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ON THE JUMP NUMBER OF LEXICOGRAPHIC SUMS OF ORDERED SETS

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Abstract. Let Q be the lexicographic sum of finite ordered sets Q_x over a finite ordered set P. For some P we can give a formula for the jump number of Q in terms of the jump numbers of Q_x and P, that is, $s(Q) = s(P) + \sum\limits_{x\in P} s(Q_x)$, where $s(X)$ denotes the jump number of an ordered set X. We first show that $w(P) - 1 + \sum\limits_{x\in P} s(Q_x) \leqslant s(Q) \leqslant s(P) + \sum\limits_{x\in P} s(Q_x)$, where $w(X)$ denotes the width of an ordered set X. Consequently, if P is a Dilworth ordered set, that is, $s(P) = w(P) - 1$, then the formula holds. We also show that it holds again if P is bipartite. Finally, we prove that the lexicographic sum of certain jump-critical ordered sets is also jump-critical.

Keywords: ordered set, jump (setup) number, lexicographic sum, jump-critical

MSC 2000: 06A07

1. Introduction

Let P be a finite ordered set (poset) and let $|P|$ be the number of elements in P. An ordered set Q is called an *induced subset* (subposet) of P provided Q is a nonempty subset of P and $x < y$ in Q if and only if $x < y$ in P for any elements x and y in Q. A *chain* C in P is an induced subset of P whose order is linear and the *length* of C is $|C| - 1$. An ordered set P is called *bipartite* if the length of every chain in P is at most one. If a and b are in P, then b *covers* a, written $a \prec b$, provided $a < b$ and $a < c \leqslant b$ implies that $c = b$. A *linear extension* of an ordered set P is a linear order L on the elements of P such that $x < y$ in P implies $x < y$ in L. Let $\mathcal{L}(P)$ be

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the set of all linear extensions of P. Szpilrajn [8] showed that $\mathcal{L}(P)$ is not empty. In this paper, every ordered set is assumed to be finite.

Let P and Q be two disjoint ordered sets. The *disjoint sum* $P+Q$ of P and Q is the ordered set on $P \cup Q$ such that $x<y$ if and only if $x, y \in P$ and $x<y$ in P or $x, y \in Q$ and $x<y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P+Q$ by adding new relations $x<y$ for all $x \in P$ and $y \in Q$.

Throughout this section, L denotes an arbitrary linear extension of P which is usually denoted by $C_1 \oplus \ldots \oplus C_m$ with chains $C_1, \ldots, C_m$ in P. A (P, L)-*chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \ldots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L. A consecutive pair (x, y) of elements in L is a *jump (setup)* of P in L if x is not comparable to y in P. The jumps induce a decomposition $L=C_1 \oplus \ldots \oplus C_m$ of L into (P, L)-chains $C_1, \ldots, C_m$, where $m=c(L)$ and $(\sup C_i, \inf C_{i+1})$ is a jump of P in L for $i=1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of P in L and let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The number $s(P)$ is called the *jump (setup) number* of P. If $s(L, P)=s(P)$ then L is called an *optimal linear extension* of P. We denote the set of all optimal linear extensions of P by $\mathcal{O}(P)$. The jump number is a kind of measure between a given ordered set and its nearest linear extensions [1]. A practical motivation for studying the jump number of an ordered set comes from scheduling problems subject to precedence constraints. Namely, no task can be scheduled until all of its predecessors are scheduled. If a certain task is scheduled not immediately after one of its predecessors, then a jump occurs.

The *width* $w(P)$ of P is the maximum number of elements of an *antichain* (mutually incomparable elements) of P. Dilworth [2] showed that $w(P)$ equals the minimum number of chains in a partition of P into chains. Since for any linear extension L of P the number of (P, L)-chains is at least as large as the minimum number of chains in a chain partition of P, it follows from Dilworth's theorem that

$$s(P) \geqslant w(P)-1. \tag{1}$$

If equality holds in (1), then P is called a *Dilworth ordered set.* It follows that if P is a Dilworth ordered set, then the (P, L)-chains in an optimal linear extension L form a minimum chain partition of P.

Now it is quite interesting to determine the jump number of variously constructed ordered sets from given ordered sets such as products, lexicographic sums, lexicographic products, etc. For a natural number n, we denote the n-element chain by $\mathbf{n}$. An ordered set is called an *upward rooted tree* if it contains a least element and no induced subset isomorphic to $(\mathbf{1}+\mathbf{1}) \oplus \mathbf{1}$. Jung [7] showed that for a natural number n there is an algorithm for finding an optimal linear extension $C_1 \oplus \ldots \oplus C_k$ of any upward rooted tree T such that

$$s(T \times \mathbf{n}) = \sum_{i=1}^{k} \min\{|C_i|, n\} - 1.$$

Let P be an ordered set and let x be a point of P. Then we can easily show that $s(P) \geqslant s(P \backslash \{x\}) \geqslant s(P) - 1$. An ordered set P is called *nontrivial* if $|P| > 1$ and *trivial* if $|P| = 1$. A nontrivial ordered set P is called *jump-critical* if $s(P \setminus \{x\}) < s(P)$ for each $x \in P$. Jump-critical ordered sets, however, are quite complicated and not yet well understood. El-Zahar and Schmerl [4] showed that a jump-critical ordered set P with jump number m has at most $(m+1)!$ elements. El-Zahar and Rival [3] showed that there are precisely seventeen jump-critical ordered sets with jump number at most three.

The *lexicographic sum* $\sum\limits_{x \in P} Q_x$ of ordered sets Q_x over an ordered set P is defined to be the ordered set on $\bigcup\limits_{x \in P} Q_x$ such that $a < b$ if and only if $a < b$ in Q_z for some $z \in P$ or $x < y$ in P when $a \in Q_x$ and $b \in Q_y$.

An ordered set P is called *series-parallel* if it can be constructed from singletons using the operations of $+$ and $\oplus$ only. Observing that $s(P \oplus Q) = s(P) + s(Q)$ and $s(P + Q) = s(P) + s(Q) + 1$, it can be easily shown that if P is series-parallel then we get $s\Big(\sum\limits_{x \in P} Q_x\Big) = s(P) + \sum\limits_{x \in P} s(Q_x)$. But, in general, this equality need not hold. In Section 2, we study the jump number of the lexicographic sum of ordered sets. In Section 3, we examine when the lexicographic sum of ordered sets is jump-critical.

2. Jump number

As mentioned in the last paragraph of the preceding section, the jump number of the lexicographic sum of arbitrary ordered sets over an arbitrary ordered set may be not so easy to estimate. Habib and Möhring [6] gave a formula for this by replacing its components by two-element antichains. In this section we give a more explicit formula in some cases. First we find some tight lower and upper bounds for this value and then show that the formula holds in one important case. To begin with we need some notation and an observation.

Let Q be the lexicographic sum of ordered sets Q_x over an ordered set P and let $L = C_0 \oplus C_1 \oplus \ldots \oplus C_n$ be a linear extension of Q. We define $Q_x \sim Q_y$ if $x \prec y$ or $x \succ y$ or $x = y$ in P and there is some C_j such that $C_j \cap Q_x \neq \emptyset$ and $C_j \cap Q_y \neq \emptyset$, and define $Q_x \approx Q_z$ if there is a sequence $x = y_0, y_1, \ldots, y_k = z$ in P such that $Q_{y_i} \sim Q_{y_{i+1}}$ for $i = 0, 1, \ldots, k-1$.

Lemma 2.1. *Let Q be the lexicographic sum of ordered sets Q_x over an ordered set P and let $L = C_0 \oplus C_1 \oplus \ldots \oplus C_n$ be a linear extension of Q. If $y \succ x \prec z$ (or dually $y \prec x \succ z$) in P then both $Q_x \sim Q_y$ and $Q_x \sim Q_z$ imply that $y = z$.*

Proof. Suppose $y \neq z$. Then some C_i has an element a in Q_x and an element b in Q_y, while another C_j has an element c in Q_x and an element d in Q_z. But this implies that C_i precedes C_j and C_j precedes C_i, which is a contradiction. $\square$

Theorem 2.2. *Let Q be the lexicographic sum of ordered sets Q_x over an ordered set P. Then*

$$w(P) - 1 + \sum_{x\in P} s(Q_x) \leqslant s(Q) \leqslant s(P) + \sum_{x\in P} s(Q_x).$$

In particular, if P is a Dilworth ordered set, then

$$s(Q) = s(P) + \sum_{x\in P} s(Q_x).$$

Proof. To prove the first inequality, let $L = C_0 \oplus C_1 \oplus \ldots \oplus C_n$ be any optimal linear extension of Q, where $n = s(Q)$. Let $A = \{a_i \in Q\colon\ a_i = \inf C_i,\ i = 0, 1, \ldots, n\}$. Observe that $s(Q) = |A| - 1$. For $x \in P$, let $P_x = \{z \in P\colon\ Q_x \approx Q_z\}$. Then P_x is a chain in P by Lemma 2.1. Let $x_* = \inf P_x$ for $x \in P$. Let $A_1 = \{a_i \in A\colon\ i$ is the least integer such that $a_i \in Q_{x_*}$ for some $x_*\}$ and $A_2 = A \setminus A_1$. Let W be a maximum size antichain in P. For each $x \in W$, Q_{x_*} contains a unique element a_x in A_1. Clearly, P_x's are disjoint, whence $x \neq y$ in W implies $a_x \neq a_y$, which means that $|A_1| \geqslant w(P)$. Since $s(Q_x) \leqslant |\{i\colon\ C_i \cap Q_x \neq \emptyset\}| - 1 = |\{i\colon\ a_i \in A_2 \cap Q_x\}|$ for $x \in P$, we have $\sum\limits_{x\in P} s(Q_x) \leqslant |A_2|$. Hence $s(Q) = |A| - 1 = |A_1| - 1 + |A_2| \geqslant w(P) - 1 + \sum\limits_{x\in P} s(Q_x)$.

However, it is quite easy to prove the second inequality. In fact, if $L \in \mathcal{O}(P)$ and $L_x \in \mathcal{O}(Q_x)$ for each $x \in P$, then $\sum\limits_{x\in L} L_x \in \mathcal{L}(P)$ and hence $s(Q) \leqslant s(P) + \sum\limits_{x\in P} s(Q_x)$. $\square$

Next we find another familiar class of ordered sets P for which $s\Big(\sum\limits_{x\in P} Q_x\Big) = s(P) + \sum\limits_{x\in P} s(Q_x)$.

Theorem 2.3. *Let Q be the lexicographic sum of ordered sets Q_x over an ordered set P. If P is bipartite, then*

$$s(Q) = s(P) + \sum_{x\in P} s(Q_x).$$

P r o o f. Let $L = C_0 \oplus C_1 \oplus \ldots \oplus C_n \in \mathcal{O}(Q)$. We define sets A, P_x, A_1, and A_2 as in the proof of Theorem 2.2. If $\{i\colon a_i \in A_1\} = \{i_1, \ldots, i_m\}$ with $i_1 < \ldots < i_m$, then for each $k = 1, \ldots, m$, $D_k = \{x \in P\colon Q_x \cap C_{i_k} \neq \emptyset\}$ is a chain in P. Note that $x \in D_{k_1}$, $y \in D_{k_2}$ and $k_1 < k_2$ imply $x \not> y$ in P, and that for each $x \in P$ there exists $a_k \in A_1 \cap Q_{x_*}$ such that $x \in D_k$. Now $D_1 \oplus \ldots \oplus D_m \in \mathcal{L}(P)$. Hence $s(P) \leqslant m - 1 = |A_1| - 1$. Consequently, we get $s(Q) = |A| - 1 = |A_1| - 1 + |A_2| \geqslant s(P) + \sum\limits_{x \in P} s(Q_x)$. $\square$

Examples. We exhibit here a series of examples to show that the lower bound of $s(Q)$ in Theorem 2.2 is tight, even though $s(P)$ gets arbitrarily large. Consider $Q = \sum\limits_{x \in P} Q_x$, where each Q_x is a two-element antichain and P is the $(4n+2)$-element ordered set K_n (see Figure 1 for $n = 2$). Then $s(Q_x) = 1$, $|K_n| = 4n+2$, $w(K_n) = 2$, $s(K_n) = n + 1$ and $s(Q) = 4n + 3 = w(K_n) - 1 + \sum\limits_{x \in K_n} s(Q_x)$. Here we shall compute $s(Q)$ and $s(K_n)$ for $n = 2$. Let $Q_{x_i} = \{a_i, b_i\}$ for $i = 1, 2, \ldots, 10$. Then $L = \{a_1\} \oplus \{a_2\} \oplus \{b_2, b_4\} \oplus \{b_1, b_3\} \oplus \{a_3, a_5\} \oplus \{a_4, a_6\} \oplus \{b_6, b_8\} \oplus \{b_5, b_7\} \oplus \{a_7, a_9\} \oplus \{a_8, a_{10}\} \oplus \{b_{10}\} \oplus \{b_9\} \in \mathcal{L}(Q)$ and so $s(Q) \leqslant 11$. By Theorem 2.2, $s(Q) = 11$. Next, $\{x_2, x_4\} \oplus \{x_1, x_3, x_6, x_8\} \oplus \{x_5, x_7, x_{10}\} \oplus \{x_9\} \in \mathcal{L}(K_2)$ but K_2 contains $\{x_1, x_2\} \oplus \{x_5, x_6\} \oplus \{x_9, x_{10}\}$ whose jump number is 3. Hence $s(K_2) = 3$.

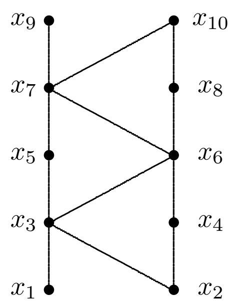

Figure 1: K_2

3. Jump criticality

It is implicit in Habib [5] that the lexicographic sum of jump-critical ordered sets Q_x over an ordered set P is itself jump-critical if $|Q_x| \geqslant 3$ for each $x \in P$. We first prove this result directly in this section.

Theorem 3.1. *Let P be an ordered set and $\{Q_x\colon x \in P\}$ be a family of jump-critical ordered sets with $|Q_x| \geqslant 3$. Then the lexicographic sum of Q_x over P is also jump-critical.*

P r o o f. Let $Q = \sum_{x \in P} Q_x$ with $s(Q) = n$ and $L = C_0 \oplus C_1 \oplus \ldots \oplus C_n$ be an optimal linear extension of Q. Observe that $s(Q_x) \geqslant 2$ for each $x \in P$. It suffices to show that $s(Q \setminus \{a\}) < s(Q)$ for each $a \in Q$.

Fix $a \in Q$. There exists an element x of P such that $a \in Q_x$. Let $L_a = L \cap Q_x = D_0 \oplus D_1 \oplus \ldots \oplus D_m$, where $D_j = C_{i_j}$ for $j = 1, \ldots, m-1$ and $D_j \subseteq C_{i_j}$ for $j = 0, m$. Let

$$
\begin{aligned}
&L_1 = C_0 \oplus \ldots \oplus C_{i_0 - 1} \oplus (C_{i_0} \setminus D_0),\\
&L_2 = L \cap \bigcup \{C_j \colon\ i_0 < j < i_m \ \text{ and } \ j \neq i_1, \ldots, i_{m-1}\},\\
&L_3 = (C_{i_m} \setminus D_m) \oplus C_{i_{m+1}} \oplus \ldots \oplus C_n.
\end{aligned}
$$

Since Q_x is jump-critical, there exists a linear extension $E_0 \oplus E_1 \oplus \ldots \oplus E_k$, $k < m$, of $Q_x \setminus \{a\}$. Let $L' = L_1 \oplus E_0 \oplus L_2 \oplus E_1 \oplus \ldots \oplus E_k \oplus L_3$. For all $c \in E_i$ we get $c \in Q_x$. For any $d \in L_2$ there exits $y \in P$ such that $d \in Q_y$. If $x > y$ or $x < y$ in P, then $c > d$ for all $c \in E_i$ or $c < d$ for all $c \in E_i$, which contradicts the fact that $c < d$ in L for $c \in D_0$ and $c > d$ in L for $c \in D_m$. Thus x and y are incomparable in P, and so c and d are also incomparable in $Q \setminus \{a\}$. Hence $L' \in \mathcal{L}(Q \setminus \{a\})$ and $c(L') \leqslant n$, which implies $s(Q \setminus \{a\}) < s(Q)$. $\square$

We have seen that if an ordered set P is series-parallel or bipartite then the following equation holds for any lexicographic sum Q of Q_x over P:

$$s(Q) = s(P) + \sum_{x \in P} s(Q_x). \tag{2}$$

A class $\mathcal{K}$ of ordered sets is said to be *lexicographic hereditary* if the following conditions hold:

(i) $P \in \mathcal{K}$ implies $P \setminus \{x\} \in \mathcal{K}$ for any $x \in P$.

(ii) The equation (2) holds for any lexicographic sum of Q_x over $P \in \mathcal{K}$.

Now we get a variation of Theorem 3.1.

Theorem 3.2. *Let P be an ordered set in a lexicographic hereditary class $\mathcal{K}$ and $\{Q_x \colon\ x \in P\}$ a family of ordered sets. If P is jump-critical and if each Q_x is jump-critical or trivial, then the lexicographic sum of Q_x over P is also jump-critical.*

P r o o f. Let $Q = \sum_{x \in P} Q_x$.

Case 1. $\{a\} = Q_y$ for some $y \in P$.

Let $P' = P \setminus \{y\}$. Now, $s(Q_y) = 0$ and $s(P') = s(P) - 1$. Then

$$s(Q \setminus \{a\}) = s(P') + \sum_{x \in P'} s(Q_x) = s(P) - 1 + \sum_{y \in P} s(Q_y) = s(Q) - 1.$$

Case 2. $\{a\} \subset Q_y$ for some $y \in P$.
Now $s(Q_y \setminus \{a\}) = s(Q_y) - 1$. Then

$$\begin{aligned} s(Q \setminus \{a\}) &= s(P) + \sum_{x \in P \setminus \{y\}} s(Q_x) + s(Q_y \setminus \{a\}) \\ &= s(P) + \sum_{x \in P} s(Q_x) - 1 = s(Q) - 1. \end{aligned}$$

□

Observing that the classes of series-parallel ordered sets and bipartite ordered sets are lexicographic hereditary, we immediately get the following corollary.

Corollary 3.3. *Let P be a series-parallel or bipartite ordered set and $\{Q_x : x \in P\}$ a family of ordered sets. If P is jump-critical and if each Q_x is jump-critical or trivial, then the lexicographic sum of Q_x over P is also jump-critical.*

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