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LASKERIAN LATTICES

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Abstract. In this paper we investigate prime divisors, B_w-primes and zs-primes in C-lattices. Using them some new characterizations are given for compactly packed lattices. Next, we study Noetherian lattices and Laskerian lattices and characterize Laskerian lattices in terms of compactly packed lattices.

Keywords: primary element, compactly packed lattice, Laskerian lattice

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By a C-lattice we mean a (not necessarily modular) complete multiplicative lattice, with a least element 0 and a compact greatest element 1 (a multiplicative identity), which is generated under joins by a multiplicatively closed subset C of compact elements. Throughout this paper L denotes a C-lattice and L_* denotes the set of all compact elements of L. For any prime element p of L, L_p denotes the localization at $F = \{x \in C \mid x \nleq p\}$. For details on C-lattices and their localization theory, the reader is referred to [10]. We note that in a C-lattice $a = b$ if and only if $a_m = b_m$ for all maximal elements m of L.

In this paper we study prime divisors, B_ω-primes and zs-primes. Next we characterize compactly packed lattices. Also we establish some equivalent conditions for a C-lattice in which every prime element is locally compact to be a Noetherian lattice. Using these results we show that if L is generated by M-principal elements, then L is a Noetherian lattice if and only if the maximal elements of L are compact and every compact element of L has a normal primary decomposition. Finally, we introduce Laskerian lattices and characterize them in terms of compactly packed lattices.

Recall that an element e of L is said to be principal if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $a \vee (b : e) = (ae \vee b) : e$. Principal elements were introduced into multiplicative lattices by R. P. Dilworth [6]. Elements satisfying (i) are called meet principal and elements satisfying (ii) are called join principal. Elements satisfying the weaker identity (i′) $a \wedge e = (a : e)e$ obtained from (i) by setting $b = 1$ are called weak meet principal, and elements satisfying the weaker identity (ii′) $a \vee (0 : e) = ae : e$ obtained from (ii) by setting $b = 0$ are called weak join principal. Elements satisfying both (i′) and (ii′) are called weak principal. An element $a \in L$ is said to be strong join principal if a is compact and join principal. An element $a \in L$ is said to be a radical element if $a = \sqrt{a}$. Following [1], a prime element p of L is said to satisfy the condition $\oplus$, if for any collection $\{p_\alpha\}$ of prime elements of L, $p \nleqslant p_\alpha$ for all α implies that there exists $x \in L_*$ such that $x \leqslant p$ and $x \nleqslant p_\alpha$ for all α. The lattice L is said to be a *compactly packed lattice if* every prime element satisfies the condition $\oplus$. L is said to be a *Noetherian lattice* if L satisfies the ascending chain condition (a.c.c.). It is well known that L is a Noetherian lattice if and only if every element is a compact element. An r-lattice is a modular multiplicative lattice that is compactly generated, principally generated and has a compact greatest element 1. An r-lattice satisfying the ascending chain condition is called a *Noether lattice*.

For general background and terminology, the reader is referred to [2], [4], [10].

An element $b \in L$ is said to be prime to a $(a, b \in L)$ if $bc \leqslant a$ implies $c \leqslant a$. For any $a \in L$ $(a < 1)$, let $H_a = \{x \in L_* \mid x \text{ is prime to } a\}$ and $\Im_a = \{x \in L \mid a \leqslant x \text{ and } H_a \cap [0, x] = \emptyset\}$. Obviously $H_a \cap [0, a] = \emptyset$ $([0, a] = \{x \in L \mid 0 \leqslant x \leqslant a\})$ and H_a is a multiplicative closed subset of L_*. So by Zorn's lemma, $\Im_a$ contains maximal elements and every maximal element is a prime element.

Definition 1. A prime element p containing a $(a, p \in L)$ is called a maximal prime divisor if $p \in \mathfrak{T}_a$ and p is a maximal element of $\mathfrak{T}_a$.

Definition 2. A prime element p containing a $(a, p \in L)$ is called a prime divisor if $p \in \Im_{(a_p)} = \{x \in L \mid a_p \leqslant x \text{ and } H_{(a_p)} \cap [0, x] = \emptyset\}$ and p is a maximal element of $\Im_{(a_p)}$.

It is well known that a prime element p containing a is a minimal prime over a if and only if for any compact element $x \leqslant p$, there exists a compact element $y \nleqslant p$ such that $x^n y \leqslant a$ for a positive integer n ([1], Lemma 3.5). Using this result, it can be easily shown that if p is a minimal prime over a, then p is a prime divisor of a and such prime elements are called minimal prime divisors of a.

We now prove several useful lemmas.

Lemma 1. *Let L satisfy the ascending chain condition* (a.c.c.) *for prime elements and suppose that each compact element has only finitely many minimal prime divisors. Then L is a compactly packed lattice.*

P r o o f. By imitating the proof of Lemma 1 of [5], we can prove that for every prime element p of L, there exists $x \in L_*$ such that $p = \sqrt{x}$. Now the result follows from the definition of a compactly packed lattice. $\square$

Lemma 2. *If every prime element of L is locally compact, then L satisfies* a.c.c. *on prime elements.*

P r o o f. The proof of the lemma is similar to that of [5, Lemma 2]. $\square$

An element $a \in L$ is said to have a primary decomposition, if there exist primary elements $q_1, q_2, \ldots, q_n$ in L such that $a = q_1 \wedge \ldots \wedge q_n$. If q is a primary element of L, then $\sqrt{q} = p$ is a prime element and it is called the prime associated with q. Note that if q_1 and q_2 are primary elements associated with the same prime, then $q_1 \wedge q_2$ is also a primary element associated with p. An element $a \in L$ is said to have a normal primary decomposition, if $a = q_1 \wedge \ldots \wedge q_n$ ($q_i'^s$ are primary elements with distinct radicals) and if no q_i contains the meet of the other primary elements. Note that if a has a primary decomposition, then this primary decomposition can be reduced to a normal primary decomposition.

Lemma 3. *Let $a \in L$ have a normal primary decomposition $a = q_1 \wedge \ldots \wedge q_n$ and put $p_i = \sqrt{q_i}$. Then a compact element x of L is non prime to a if and only if $x \leqslant p_i$ for some i.*

P r o o f. If x is non prime to a, then $xy \leqslant a$ for a compact element $y \nleqslant a$. So $y \nleqslant q_i$ for some i. Since $xy \leqslant a \leqslant q_i$, $y \nleqslant q_i$ and q_i is primary, it follows that $x \leqslant \sqrt{q_i} = p_i$.

Conversely, assume that $x \leqslant p_i$ for some i. Since $\bigwedge\limits_{i=1}^{n} q_i$ is a normal primary decomposition of a, it follows that $a < \bigwedge\limits_{j \neq i} q_j$. Choose any compact element $y \leqslant \bigwedge\limits_{j \neq i} q_j$ such that $y \nleqslant a$. As $x \leqslant p_i = \sqrt{q_i}$, $x^k \leqslant q_i$ for a positive integer k and so $x^k y \leqslant a$. Let i be the smallest integer such that $x^i y \leqslant a$. Then $x(x^{i-1}y) \leqslant a$ and $x^{i-1}y \nleqslant a$ and hence x is non prime to a. $\square$

Lemma 4. *Let $a \in L$ have a normal primary decomposition $a = q_1 \wedge \ldots \wedge q_m$ and put $p_i = \sqrt{q_i}$. Let p be a prime element of a. Then $a_p = \bigwedge\{q_i \mid p_i \leqslant p\}$.*

P r o o f. The proof of the lemma follows from [10, Properties 0.7 and 0.8]. $\square$

Lemma 5. *Let $a \in L$ have a normal primary decomposition $a = q_1 \wedge \ldots \wedge q_m$ and put $p_i = \sqrt{q_i}$. If p is a prime element containing a, then $p = p_i$ for some i if and only if p is a prime divisor of a.*

P r o o f. Suppose $p = p_k$ for some k $(1 \leqslant k \leqslant m)$. Then by Lemma 4, $a_p = \bigwedge\{q_i \mid p_i \leqslant p_k\}$. As $\bigwedge_{i=1}^{m} q_i$ is a normal primary decompostion of a, it follows that $\bigwedge\{q_i \mid p_i \leqslant p_k\}$ is a normal primary decomposition of a_p. By Lemma 3, $p \in \Im_{(a_p)}$ and it is not hard to show that p is a maximal element of $\Im_{(a_p)}$. Therefore p is a prime divisor of a.

Conversely, assume that p is a prime divisor of a. Since $a \leqslant p$, it follows that $p_i \leqslant p$ for some i. Note that $a_p = \bigwedge\{q_i \mid p_i \leqslant p\}$ is a normal primary decomposition of a_p. By Lemma 3, each p_i $(p_i \leqslant p)$ is an element of $\Im_{(a_p)}$. Since $p \in \Im_{(a_p)}$ for any compact element $x \leqslant p$, x is non prime to a_p and so by Lemma 3, $x \leqslant p_i$ $(p_i \leqslant p)$ for some i. This show that $p = p_i$ for some i. □

Definition 3. A prime element p containing a is called a B_w-prime of a if p is a minimal prime divisor of $(a : x)$ for some $x \in L_*$.

Definition 4. A prime element p containing a $(a, p \in L)$ is said to be a zs-prime of a if $p = \sqrt{(a : x)}$ for some $x \in L_*$.

Remark 1. Clearly if p is a zs-prime of a, then p is a B_w-prime of a and it is not hard to show that every B_w-prime of a is a prime divisor of a. Also it should be mentioned that if R is a commutative ring with identity and $L(R)$ is the lattice of all ideals of R, then a prime ideal P containing an ideal I of R is a B_w-prime (zs-prime) of I if and only if P is a B_w-prime (zs-prime) of I in the sense of [8].

Theorem 1. *Let $a \in L$ have a normal primary decomposition $a = q_1 \wedge \ldots \wedge q_m$ and put $p_i = \sqrt{q_i}$. Suppose p is a prime element containing a. Then the following statements are equivalent:*

(i) $p = p_i$ *for some* i $(1 \leqslant i \leqslant m)$.

(ii) p *is a* zs*-prime of* a.

(iii) p *is a* B_w*-prime of* a.

(iv) p *is a prime divisor of* a.

P r o o f. (i)⇒(ii). Suppose (i) holds. Since $\bigwedge_{i=1}^{m} q_i$ is a normal primary decomposition of a, it follows that $\bigwedge_{j \neq i} \sqrt{q_j} \nleqslant \sqrt{q_i}$, so there exists $x \in L_*$ such that $x \nleqslant \sqrt{q_i}$ and $x \leqslant \bigwedge_{j \neq i} \sqrt{q_j}$. Therefore $x^k \leqslant \bigwedge_{j \neq i} q_j$ for a positive integer k. Consequently $p_i = \sqrt{(a : x^k)}$. Hence p is a zs-prime of a and (ii)⇒(iii)⇒(iv) follows from Remark 1 while (iv)⇒(i) follows from Lemma 5. This completes the proof of the theorem. □

Lemma 6. *Let $p \leqslant q$ be prime elements of L and let a be an element of L. Then the following statements hold.*

(i) *p is a minimal prime over a if and only if p_q is a minimal prime over a_q in L_q.*
(ii) *p is a B_w-prime of a in L if and only if p_q is a B_w-prime of a_q in L_q.*
(iii) *If p is the unique B_w-prime of a, then a is p-primary.*
(iv) *If $x \in L_*$, $x_p = p_p$ and x_q is a p_q-primary element of L_q, then $x_q = p_q$.*
(v) *If $\{\sqrt{(a:x)} \mid x \in L_*\}$ satisfies* a.c.c., *then every B_w-prime of a is also a zs-prime of a.*
(vi) *Let $a \in L$. If a has only finitely many B_w-primes, then $\{zs\text{-primes of } a\} = \{B_w\text{-primes of } a\} = \{\text{prime divisors of } a\}$.*

P r o o f. (i) and (iv) follow from [10, Properties 0.5, 0.7 and 0.8]. The proof of (ii) is a direct consequence of (i) and the proofs of (iii), (v) and (vi) are similar to those of [8, Lemma 1.1, Lemma 3.2 and Proposition 3.5]. □

Theorem 2. *The following statements on L are equivalent:*
(i) *L satisfies* a.c.c. *on radical elements.*
(ii) *For every $a \in L$, there exists $x \in L_*$ such that $\sqrt{a} = \sqrt{x}$.*
(iii) *L is a compactly packed lattice.*
(iv) *Every $a \in L$ has only finitely many minimal prime divisors and L satisfies* a.c.c. *on prime elements.*
(v) *Every compact element has only finitely many minimal prime divisors and L satisfies* a.c.c. *on prime elements.*

P r o o f. (i)⇒(ii). Suppose (i) holds and let $a \in L$. Then $\{\sqrt{x} \mid x \in L_*$ and $x \leqslant a\}$ has a maximal element, say $\sqrt{y}$. Obviously $\sqrt{a} = \sqrt{y}$. (ii)⇒(iii) follows from [1, Theorems 6.1, 6.2 and 6.5]. We show that (iii)⇒(iv). Suppose (iii) holds. Note that by [1, Theorems 6.1, 6.2 and 6.5], if p is a prime element, then $p = \sqrt{a}$ for some $a \in L_*$. Again by Zorn's lemma, for every $a \in L$, $\sqrt{a} = \sqrt{x}$ for some $x \in L_*$. Therefore by [1, Theorem 6.1], every element has only finitely many minimal prime divisors. Obviously, L has a.c.c on prime elements. (iv)⇒(v) is obvious. (v)⇒(i) follows from Lemma 1 and the fact that if every prime element is the radical of some compact element, then every radical element is the radical of some compact element. □

Remark. If R is a commutative ring with identity, then $L(R)$, the lattice of all ideals of R, is a compactly packed lattice if and only if R has a Noetherian spectrum (in the sense of [11]).

Theorem 3. *Suppose every prime element of L is locally compact. If L satisfies any one of the following conditions:*
(i) *every compact element of L has a normal primary decomposition;*
(ii) *every compact element of L has only finitely many B_w-primes;*

(iii) *every compact element of* L *has only finitely many prime divisors;*
(iv) *each* $x \in L_*$ *has only finitely many minimal prime divisors and* $\sqrt{x}$ *is compact, then every prime element is compact.*

Proof. Note that (i)⇒(ii) follows from Theorem 1. If L satisfies (iv), then by Lemma 1, every prime element is compact. Now by Remark 1 and Lemma 6 (vi), it suffices to show that if L satisfies the condition (iii), then every prime element is compact. Suppose every compact element has only finitely many prime divisors. Let p be a prime element of L. By Lemma 1 and Lemma 2, $p = \sqrt{x}$ for some $x \in L_*$. By hypothesis $p = p_p = a_p$ for some $a \in L_*$. Note that $p = \sqrt{x \vee a}$ and $(x \vee a)_p = p_p$. Let $x_1 = x \vee a$ and let $p, p_1, \ldots, p_n$ be the prime divisors of x_1. Without loss of generality assume that $p < p_i$ for $i = 1, 2, \ldots, n$. Again by hypothesis, there exist $\gamma_i \in L_*$ $(i = 1, 2, \ldots, n)$ such that $(p)_{p_i} = (\gamma_i)_{p_i}$ for $i = 1, 2, \ldots, n$. Let $x_2 = x_1 \vee \gamma_1 \vee \gamma_2 \vee \ldots \vee \gamma_n$. Then $p = \sqrt{x_2}$ and $(x_2)_{p_i} = (p)_{p_i}$ for $i = 1, 2, \ldots, n$. We show that for $1 \leqslant i \leqslant n$, p_i is not a prime divisor of x_2. Choose any $y_i \in L_*$ such that $y_i \leqslant p_i$ and $y_i \nleqslant p$. Then each y_i is prime to p and each y_i is prime to $(p)_{p_i} = (x_2)_{p_i}$. This shows that $H_{(x_2)_{p_i}} \cap [0\ p_i] \neq \emptyset$. Consequently, no p_i is a prime divisor of x_2. Suppose that q $(q \neq p)$ is any prime which contains x_2 and suppose that $p_i \nleqslant q$ for any i. Since $x_1 \leqslant x_2 \leqslant q$, we have $p < q$. Again since $p = \sqrt{x_1}$, it follows that p is the unique minimal prime divisor of x_1, so p_q is the unique minimal prime divisor of $(x_1)_q$ (by Lemma 6 (i)) in L_q. So p_q is a B_w-prime of $(x_1)_q$. Again if q'_q is a B_w-prime of $(x_1)_q$ in L_q (q' is a prime element and $q' \leqslant q$), then by Lemma 6 (ii), q' is a B_w-prime of x_1 in L, so q' is a prime divisor of x_1 and hence $q' = p$ (since $p_i \nleqslant q$ for any i). Therefore p_q is the unique B_w-prime of $(x_1)_q$, so by Lemma 6 (iii), x_{1_q} is p_q-primary and again by Lemma 6 (iv), $(x_2)_q = p_q$. As $p < q$, q is not a prime divisor of x_2. Therefore if $p, p'_1, p'_2, \ldots, p'_m$ are the prime divisors of x_2, then for $1 \leqslant i \leqslant m$, $p'_i > p_j$ for some j, $1 \leqslant j \leqslant n$. As L satisfies a.c.c. for prime elements, a finite number of repetitions of the above procedure yields a compact element $x_3 \in L_*$ such that $(x_3)_p = p_p$ and p is the unique prime divisor of x_3. So by Lemma 6 (iii), x_3 is p-primary and hence $x_3 = p$. Consequently, p is compact. Thus every prime element is compact and the proof is complete. □

Definition 5. An element $x \in L$ is said to be a modular element (or an m-element) if for any $a, b \in L$, $a \geqslant b$ implies $a \wedge (x \vee b) = (a \wedge x) \vee b$.

Definition 6. An element $x \in L$ is said to be an M-element if x^n is an m-element for every positive integer n.

Note that L is a modular lattice if and only if every element is an m-element. Also it is not hard to show that L is a modular lattice if and only if every compact element is a modular element.

A weak meet principal (meet principal, principal) element x is said to be m-weak meet principal (m-meet principal, m-principal) if x is a modular element.

Theorem 4. *Suppose L is generated by compact m-weak meet principal elements. If every prime element is compact, then every element is compact.*

P r o o f. Suppose every prime element is compact and let $\Psi = \{x \in L \mid x \text{ is not compact}\}$ be a non empty set. By Zorn's lemma, Ψ has a maximal element, say p. By hypothesis p is not prime, so there exist compact m-weak meet principal elements $x, y \in L$ such that $xy \leqslant p$, $x \nleqslant p$ and $y \nleqslant p$. So $p < p \vee x$, $p < p : x$ and hence $p \vee x$ and $p : x$ are compact elements. Since $p \vee x$ is compact, it follows that $p \vee x = p_1 \vee x$ for a compact element $p_1 \leqslant p$. Observe that $p \leqslant p_1 \vee x$, so $p = p \wedge (x \vee p_1) = p_1 \vee (p \wedge x)$ (as x is an m-element) $= p_1 \vee ((p : x)x)$ (as x is weak meet principal) and therefore p is compact as $p_1, x, (p : x) \in L_*$. This contradiction shows that every element is compact. □

An element $a \in L$ is said to be meet irreducible if $a = b \wedge c$ implies either $a = b$ or $a = c$. It is well known that if L satisfies a.c.c, then every element is a finite meet of meet irreducible elements.

Lemma 7. *Suppose L is generated by M-meet principal elements and let $a \in L$ be a meet irreducible element. If $\{(a : x) \mid x \in L\}$ satisfies* a.c.c., *then a is primary.*

P r o o f. The proof of the lemma is similar to that of [6, Theorem 3.1]. □

Theorem 5. *Suppose L is generated by compact M-meet principal elements. If L is a Noetherian lattice, then L satisfies the conditions* (i)–(iv) *of Theorem* 3. *Conversely, if every prime element is locally compact and L satisfies the conditions of Theorem* 3, *then L is a Noetherian lattice.*

P r o o f. The proof of the theorem follows from Theorems 1, 3, 4 and Lemma 7. □

Theorem 6. *Let L be a quasi-local lattice generated by M-principal elements. Suppose the maximal element m is compact. Then the following statements are equivalent:*

(i) *L is a Noetherian lattice.*
(ii) *Every compact element of L has a normal primary decomposition.*
(iii) *For any two compact elements a and b of L, there exists an integer n such that $(a \vee b^{\ell}) \wedge (a : b^{\ell}) = a$ for $\ell \geqslant n$.*
(iv) *$\bigwedge_{n=1}^{\infty} (m^n \vee a) = a$ for all compact elements a of L.*
(v) *If $b = a \vee mb$ and $a \in L_*$, then $a = b$.*

P r o o f. (i) $\Rightarrow$ (ii) follows from Lemma 7 and by imitating the proof of [3, Theorem 4.1], it can be easily shown that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v) follow from [1, Corollary 1.4 and Theorem 1.1]. Now we prove that (iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i). Suppose L is not Noetherian. By the proof of Theorem 4, there exists a prime element p such that p is maximal among the set of all non compact elements. Clearly $p \neq m$. Choose any M-principal element $x \leqslant m$ such that $x \nleqslant p$. Then $x^n \nleqslant p$ for all $n \in \mathbb{Z}^+$. Let $n \geqslant 1$. Then $p < p \vee x^n$, so $p \vee x^n$ is compact and hence $p \vee x^n = p_1 \vee x^n$ for a compact element $p_1 \leqslant p$. If $a \leqslant p$ is any principal element, then $a \vee p_1 = (a \vee p_1) \wedge (p_1 \vee x^n) = p_1 \vee ((a \vee p_1) \wedge x^n) = p_1 \vee (((a \vee p_1) : x^n)x^n)$ as x^n is an m-principal element. Since $(a \vee p_1) : x^n)x^n \leqslant p$, $x^n \nleqslant p$ and p is prime, it follows that $(a \vee p_1) : x^n) \leqslant p$. So $a \leqslant p_1 \vee x^n p \leqslant p_1 \vee m^n p$ and therefore $p = p_1 \vee m^n p$ and this is true for all $n \in \mathbb{Z}^+$. Consequently, either (iv) or (v) implies that $p = p_1$, a contradiction. This shows that L is a Noetherian lattice and the proof is complete. □

Theorem 7. *Suppose L is generated by M-principal elements. Then the following statements are equivalent:*

(i) *L is a Noetherian lattice.*

(ii) *The maximal elements of L are compact and every compact element of L has a normal primary decomposition.*

P r o o f. (i)$\Rightarrow$(ii) follows from Lemma 7. Suppose (ii) holds. By hypothesis and Lemma 4, every compact element of L_m (m is a maximal element) has a normal primary decomposition, so by Theorem 6, L is a locally Noetherian lattice. Again by Theorem 5, L is a Noetherian lattice. This completes the proof of the theorem. □

Corollary 1. *Suppose L is an r-lattice in which every compact element is a finite meet of primary elements. If p is a compact prime element minimal over a principal element, then* $\operatorname{rank} p \leqslant 1$.

P r o o f. The proof of the theorem follows from Theorem 6 and [6, Theorem 6.4]. □

Corollary 2. *Suppose L is an r-lattice in which every compact element has a normal primary decomposition. If the prime elements are comparable and the maximal element is compact, then* $\dim L \leqslant 1$.

Definition 7. L is said to be a Laskerian lattice if every element is a finite meet of primary elements.

Noether lattices [6] are Laskerian lattices. If R is a Laskerian ring (see [7], [9]), then the lattice $I(R)$ of all ideals of R is a Laskerian r-lattice. If L is an idempotent (i.e., $a^2 = a$ for all $a \in L$) distributive lattice satisfying the ascending chain condition, then L is a Laskerian lattice ([1, Theorem 6.1]).

We need the following lemma.

Lemma 8. *Let L be a Laskerian lattice generated by strong join principal elements. If p is a prime element containing a, then $a_p = \bigwedge\{q \mid a \leqslant q$ and q is p-primary$\}$.*

P r o o f. Let $b = \bigwedge\{q \mid a \leqslant q$ and q is *p-primary*$\}$. Clearly $a_p \leqslant b$. Suppose $a_p < b$. Then there exists a strong join principal element $x \leqslant b$ such that $x \nleqslant a_p$. As L is Laskerian, it follows that $a \vee xp$ has a normal primary decomposition, say $a \vee xp = q_1 \wedge \ldots \wedge q_n$, and $p_i = \sqrt{q_i}$ ($q_i'^s$ are p_i-primary). By Lemma 4, $(a \vee xp) = \bigwedge\{q_i \mid p_i \leqslant p\}$. By Theorem 1.4 of [2], $x_p \nleqslant (a \vee xp)_p$, so $x_p \nleqslant q_i$ ($p_i \leqslant p$) for some i and hence $x \nleqslant q_i$. Again since $xp \leqslant q_i$, it follows that $p \leqslant p_i$, so q_i is p-primary. This contradiction shows that $b = a_p$ and the proof is complete. □

Theorem 8. *Suppose L is generated by strong join principal elements. If L is Laskerian, then L is a compactly packed lattice.*

P r o o f. Suppose L is Laskerian. Then clearly L contains only finitely many minimal primes. So by Theorem 2, it is enough if we show that L satisfies a.c.c. on prime elements. Let $p_0 < p_1 < p_1' < p_2 < p_2' < p_3 < p_3' < \ldots$ be a chain of prime elements. By Theorem 1, every element has only finitely many zs-primes. We show that there is an element $a \in L$ such that a has infinitely many zs-primes. First we show by induction that for $n \in \mathbb{Z}^+$ there exist $q_1, \ldots, q_n \in L$, a_n, b_n and strong join principal elements $x_1, x_2, ..., x_n$ in L such that

(i) q_i is p_i-primary for i and $a_n = q_1 \wedge \ldots \wedge q_n$,

(ii) for $1 \leqslant i \leqslant n$ we have $x_i \leqslant \bigwedge\limits_{j \neq i} q_j$ and $x_i \nleqslant q_i$,

(iii) $x_1 \vee x_2 \vee \ldots \vee x_n \leqslant b_n$, $a_n \nleqslant b_n$ and every zs-prime of b_n is contained in p_n'.

Suppose $n = 1$. Then take $q_1 = p_1$. Since $p_1 < p_1'$ and p_1 is nonminimal, it follows that $0_{p_1'} < p_1$, so by Lemma 8, $p_1 \nleqslant q_1'$ for some p_1'-primary element q_1'. Choose any strong join principal element $x_1 \leqslant q_1'$ such that $x_1 \nleqslant q_1$. Let $b_1 = (x_1)_{p_1'}$. Clearly $q_1 = p_1 \nleqslant b_1$. As L is Laskerian, b_1 has a normal primary decomposition, say $b_1 = h_1 \wedge \ldots \wedge h_n$, $r_i = \sqrt{h_i}$ ($h_i'^s$ are r_i-primary elements). Since $b_1 = (b_1)_{p_1'}$, by Lemma 4 we have $r_i \leqslant p_1'$ for $i = 1, 2, \ldots, n$. Again by Theorem 1, $r_i'^s$ ($i = 1, 2, \ldots, n$) are the only zs-primes of b_1. Therefore each zs-prime of b_1 is contained in p_1'. Thus the conditions (i), (ii) and (iii) are satisfied.

Suppose we have $q_1, \dots, q_n$, a_n, b_n and strong join principal elements satisfying (i)–(iii). Since $a_n \not\leqslant b_n$, there exists a strong join principal element y_{n+1} such that $y_{n+1} \leqslant a_n$ and $y_{n+1} \not\leqslant b_n$. Since $p'_n < p_{n+1}$ and $b_n < p_{n+1}$, by Lemma 8 there exists a p_{n+1}-primary element q_{n+1} such that $b_n \leqslant q_{n+1}$ and $y_{n+1} \not\leqslant q_{n+1}$. Define $a_{n+1} = a_n \wedge q_{n+1}$. We show that $a_{n+1} \not\leqslant b_n$. As L is Laskerian, b_n has a normal primary decomposition, say $b_n = h_1 \wedge \ldots \wedge h_k$, $r_i = \sqrt{h_i}$ $(1 \leqslant i \leqslant k)$ where $r_i'^s$ are zs-primes of b_n. By (iii), each $r_i \leqslant p'_n$ and therefore $q_{n+1} \not\leqslant r_i$ for $i = 1, 2, \dots, k$. If $a_{n+1} \leqslant b_n$, then $a_n \wedge q_{n+1} \leqslant b_n \leqslant h_i$ for $i = 1, 2, \dots, k$. Since $q_{n+1} \not\leqslant r_i$ for $i = 1, 2, \dots, k$ and $h_i'^s$ are r_i-primary elements, it follows that $a_n \leqslant \bigwedge\limits_{i=1}^{k} h_i = b_n$, a contradiction. This shows that $a_{n+1} \not\leqslant b_n$. Note that $b_n = (b_n)_{p'_{n+1}}$ since each $r_i \leqslant p'_n < p'_{n+1}$ and by Lemma 8, $b_n = \bigwedge\limits_{\lambda \in \Delta} \{c_\lambda \mid b_n \leqslant c_\lambda \text{ and } c_\lambda \text{ is a } p'_{n+1}\text{-primary element}\}$. Since $a_{n+1} \not\leqslant b_n$, it follows that $a_{n+1} \not\leqslant c_\lambda$ for some $\lambda \in \Delta$. Consequently, $a_{n+1} \not\leqslant (b_n \vee y_{n+1} c_\lambda)_{p'_{n+1}}$ as $(b_n \vee y_{n+1} c_\lambda)_{p'_{n+1}} \leqslant c_\lambda$. As $p_{n+1} < p'_{n+1}$, we have $c_\lambda \not\leqslant p_{n+1}$, so there exists a strong join principal element $r \leqslant c_\lambda$ such that $r \not\leqslant p_{n+1}$. Define $x_{n+1} = y_{n+1} r$ and $b_{n+1} = (b_n \vee x_{n+1})_{p'_{n+1}}$. Observe that x_{n+1} is a strong join principal element. Since $y_{n+1} \not\leqslant q_{n+1}$ and $r \not\leqslant p_{n+1}$, it follows that $x_{n+1} \not\leqslant q_{n+1}$. Thus (i) and (ii) are satisfied for $q_1, q_2, \dots, q_{n+1}$ and $x_1, x_2, \dots, x_{n+1}$. Moreover, (iii) is satisfied for b_{n+1}, by the choice of x_{n+1} and b_{n+1}. Therefore, we conclude by induction that there exist infinite sequences $\{q_i\}_{i=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $\{x_i\}_{i=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that the conditions (i), (ii) and (iii) are satisfied for all n. Now let us define $a = \bigwedge\limits_{n=1}^{\infty} a_n$. Since $x_n \leqslant \bigwedge\limits_{j \neq n} q_j$ and $x_n \not\leqslant q_n$, it follows that $(a : x_n) = (a_n : x_n) = (q_n : x_n)$ is p_n-primary, so p_n is a zs-prime of a and this is true for all n. Therefore a has infinitely many zs-primes. This contradiction shows that L satisfies a.c.c. on prime elements and the proof is complete. □

Theorem 9. *Suppose L is generated by M-principal elements. Then L is Laskerian if and only if L satisfies the following conditions:*

(i) *L is a compactly packed lattice.*

(ii) *For each $a \in L$, there is a prime element p minimal over a and an M-principal element $x \not\leqslant p$ such that $(a : x)$ is p-primary.*

P r o o f. Suppose L is a Laskerian lattice. By Theorem 2 and Theorem 8, L is a compactly packed lattice. Again by imitating the proof of Theorem 1 ((i) $\Rightarrow$ (ii)), it can be easily shown that L satisfies the condition (ii).

Conversely, assume that L satisfies (i) and (ii). Let $a \in L$ and let p be a minimal prime over a such that $(a : x)$ is p-primary for some M-principal element $x \not\leqslant p$. Then $(a : x) \wedge (a \vee x) = ((a : x) \wedge a) \vee ((a : x) \wedge x)$ (x is a modular element) $= a \vee ((a : x) \wedge x) = a \vee ((a : x^2)x)$ (as x is weak meet principal). Note that $(a : x^2) \leqslant (a : x)$

since $x \nleqslant p$ and $(a : x)$ is p-primary. Therefore $(a : x^2)x \leqslant (a : x)x \leqslant a$ and hence $a = (a : x) \wedge (a \vee x)$. Put $a_1 = (a \vee x)$ and $q_1 = (a : x)$. Then $a = q_1 \wedge a_1$ where $\sqrt{a} < \sqrt{a_1}$ since $x \leqslant \sqrt{a_1}$. Similarly $a_1 = q_2 \wedge a_2$ where $q_2 = (a_1 : y)$ is p_1-primary, p_1 is a minimal prime over a_1, $y \nleqslant p_1$ is an m-principal element and $\sqrt{a_1} < \sqrt{a_2}$. By continuing this process, we get sequences of elements $q_1, q_2, \ldots, q_n$ and $a_1, a_2, \ldots, a_n$ such that $a_{i-1} = q_i \wedge a_i$, q_i is primary for $i = 1, 2, \ldots, n$ $(a_0 = a)$ and $\sqrt{a_0} < \sqrt{a_1} < \sqrt{a_2} < \ldots < \sqrt{a_n}$. Since L satisfies a.c.c. on radical elements, it follows that $\sqrt{a_0} < \sqrt{a_1} < \sqrt{a_2} < \ldots < \sqrt{a_n}$ is a finite chain with $\sqrt{a_n}$ as a maximal element. Then $a_n = 1$ and hence $a = q_1 \wedge \ldots \wedge q_n$. This shows that L is Laskerian and the proof is complete. □

Lemma 9. *Suppose L is a compactly packed lattice. Let $a \in L$ and let p be a minimal prime over a. Then $p = \sqrt{(a : x)}$ for a compact element $x \nleqslant p$.*

Proof. Let $a \in L$ and let p be a minimal prime over a. Since L satisfies a.c.c. on radical elements, it follows that $\Gamma = \{\sqrt{(a : x)} \mid x \in L_*,\ x \nleqslant p$ and p is a minimal prime over $\sqrt{(a : x)}\}$ has a maximal element, say $\sqrt{(a : x)}$. Suppose p_0 is any other minimal prime over $\sqrt{(a : x)}$. Choose any element $y \leqslant p_0$ such that $y \nleqslant p$. Since $xy \nleqslant p$ and $\sqrt{(a : x)} \leqslant \sqrt{(a : xy)}$, it follows by the maximality that $\sqrt{(a : x)} = \sqrt{(a : x^n)} = \sqrt{(a : xy)} \leqslant \sqrt{(a : xy^m)}$ for all $m, n \in \mathbb{Z}^+$. Since $y \leqslant p_0$ and p_0 is any other minimal prime over $\sqrt{(a : x)}$, it follows that there exists $z \nleqslant p_0$ such that $y^n z \leqslant \sqrt{(a : x)}$, so $z \leqslant \sqrt{(a : x)} \leqslant p_0$, a contradiction. This shows that p is the unique minimal prime over $\sqrt{(a : x)}$ and hence $p = \sqrt{(a : x)}$. □

Lemma 10. *Suppose L is a compactly packed lattice in which every primary element with non maximal prime radical is compact. Then for each $a \in L$, there is a prime element p minimal over a and a compact element $x \nleqslant p$ such that $(a : x)$ is p-primary.*

Proof. Let $a \in L$ and let $p \in L$ be a minimal prime over a. By Lemma 9, $p = \sqrt{(a : x)}$ for some $x \nleqslant p$. If p is maximal, then $(a : x)$ is p-primary. Suppose p is non maximal. Note that $q = a_p$ is p-primary. Again by hypothesis, $xq \leqslant a$ for a compact element $x \nleqslant p$. As q is p-primary, it follows that $q = (a : x)$. □

Theorem 10. *Suppose L is a compactly packed lattice generated by M-principal elements. If every primary element with non maximal prime radical is compact, then L is a Laskerian lattice.*

Proof. Suppose every primary element with non maximal prime radical is compact. Let $a \in L$ and let p be a minimal prime over a. Then by Lemma 9 and Lemma 10, $(a : x)$ is p-primary for a compact element $x \nleqslant p$. As L is generated by M-principal elements, it follows that there is an M-principal element $x_1 \leqslant x$ such that $x_1 \nleqslant p$. Since $(a : x) \leqslant (a : x_1)$ and $(a : x)$ is p-primary, it follows that $(a : x) = (a : x_1)$. Now the result follows from Theorem 9. $\square$

Let $r^* = \bigwedge\{m \in L \mid m$ is a maximal element of $L\}$. The element r^* is called the *Jacobson radical of* L. The following theorem gives some of the properties of Laskerian lattices.

Theorem 11. *Suppose L is a Laskerian lattice generated by compact join principal elements. Let $a, c \in L$ and let $b = \bigwedge_{n=1}^{\infty} (a^n \vee c)$. Then the following statements hold.*

(i) *If a is compact and $a \leqslant r^*$, then $b = c$.*

(ii) *$0 = \bigwedge\{q \in L \mid q$ is m-primary for a maximal element m of $L\}$.*

(iii) *If both a and b are compact elements of L, then $b = \vee\{r \in L \mid r$ is join principal, $a \vee (c : r) = 1\}$.*

(iv) *If both a $(a < 1)$ and $b' = \bigwedge_{n=1}^{\infty} a^n$ are compact elements of L, then $\bigwedge_{n=1}^{\infty} a^n = 0$ if and only if there is no zero divisor r $(\neq 0)$ such that $a \vee r = 1$.*

Proof. (i) Suppose a is compact and let $a \leqslant r^*$. Let m be any maximal element of L. Note that for any m-primary element q of L $b \leqslant q$ if and only if $c \leqslant q$. Therefore by Lemma 8, $b_m = c_m$ and hence $b = c$.

(ii) Let x be any compact join principal element such that $x \leqslant \bigwedge\{q \in L \mid q$ is m-primary for a maximal element m of $L\}$. Then by Lemma 8, $x_m = 0_m$ for every maximal element m of L. Consequently, $x = 0$.

(iii) By imitating the proof of [1, Theorem 1.2], we can get the result and (iv) directly follows from (iii). This completes the proof of the theorem. $\square$

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