE,\mathbb{C})$ and $G(K)$ for $G(K,\mathbb{C})$. Another description of a predual of $(\mathscr{H}(K),\tau_\omega)$ is given in [12]. It is shown in [13] that $Q(^nE)$ is isomorphic to $\widehat{\bigotimes}_{s,n,\pi} E$. In his thesis [35] Ryan showed that $\widehat{\bigotimes}_{s,n,\pi} E$ is a predual of $(P(^nE),\tau_\omega)$.

Section 2 of [14] is easily adapted to show that if U is a balanced open subset of a Fréchet space E then $\{Q(^nE,F)\}_n$ is an $\mathscr{S}$-decomposition for $G(U,F)$.

Since K is a compact subset of a Fréchet space E and F a Banach space which is 1-complemented in its bidual it follows as in Theorem 2 that τ_ω is the barrelled topology associated with τ_o on $\mathscr{H}(K,F)$. This result has also been obtained by Bonet, Domański and Mujica, [12] with help of an abstract technical condition on spaces of Banach-valued linear maps on Fréchet spaces.

If E is a quasinormable Fréchet space we can show, as in [5, Lemma 4], that $(P(^nE,F),\tau_\omega)$ satisfies the strict Mackey convergence condition. Since the inductive limit $(P(^nE,F),\tau_\omega) = \operatorname{ind}_m(P(^nE_m,F),\|\cdot\|)$ is regular, [10, p. 123] implies that $(P(^nE,F),\tau_\omega)$ is sequentially retractive. From Theorem in [11] we conclude that $Q(^nE,F)$ is a quasinormable Fréchet space. (This result can also be obtained from [7, Corollary 5].) In particular, if E is quasinormable, we have $Q(^nE,F)'_b = (P(^nE,F),\tau_\omega)$ for each n. It is also possible to prove, using the methods of [15], that if U is balanced open in a Fréchet space E and F is a dual-Banach space then

$(\mathscr{H}(U,F),\tau_\delta)$ satisfies the strict Mackey convergence condition if and only if E is quasinormable.

3. Tensor representation of preduals

In this section, we construct preduals of spaces of vector-valued holomorphic functions, germs and polynomials using tensor products. These results allow us to give necessary and sufficient conditions for spaces of vector-valued holomorphic functions to be reflexive. We begin with the case of homogeneous polynomials.

Theorem 3. *Let E be a Fréchet space and $F = V'_b$ a dual Banach space, then $Q(^nE,F) = Q(^nE)\mathbin{\widehat{\bigotimes}_\pi} V$ for each integer n.*

Proof. For each integer n we have

$$(P(^nE,F),\tau_\gamma) = (P(^nE),\tau_o)\varepsilon V'_\gamma = Q(^nE)'_\gamma \varepsilon V'_\gamma.$$

By Buchwalter's Duality Theorem, Theorem 16.1.7 of [26],

$$(P(^nE,F),\tau_\gamma)'_b = (Q(^nE)'_\gamma \varepsilon V'_\gamma)'_\gamma = Q(^nE)\mathbin{\widehat{\bigotimes}_\pi} V.$$

Since this space is complete we see that $Q(^nE,F) = Q(^nE)\mathbin{\widehat{\bigotimes}_\pi} V$. □

We note that it follows from [19, Theorem 2.2] that

$$Q(^nE,F) = \mathscr{N}((P(^nE),\tau_o),V),$$

the space of nuclear maps from $(P(^nE),\tau_o)$ into V.

We now look at holomorphic germs.

Lemma 4. *Let K be a compact subset of a Fréchet space E and let F be a* gDF *space. Then $(\mathscr{H}(K,F),\tau_o) = (\mathscr{H}(K),\tau_o)\varepsilon F$ algebraically and both spaces have the same bounded sets.*

Proof. The algebraic equivalence of $(\mathscr{H}(K,F),\tau_o)$ and $(\mathscr{H}(K),\tau_o)\varepsilon F$ follows from [9, Lemma 4]. Let $\{V_j\}_j$ be a neighbourhood basis of 0 in E and B a bounded subset of $F\varepsilon(\mathscr{H}(K),\tau_o)$. By [34, Proposition 1.9], $B(G)$ is compact in $(\mathscr{H}(K),\tau_o)$ for every equicontinuous subset G of F'. By the argument of [9, Lemma 4] it follows that there is an integer n_o such that $B(G)$ is contained and bounded in $(\mathscr{H}(K+V_{n_o}),\tau_o)$. Applying [34, Proposition 1.9] again we see that B is contained and bounded in $F\varepsilon(\mathscr{H}(K+V_{n_o}),\tau_o)$. □

Since the τ_o bounded subsets of $\mathscr{H}(K+V_{n_o},F)$ are locally bounded when F is a Banach space we get the following result.

Corollary 5. *Let K be a compact subset of a Fréchet space E and let F be a dual-Banach space. Then each bounded subset of $(\mathscr{H}(K,F),\tau_\gamma)$ is locally bounded and the inductive limit $(\mathscr{H}(K,F),\tau_\gamma)=\mathrm{ind}_j(\mathscr{H}(K+V_j,F),\tau_\gamma)$ is regular, moreover, $(\mathscr{H}(K,F),\tau_o)$ and $(\mathscr{H}(K,F),\tau_\omega)$ have the same bounded sets.*

Theorem 6. *Let K be a compact subset of a Fréchet space E and $F=V_b'$ a dual Banach space. Then $G(K,F)=G(K)\mathbin{\widehat{\bigotimes}_\pi} V=((\mathscr{H}(K),\tau_o)\varepsilon F_\gamma)_b'$.*

Proof. By definition $G(K,F)$ is the space of all linear maps from $\mathscr{H}(K,F)$ to $\mathbb{C}$ whose restriction to each locally bounded set is τ_γ continuous. Each locally bounded subset of $\mathscr{H}(K,F)$ is compact for τ_γ and the topology induced from $(\mathscr{H}(K),\tau_o)\varepsilon F_\gamma$. Since the latter topology is weaker than τ_γ it coincides with τ_γ on the locally bounded sets. Therefore, $G(K,F)$ may be regarded as the space of linear maps from $\mathscr{H}(K,F)$ into $\mathbb{C}$ whose restriction to each locally bounded set is continuous for the topology $(\mathscr{H}(K),\tau_o)\varepsilon F_\gamma$. Applying Grothendieck's Completeness Theorem, [25, Theorem 3.11.1], and Corollary 5, we see that $G(K,F)$ is the completion of $((\mathscr{H}(K),\tau_o)\varepsilon F_\gamma)_b'$. By Buchwalter's Duality Theorem, [26, Theorem 16.1.7],

$$((\mathscr{H}(K),\tau_o)\varepsilon F_\gamma)_b'=G(K)\mathbin{\widehat{\bigotimes}_\pi} V.$$

Since this space is complete we see that

$$G(K,F)=G(K)\mathbin{\widehat{\bigotimes}_\pi} V=((\mathscr{H}(K),\tau_o)\varepsilon F_\gamma)_b'.$$

$\square$

We again note that [19, Theorem 2.2] implies that $G(K,F)=\mathscr{N}((\mathscr{H}(K),\tau_o),V)$, the space of nuclear maps from $(\mathscr{H}(K),\tau_o)$ into V.

If E and F are as in Theorem 6 then we can write the equation

$$G(K,F)=G(K)\mathbin{\widehat{\bigotimes}_\pi} V$$

as

$$(\mathscr{H}(K,V_\gamma),\tau_o)_b'=(\mathscr{H}(K),\tau_o)_b'\mathbin{\widehat{\bigotimes}_\pi} V,$$

which generalises [9, Corollary 7].

Theorem 7. *Let U be an open subset of a Fréchet space E and $F = V_b'$ a dual-Banach space, then $(G(U) \widehat{\bigotimes_\pi} V)' = \mathscr{H}(U, F)$.*

Proof. Denote by $G_c(U)$ the space $(\mathscr{H}(U), \tau_o)_b' = \mathrm{ind}_{K \subset U} G(K)$. It follows from [13, Proposition 2.6] that $G(U)$ is the completion of $G_c(U)$. By Corollary 15.5.4 of [25] we have the algebraic equivalence

$$\begin{aligned}(G(U) \widehat{\bigotimes_\pi} V)' = (G_c(U) \widehat{\bigotimes_\pi} V)' &= ((\operatorname*{ind}_{K \subset U} G(K)) \widehat{\bigotimes_\pi} V)' \\ &= \operatorname*{proj}_{K \subset U} (G(K) \widehat{\bigotimes_\pi} V)' = \operatorname*{proj}_{K \subset U} G(K, F)' \\ &= \operatorname*{proj}_{K \subset U} \mathscr{H}(K, F) = \mathscr{H}(U, F).\end{aligned}$$

□

The proof of [15, Theorem 3] is easily adapted to give the following result.

Theorem 8. *Let E be a Fréchet space and F a reflexive Banach space. Then the following conditions are equivalent:*

(a) *$(P(^nE, F), \tau_\omega)$ is reflexive for every positive integer n.*
(b) *$(\mathscr{H}(U, F), \tau_\omega)$ is semi-reflexive for one (and hence every) balanced open subset U of E.*
(c) *$(\mathscr{H}(U, F), \tau_\gamma)' = (\mathscr{H}(U, F), \tau_\omega)'$ for one (and hence every) balanced open subset U of E.*
(d) *$G(U, F) = (\mathscr{H}(U, F), \tau_\delta)_b'$ for one (and hence every) balanced open subset U of E.*

One can replace holomorphic functions by homogeneous polynomials or holomorphic germs in Theorem 8.

It follows from Theorem 8 and [2] that $(\mathscr{H}(U, F), \tau_o)' = (\mathscr{H}(U, F), \tau_\omega)'$ for U balanced open in Tsirelson's space T^* and F any reflexive Banach space with nontrivial Radamacher type or cotype. When E and F are reflexive Banach spaces with the approximation property, Alencar, [1], showed that $(P(^nE, F), \tau_\omega)$ is reflexive if and only if every n-homogeneous polynomial from E into F is of compact type.

For U balanced open in a Fréchet Schwartz space E it is shown in [28] that $\tau_o = \tau_\omega$ on $\mathscr{H}(U)$. This equivalence is not carried over to the vector-valued case, as it follows from [33] that there is a Fréchet Schwartz space E such that $\tau_o \neq \tau_\omega$ on $\mathscr{H}(E, C_2)$. By [17] Fréchet Schwartz spaces E with the property that $\tau_o = \tau_\omega$ on $\mathscr{H}(U, F)$ for every balanced open subset U of E and every Banach space F, are those which are quasinormable by operators [32]).

Let F be a reflexive Banach space. Since $Q({}^nE)$ is Schwartz, it is a projective limit of Banach spaces with compact linking maps. As the tensor product of a compact map and a weakly compact map is weakly compact it will follow that $Q({}^nE) \mathbin{\widehat{\bigotimes}_\pi} F'$ is a quasinormable infra-Schwartz Fréchet space and therefore must be reflexive. Hence $(P({}^nE, F), \tau_\omega)$ is reflexive and from this we conclude:

Corollary 9. *Let U be a balanced open subset of a Fréchet Schwartz space E and F a reflexive Banach space. Then $(\mathscr{H}(U,F), \tau_o)' = (\mathscr{H}(U,F), \tau_\omega)'$.*

4. Equivalence of topologies for spaces of vector valued holomorphic functions

We now take a brief look at the coincidence of the compact open and the τ_ω topologies on spaces of homogeneous polynomials and holomorphic germs and for the equivalence of the compact open and the τ_δ-topology on spaces of vector-valued holomorphic functions on Fréchet Schwartz spaces.

Lemma 10. *Let E be a Fréchet space with $\tau_o = \tau_\omega$ on $P({}^nE)$ and $F = V_b'$ a dual-Banach space. Then $\tau_o = \tau_\omega$ on $P({}^nE, F)$ if and only if $(Q({}^nE), V)$ has* BB.

Proof. If $(Q({}^nE), V)$ has BB then it follows from [8, Corollary 7] that $Q({}^nE) \mathbin{\widehat{\bigotimes}_\pi} V$ satisfies the density condition and therefore is distinguished. Thus we have

$$(P({}^nE, F), \tau_o) = (P({}^nE), \tau_o)\varepsilon F = Q({}^nE)_b'\varepsilon F = (Q({}^nE) \mathbin{\widehat{\bigotimes}_\pi} V)_b' = (P({}^nE, F), \tau_\omega).$$

Conversely, if $\tau_o = \tau_\omega$ on $P({}^nE, F)$ then $(Q({}^nE) \mathbin{\widehat{\bigotimes}_\pi} V)_b' = Q({}^nE)_b'\varepsilon V_b'$ and hence $(Q({}^nE), V)$ has BB. $\square$

For holomorphic germs we have the following result.

Theorem 11. *Let K be a compact subset of a Fréchet space E with $\tau_o = \tau_\omega$ on $\mathscr{H}(K)$ and let $F = V_b'$ be a dual-Banach space. If $(G(K), V)$ has* BB *then $\tau_o = \tau_\omega$ on $\mathscr{H}(K, F)$.*

Proof. Since $\tau_o = \tau_\omega$ on $\mathscr{H}(K)$, $G(K)$ is a Fréchet Montel space. By Theorem 6 and [8, Corollary 7] it follows that $G(K,F) = G(K) \mathbin{\widehat{\bigotimes}_\pi} V$ satisfies the density condition and hence is distinguished. Thus we have

$$(G(K) \mathbin{\widehat{\bigotimes}_\pi} V)_b' = (G(K) \mathbin{\widehat{\bigotimes}_\pi} V)_i' = (\mathscr{H}(K,F), \tau_\omega).$$

Since $(G(K), V)$ has BB it follows from the proof of [26, 16.1.7] (replacing compact sets by bounded sets) that $(G(K) \widehat{\bigotimes_\pi} V)'_b = (\mathscr{H}(K), \tau_o)\varepsilon F$. As $(\mathscr{H}(K), \tau_o)\varepsilon F$ is a weaker topology on $\mathscr{H}(K, F)$ than τ_o, we have that $\tau_o = \tau_\omega$ on $\mathscr{H}(K, F)$. $\square$

Let us now turn to the equivalence of τ_o and τ_δ.

Theorem 12. *Let U be a balanced open subset of a Fréchet space E such that $\tau_o = \tau_\delta$ on $\mathscr{H}(U)$ and let F be a reflexive Banach space. Then $\tau_o = \tau_\delta$ on $\mathscr{H}(U, F)$ if and only if $(G(U), F'_b)$ has BB.*

Proof. Let $B \in (G(U) \widehat{\bigotimes_\pi} F'_b)' = \mathscr{B}(G(U), F'_b)$ (the space of bilinear maps on $G(U) \times F'_b$). Then the map $B \to T_B$, $T_B\colon\ G(U) \to F\colon\ x \to B(x, \cdot)$, gives an injective linear map from $(G(U) \widehat{\bigotimes_\pi} F'_b)'_b$ into

$$\mathscr{L}(G(U), F) = G(U)'_b \varepsilon F = (\mathscr{H}(U), \tau_o)\varepsilon F = (\mathscr{H}(U, F), \tau_o).$$

Conversely, given $T \in \mathscr{L}(G(U), F)$, the map $B\colon\ G(U) \times F'_b \to \mathbb{C}$ defined by $(x, y) \to \langle Tx, y\rangle$ is bilinear and separately continuous and hence is continuous by Proposition 11.3.7 (i) and (ii) of [31]. Thus the map $B \to T_B$ is surjective.

Let us suppose that $(G(U), F'_b)$ has BB. Then sets of the form

$$\{(A \bigotimes A')^\circ\colon\ A \subset G(U) \text{ and } A' \subset F'_b \text{ are bounded}\}$$

are a neighbourhood basis for $(G(U) \widehat{\bigotimes_\pi} F'_b)'_b$. As $B \in (A \bigotimes A')^\circ$ if and only if $T_B \in W_{A,A'^\circ} := \{T\colon\ G(U) \to F'_b\colon\ T(A) \subset A'^\circ\}$, the map $B \to T_B$ is a homeomorphism and we have

$$(G(U) \widehat{\bigotimes_\pi} F'_b)'_b = G(U)'_b \varepsilon F = (\mathscr{H}(U), \tau_o)\varepsilon F = (\mathscr{H}(U, F), \tau_o).$$

By [29], $G(U)$ is a complete bornological space and hence an inductive limit of Banach spaces. From [20, Theorem 6] we have that $G(U) \widehat{\bigotimes_\pi} F'_b$ is a complete barrelled space. Using [16, Proposition 12] it follows that $\{(P(^nE), \tau_0)\varepsilon F\}_n$ is an $\mathscr{S}$-absolute decomposition for $(\mathscr{H}(U), \tau_o)\varepsilon F$. Since $(\mathscr{H}(U), \tau_o)\varepsilon F$ is complete it follows from [27, Theorem 3.2] that $(G(U) \widehat{\bigotimes_\pi} F'_b)'_b = (\mathscr{H}(U), \tau_o)\varepsilon F$ is semi-reflexive, [36, IV.18. (b)] implies that $G(U) \widehat{\bigotimes_\pi} F'_b$ is reflexive and hence $(\mathscr{H}(U, F), \tau_o)$ is barrelled. It now follows from the comment following Theorem 2 that $\tau_o = \tau_\delta$ on $\mathscr{H}(U, F)$.

Conversely if $\tau_o = \tau_\delta$ on $\mathscr{H}(U, F)$ then

$$(G(U) \widehat{\bigotimes_\pi} F'_b)'_i = (G(U) \widehat{\bigotimes_\pi} F'_b)'_b = \mathscr{L}_e(G(U), F)$$

and $(G(U), F'_b)$ has BB. $\square$

We now give some examples where $\tau_o = \tau_\delta$ on $\mathscr{H}(U,F)$.

Theorem 13. *Let U be a balanced open subset of a Fréchet nuclear space E and let F be a Banach space. If $\tau_o = \tau_\delta$ on $\mathscr{H}(U)$ then $\tau_o = \tau_\delta$ on $\mathscr{H}(U,F)$.*

Proof. By Proposition 1.1 of [4] we have

$$(\mathscr{H}(U,F),\tau_o) = (\mathscr{H}(U),\tau_o)\varepsilon F = (\mathscr{H}(U),\tau_\delta)\varepsilon F = (\operatorname*{ind}_B \mathscr{H}(U)_B)\varepsilon F,$$

where the inductive limit is taken over all locally bounded subsets B of $\mathscr{H}(U)$. Since $(\mathscr{H}(U),\tau_o)$ is a nuclear space, we have

$$(\mathscr{H}(U,F),\tau_o) = (\operatorname*{ind}_B \mathscr{H}(U)_B)\mathbin{\widehat{\bigotimes}_\pi} F.$$

By [20, Theorem 6], $(\mathscr{H}(U,F),\tau_o)$ is barrelled.

Since E is a Fréchet nuclear space, [17, Theorem 11] implies that $\tau_o = \tau_\omega$ on $P({}^nE,F)$. Applying [21, Proposition 3.12] we conclude that $\tau_o = \tau_\delta$ on $\mathscr{H}(U,F)$. $\square$

In particular, Theorem 13 and [22] show that if E is a Fréchet nuclear DN space then $\tau_0 = \tau_\delta$ on $\mathscr{H}(E,F)$ for every Banach space F.

By taking projective limits we see that Theorem 13 extends to holomorphic functions with values in any locally convex space.

For Fréchet Schwartz spaces we have

Theorem 14. *Let U be a balanced open subset of a Fréchet Schwartz space E such that $\tau_o = \tau_\delta$ on $\mathscr{H}(U)$ and let F be a $\mathscr{L}^\infty$-space. Then $\tau_o = \tau_\delta$ on $\mathscr{H}(U,F)$ if and only if $(G(U),\ell_1)$ has* BB.

Proof. It follows by [6, Proposition 3.A.7] and [16, Corollary 4] that $(\mathscr{H}(U),\tau_o) = (\mathscr{H}(U),\tau_\delta) = \operatorname{ind}_B \mathscr{H}(U)_B$ is a boundedly retractive inductive limit (where the inductive limit is taken over all locally bounded sets) and therefore is compactly regular. It now follows by [20, Corollary 12] that

$$(\mathscr{H}(U,F),\tau_o) = (\mathscr{H}(U),\tau_o)\varepsilon F$$

is bornological if and only if $G(U) = (\mathscr{H}(U),\tau_o)'_b$ has Property B of Pietsch. $\square$

Clearly $\tau_o = \tau_\delta$ implies that E is Montel; however there is no known example of a Fréchet Montel space which is not Schwartz where $\tau_o = \tau_\delta$.

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