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FLOW COMPACTIFICATIONS OF NONDISCRETE MONOIDS, IDEMPOTENTS AND HINDMAN'S THEOREM

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Abstract. We describe the extension of the multiplication on a not-necessarily-discrete topological monoid to its flow compactification. We offer two applications. The first is a nondiscrete version of Hindman's Theorem, and the second is a characterization of the projective minimal and elementary flows in terms of idempotents of the flow compactification of the monoid.

Keywords: flow, Stone-Čech compactification, Hindman's theorem

MSC 2000: 54H20, 37B05, 11B75

1. Introduction

Suppose a discrete monoid T is colored with finitely many colors. A beautiful theorem of Hindman [9] says that for one of the colors, say green, there is a sequence of green points all of whose ordered products are green. Now let us color an arbitrary topological monoid T with finitely many colors, subject only to the requirement that the set of points of each color is an open set in T. We seek a color, say green, such that the set of green points contains a sequence of pairwise disjoint nonempty open sets, all of whose ordered products are green. However, no such sequence may exist, as Example 4.6 below demonstrates. Nevertheless we show in Theorem 4.5 that, under weak conditions on T, such a sequence can always be found if we ask only that all ordered products be "near" green points.

A preliminary version of these results was presented to the seminar of B. Balcar and P. Simon of Charles University. The first author would like to express his deep appreciation for the hospitality extended to him during his 1997–98 sabbatical year in Prague.

Our approach is to exploit the properties of the flow compactification $\hat{X}$ of a flow X with action monoid T. (See Section 2 for careful definitions.) We show that the resulting flow compactification $\hat{T}$ of T inherits an associative multiplication from T and that $\hat{T}$, with this operation, is a semitopological monoid. From this we show that many properties which arise in topological dynamics, most notably recurrent and uniformly recurrent points, can be described most naturally by considering $\hat{T}$, and that existence of nontrivial idempotent points in $\hat{T}$ can be determined from natural conditions on T.

After developing this machinery, we use it to generalize Hindman's theorem in Section 4. This leads to a straightforward proof of the classical result, Corollary 4.12. In the final section, we study small projective flows and use the machinery on idempotents to characterize elementary and minimal projective compact flows.

The existence and uniqueness of $\hat{T}$ is due to deVries [15] in case T is a topological group. Aside from the additional generality of our treatment for topological monoids T, what distinguishes our development of $\hat{T}$ is that it is entirely topological. Our approach gives, for example, proofs of quite general versions of Hindman's Theorem which never mention a formula for the multiplication in $\hat{T}$. Thus our original aim in writing this article was to replace the combinatorial complexity of the existing proofs of Hindman's Theorem with appeals to the simplest underlying topological features of the situation. The reader will have to judge the success of our efforts.

All topological spaces are assumed to be Tychonoff. An encyclopedic reference to topics in topological dynamics may be found in [14].

2. Flows and the flow compactification

Let T be a fixed monoid. We reserve the letters t and s, sometimes adorned with primes and subscripts, for its elements. An *action* of T on a space X is a map $e\colon T \times X \to X$ such that, writing $e(t,x)$ as tx, we have

$$1x = x \text{ and } (t_1 t_2)x = t_1(t_2 x)$$

for all $t_i \in T$. We say that T *acts on* X if there is an action of T on X. Now suppose T is a topological monoid. A *flow* is a pair (X, e_X), where e_X is a *continuous* action of T on X. A *flow map* $f\colon (X, e_X) \to (Y, e_Y)$ is a continuous function $f\colon X \to Y$ which commutes with the actions, i.e., $f(tx) = tf(x)$ for all $x \in X$ and $t \in T$.

An important type of flow is a *left ideal* of T, i.e., a subset $S \subseteq T$ for which $ts \in S$ for all $s \in S$ and $t \in T$. The action on left ideals is always by left multiplication.

Definition. Suppose T acts on a space X. A real-valued function $g\colon X \to \mathbb{R}$ is T*-uniformly continuous* if it is bounded, continuous, and satisfies the following condition. For all $t \in T$ and $\varepsilon > 0$ there is some $T_t \in \mathcal{N}_t$ (the open neighborhoods of t) such that for all $t' \in T_t$ and $x \in X$ we have

$$|g(tx) - g(t'x)| < \varepsilon.$$

We use $C^T(X)$ to designate the set of T-uniformly continuous functions on X.

Routine calculation establishes the following result, which is important for our purposes.

2.1. Theorem. *Suppose T acts on a space X. Then $C^T(X)$ is a subring and vector sublattice of $C(X)$ which is uniformly closed. And if X is compact then $C^T(X) = C(X)$.*

For every flow X there is a compact flow $\hat{X}$ which is analogous to the Stone-Čech compactification of X in the sense that all T-uniformly continuous functions on X extend to $\hat{X}$. This result is due to de Vries [15] in the case that T is a topological group. A self contained proof, which includes the case where T is a monoid, may be found in [3].

2.2. Proposition. *Let Y be a flow. Then there exists $\hat{Y}$ with the following properties.*

(1) *$\hat{Y}$ is a compact flow.*
(2) *There is a flow map $i\colon Y \to \hat{Y}$ with dense range.*
(3) *For every compact flow X and flow map $g\colon Y \to X$ there exists a unique flow map $f\colon \hat{Y} \to X$ such that $f \circ i = g$.*
(4) *$C^T(Y) = C^T(\hat{Y}) = C(\hat{Y})$. More precisely, every T-uniformly continuous function f on Y extends uniquely to a continuous function $\hat{f}$ on $\hat{Y}$ which satisfies $f = \hat{f} \circ i$.*

We are especially interested in the compact flow $\hat{T}$ which results by taking $Y = T$ in Proposition 2.2, where T acts on itself by left multiplication. We state explicitly the properties of $\hat{T}$ and then use them to show that $\hat{T}$ is itself a semitopological monoid.

2.3. Corollary. *Let T be a topological monoid. Then there exists $\hat{T}$ with the following properties:*

(1) *$\hat{T}$ is a compact flow.*
(2) *There is a flow map $i\colon T \to \hat{T}$ with dense range.*

(3) *For every compact flow X and flow map $g\colon T \to X$ there exists a unique flow map $f\colon \hat{T} \to X$ such that the diagram*

$$\begin{tikzcd} \hat{T} \arrow[dr, "f"] & \\ T \arrow[u, "i"] \arrow[r, "g"'] & X \end{tikzcd}$$

commutes. That is, $f \circ i = g$.

(4) $C^T(T) = C^T(\hat{T}) = C(\hat{T})$. *More precisely, every T-uniformly continuous function f on T extends uniquely to a continuous function $\hat{f}$ on $\hat{T}$ which satisfies $f = \hat{f} \circ i$.*

We say that a flow Y is *compactifiable* if the mapping i from Proposition 2.2 (2) is a homeomorphic embedding. It is known [13] that every topological group, acting on itself by left multiplication, is compactifiable. Also, every flow with a *locally compact group* of actions is compactifiable [15]. But there are Tychonoff flows acted on by a topological group which are not compactifiable [11]. Here is an example of a nontrivial monoid T with $\hat{T}$ a singleton.

2.4. Example. Let $T = \{t \in \mathbb{N}^{\mathbb{N}} : \exists\, m\, \forall\, n \geqslant m \ (tn = n)\}$. Topologize T by neighborhoods of $t \in T$ of the form

$$T(t, m) = \{t' \in T\colon\ \forall\, n \leqslant m \ \ (t'n = tn)\}.$$

In [3] it is shown that $\hat{T}$ is a singleton and that the only T-uniformly continuous functions on T are the constant functions.

Now let X be a compact flow and let $x \in X$. Then the map $\tilde{x}\colon T \to X$ given by $\tilde{x}(t) = tx$ is a flow map, and by (3) of Corollary 2.3 has a unique extension to a map $\hat{x}\colon \hat{T} \to X$, as summarized in the following commuting diagram.

$$\begin{tikzcd} \hat{T} \arrow[dr, "\hat{x}"] & \\ T \arrow[u, "i"] \arrow[r, "\tilde{x}"'] & X \end{tikzcd}$$

This leads to a map $\hat{e}\colon \hat{T} \times X \to X$ defined by $\hat{e}(p, x) = \hat{x}(p)$. For simplicity and consistency of notation, we denote $\hat{e}(p, x)$ by px. This map extends the action of T on X since

$$\hat{e}(i(t), x) = \hat{x}(i(t)) = \tilde{x}(t) = tx = e(t, x).$$

If $x \in X$ is fixed, then this shows that the map $p \longmapsto px$ is continuous. But if $p \in \hat{T}$ is fixed then the map $x \longmapsto px$ is not in general continuous. Also, when $X = \hat{T}$ we get a binary operation on $\hat{T}$ defined by $pq = \hat{q}(p)$. We proceed to show that $\hat{T}$ has a monoid structure and that the map $\hat{e}$ gives an action of $\hat{T}$ on X.

2.5. Proposition. *Let X be a compact flow and $p, q \in \hat{T}$. Then for all $x \in X$*

$$(pq)x = p(qx).$$

P r o o f. First we unwind the definitions of the left and the right sides, respectively.

$$(pq)x = \hat{x}(pq) = \hat{x}(\hat{q}(p))$$

and

$$p(qx) = (\hat{qx})(p) = \widehat{(\hat{x}(q))}(p).$$

Now both $\hat{x} \circ \hat{q}$ and $\widehat{(\hat{x}(q))}$ are continuous flow maps from $\hat{T}$ into X, so to show they are equal it is enough to show that they agree on the dense subset $i(T) \subseteq \hat{T}$, i.e., that

$$\hat{x} \circ \hat{q} \circ i = (\hat{x}\hat{(q)}) \circ i.$$

To do this let $t \in T$. Then

$$\begin{aligned} \hat{x}(\hat{q}(i(t))) &= \hat{x}(\tilde{q}(t)) = \hat{x}(tq) \\ &= t\hat{x}(q) \text{ since } \hat{x} \text{ is a flow map} \\ &= \widetilde{(\hat{x}(q))}(t) = \widehat{(\hat{x}(q))}(i(t)) \end{aligned}$$

and we are done. □

The next result is of central importance in what follows. Here, in case T is a topological group, this result is known and is discussed in detail in DeVries [14], Appendix D, pp. 662–668.

2.6. Corollary. *$\hat{T}$ is a semitopological monoid whose identity is $i(1)$.*

P r o o f. Proposition 2.5 (with $X = \hat{T}$) shows that the binary operation on $\hat{T}$ is associative. Also, for fixed $q \in \hat{T}$, the fact that the map $p \mapsto pq$ is continuous is just a reference to the continuity of $\hat{q}$. Lastly, we show that $i(1)$ is the identity of $\hat{T}$. Let $q \in \hat{T}$. Then

$$(i(1))q = \hat{q}(i(1)) = \tilde{q}(1) = 1q = q.$$

To show that $q(i(1)) = q$, observe that $\widehat{i(1)}\colon \hat{T} \to \hat{T}$ is a continuous flow map and that

$$\begin{aligned}(\widehat{i(1)} \circ i)(t) &= \widetilde{i(1)}(t) = t(i(1)) \\ &= i(t1) \ \text{ since } i \text{ is a flow map} \\ &= i(t) = (\mathrm{id} \circ i)(t).\end{aligned}$$

Thus $\widehat{i(1)} = \mathrm{id}$, since these two continuous flow maps agree on the dense subflow $i(T) \subseteq \hat{T}$. That is $q(i(1)) = q$. □

The action of $\hat{T}$ on compact flows commutes with flow maps.

2.7. Proposition. *Let X and Y be compact flows and $f\colon X \to Y$ be a flow map. Then $f(px) = pf(x)$ for all $x \in X$ and $p \in \hat{T}$.*

Proof. Fix $x \in X$ and let y designate $f(x)$. Let $\tilde{x}$, $\hat{x}$, $\tilde{y}$, and $\hat{y}$ have the meanings assigned to them in the discussion leading up to Proposition 2.5. Then $f\tilde{x} = \tilde{y}$ because

$$f\tilde{x}(t) = f(tx) = tf(x) = \tilde{y}(t)$$

for all $t \in T$. Therefore $f\hat{x}i = f\tilde{x} = \tilde{y} = \hat{y}i$, so that $f\hat{x} = \hat{y}$ by the uniqueness clause of Corollary 2.3. We have

$$f(px) = f(\hat{x}(p)) = \hat{y}(p) = py = pf(x).$$

□

3. Idempotents, proximal points, and recurrent points in flows

In a compact flow X, the classical notions of proximal and recurrent points can be characterized in terms of the action of $\hat{T}$ on X. Though our context is considerably more general, this section and the next owe a considerable debt to Blass' development in the admirable expository article [4].

Definition. Two points x_1 and x_2 in a flow X are *proximal* if for every neighborhood u of the diagonal $\Delta \equiv \{(x, x)\colon x \in X\}$ in $X \times X$ there is some action t for which $(tx_1, tx_2) \in u$.

3.1. Proposition. *Two points x_i of a compact flow X are proximal if and only if there is some $p \in \hat{T}$ such that $px_1 = px_2$.*

Proof. ($\Rightarrow$) Let

$$\mathcal{P} = \left\{ \mathbf{b} = \{b_i\colon\ i \in F\}\colon\ F \text{ finite, each } b_i \text{ is open and } \bigcup_{b \in \mathbf{b}} b = X \right\}.$$

($\mathcal{P}$ is just the set of finite open covers of X.) Then the assumptions yield that if $\mathbf{b} \in \mathcal{P}$,

$$a_{\mathbf{b}} = \bigcup_{b \in \mathbf{b}} \{t\colon\ tx_1, tx_2 \in b\} \neq \emptyset.$$

Now it is clear that the family $\{a_{\mathbf{b}}\colon\ \mathbf{b} \in \mathcal{P}\}$ has the finite intersection property, so

$$\bigcap \{\mathrm{cl}(i(a_{\mathbf{b}}))\colon\ \mathbf{b} \in \mathcal{P}\} \neq \emptyset.$$

Let p be any point of this intersection. We claim that $px_1 = px_2$. To prove the claim we show that (px_1, px_2) lies in every neighborhood of Δ. For consider the continuous function $(\hat{x_1}, \hat{x_2})\colon\ \hat{T} \to X \times X$ defined by

$$(\hat{x_1}, \hat{x_2})(q) = (\hat{x_1}(q), \hat{x_2}(q)), \quad q \in \hat{T}.$$

If u is a neighborhood of Δ then an easy compactness argument yields $\mathbf{b} \in \mathcal{P}$ so that $\bigcup_{b \in \mathbf{b}} \mathrm{cl}(b \times b) \subseteq u$. But $p \in \mathrm{cl}(i(a_{\mathbf{b}}))$ implies

$$(px_1, px_2) = (\hat{x_1}, \hat{x_2})(p) \in \mathrm{cl}((\hat{x_1}, \hat{x_2})(i(a_{\mathbf{b}}))),$$

and $(\hat{x_1}, \hat{x_2})(i(a_{\mathbf{b}})) \subseteq \bigcup_{b \in \mathbf{b}} (b \times b)$ because for each $t \in a_{\mathbf{b}}$ there is a $b \in \mathbf{b}$ such that

$$(\hat{x_1}, \hat{x_2})(i(t)) = (\hat{x_1}(i(t)), \hat{x_2}(i(t))) = (\widetilde{x_1}(t), \widetilde{x_2}(t)) = (tx_1, tx_2) \in b \times b.$$

It follows that $(px_1, px_2) \in u$.

($\Leftarrow$) Suppose that $px_1 = px_2$ and let u be a neighborhood of Δ. Then the map $(\hat{x_1}, \hat{x_2})$ is continuous at p and $i(T)$ is dense in $\hat{T}$, so there is a $t \in T$ so that $(\hat{x_1}, \hat{x_2})(i(t)) = (tx_1, tx_2) \in u$. $\square$

Definition. A point x in a flow X is *recurrent* if there is some a in $\mathcal{N}_1$ such that for every $b \in \mathcal{N}_x$ there is some $t \notin a$ with $tx \in b$.

If $px = x$ for some point x in a flow X and some $p \in \hat{T}$ with $p \neq 1$ then x is recurrent. The converse, however, is not exactly true. Let T be a topological monoid having a nontrivial neighborhood of 1 such that $\hat{T}$ is a singleton (see Example 2.4 above). Let T act on the one point space $\{\bullet\}$ in the only way it can. Then every $t \in T$ satisfies $t\bullet = \bullet$, so $\bullet$ is a recurrent point. But, although every point of $\hat{T}$

fixes $\bullet$, in fact there is only one such point and it is $i(1)$. This shows that recurrence of a point x does not imply in general that a non-identity $p \in \hat{T}$ fixes x. The next result, Proposition 3.2, precisely clarifies these ideas.

3.2 Proposition. *Let X be a compact flow and $x \in X$. Then the following are equivalent:*

(1) *There exists a $p \neq i(1) \in \hat{T}$ such that $px = x$.*

(2) *There is a neighborhood a of 1 in T such that*

  (a) *if $b \in \mathcal{N}_x$ there is some $t \notin a$ with $tx \in b$, i.e., x is recurrent; and*

  (b) *$i(1) \notin \mathrm{cl}(i(T\backslash a))$.*

Proof. (1) $\Rightarrow$ (2) Pick open neighborhoods U and V of p so that $i(1) \notin \mathrm{cl}(V)$ and $\mathrm{cl}(U) \subseteq V$. Put $W = \hat{T}\backslash \mathrm{cl}(U)$ and $a = i^{-1}(W)$. Since $V \cup W = \hat{T}$, it follows that $T\backslash a \subseteq i^{-1}(V)$, so $i(T\backslash a) \subseteq V$. Thus $\mathrm{cl}(i(T\backslash a)) \subseteq \mathrm{cl}(V)$ and since $i1 \notin \mathrm{cl}(V)$, it is obvious that $i1 \notin \mathrm{cl}(i(T\backslash a))$. This shows that the neighborhood a satisfies property (b).

It is also easy to see that a satisfies property (a). Let $b \in \mathcal{N}_x$. Since $\hat{x}$ is continuous at p and $\hat{x}p = x$, $p \in \hat{x}^{-1}(b)$. So $\hat{x}^{-1}(b) \cap U \neq \emptyset$, and since $i(T)$ is dense in $\hat{T}$, any $t \in i^{-1}(\hat{x}^{-1}(b) \cap U)$ satisfies $t \notin a$ and $tx = \tilde{x}(t) = \hat{x}(i(t)) \in b$. This establishes (a).

(2) $\Rightarrow$ (1) Consider $\hat{x}\colon \mathrm{cl}(i(T\backslash a)) \to X$. Compactness and (a) imply that $x \in \hat{x}(\mathrm{cl}(i(T\backslash a)))$ so there is a $p \in \mathrm{cl}(i(T\backslash a))$ such that $\hat{x}(p) = x$. By (b), $p \neq i(1)$. □

This next simple lemma is needed in the proof of Corollary 3.5.

3.3. Lemma. *Let Z be compact, D dense in Z and U a regular open set in Z. Then $\mathrm{cl}(D\backslash U) = Z\backslash U$.*

Proof. Put $K = \mathrm{cl}(D\backslash U)$ and suppose that $z \notin K$. Since $z \in \mathrm{cl}(D)$, it is easy to see that $z \in \mathrm{cl}(D \cap U)$, so $z \in \mathrm{cl}(U)$. Thus

$$U \subseteq Z\backslash K \subseteq \mathrm{cl}(U).$$

Since U is regular, $U = Z\backslash K$, or $K = Z\backslash U$. □

3.4. Example. This lemma fails if U is not regular. To see this, let Z be $[0,2]$, D be the irrationals in Z, and U be $Z\backslash\{1\}$. Then $Z\backslash U = \{1\}$, $D\backslash U = \emptyset$, and $\mathrm{cl}(D\backslash U) = \emptyset \neq Z\backslash U$. It is clear, however, that $\mathrm{cl}(D\backslash U) \subseteq Z\backslash U$. It is easy to produce examples where $\mathrm{cl}(D\backslash U) \neq \emptyset$.

3.5. Corollary. *Suppose that T is nontrivial. Then (2) of Proposition 3.2 is satisfied if x is recurrent and T is compactifiable, i.e., if i is a homeomorphism.*

P r o o f. We need to produce the neighborhood a of 1 satisfying (2) of Proposition 3.2. Let a_1 be a neighborhood of 1 in T satisfying the recurrence definition for x. Then since $i(T)$ has the relative topology in $\hat{T}$ and $i(a_1)$ is open in $i(T)$, there is an open V in $\hat{T}$ such that $V \cap i(T) = i(a_1)$. So there exists regular open U such that $i(1) \in U \subseteq V$.

Set $a = i^{-1}(U)$. Note that $a \subseteq a_1$ so that a satisfies property (2a) of Proposition 3.2. Now apply Lemma 3.3 with $Z = \hat{T}$, $D = i(T)$ and U to obtain

$$\mathrm{cl}(i(T\backslash a)) = \mathrm{cl}(i(T)\backslash U) = \hat{T}\backslash U.$$

Since $i(1) \in U$, $i(1) \notin \mathrm{cl}(i(T\backslash a)$ and (2b) of Proposition 3.2 is satisfied. □

We would like to know if (2) of Proposition 3.2 is satisfied when i is just one-to-one.

Definition. A point x of a flow X is *uniformly recurrent* if for all $b \in \mathcal{N}_x$ there is a finite subset $T_0 \subseteq T$ such that for every $t \in T$ there is some $t_0 \in T_0$ with $t_0tx \in b$, i.e.,

$$\bigcup_{T_0} t_0^{-1}\tilde{x}^{-1}b = T.$$

3.6. Proposition. *A point x of a compact flow X is uniformly recurrent if and only if for every $q \in \hat{T}$ there is some $p \in \hat{T}$ such that $(pq)x = x$.*

P r o o f. ($\Rightarrow$) Let $z = qx$ for x and q as in the statement of the proposition. Now $\hat{z}\colon \hat{T} \to X$ is continuous and $\hat{T}$ is compact, so to show the existence of p we need only show that

$$x \in \hat{z}(\hat{T}) = \hat{z}(\mathrm{cl}(i(T))) = \mathrm{cl}(\hat{z}(i(T))) = \mathrm{cl}(\tilde{z}(T)).$$

To this end, let $c \in \mathcal{N}_x$. Pick $b \in \mathcal{N}_x$ so that $\mathrm{cl}(b) \subseteq c$. Pick a finite set $T_0 \subseteq T$ such that for every $t \in T$ there is some $t_0 \in T_0$ with $t_0tx \in b$. We claim that $t_0z \in c$ for some $t_0 \in T_0$. If not, then because T_0 is finite there exists a neighborhood a of z such that

$$t_0a \cap \mathrm{cl}(b) = \emptyset$$

for each t_0. Now since $\hat{x}$ is a continuous function which takes q to z and since $i(T)$ is dense in $\hat{T}$, $tx \in a$ for some $t \in T$. But then $t_0tx \in b$ for some t_0, a contradiction to $t_0a \cap \mathrm{cl}(b) = \emptyset$.

($\Leftarrow$) Let b be a neighborhood of x. We first observe that for each $q \in \hat{T}$ there exists $t \in T$ such that $tqx \in b$. Indeed, since

$$\widehat{qx}\colon \hat{T} \to X$$

is continuous and $\widehat{qx}(p) = x$ for some p, the density of $i(T)$ gives t such that $tqx = \widehat{qx}(i(t)) \in b$. So for each q there is a t and a neighborhood $c_{t,q}$ of qx so that $tc_{t,q} \subseteq b$. (The notation just means that c depends on t and q.) These c's cover $\hat{T}$, so finitely many do, say $\{c_{t,q}\colon\ t \in T_0\}$, and this set T_0 is what we are looking for. To verify, let $t \in T$. Then $tx \in c_{t_0,q}$ for some t_0 and so $t_0tx \in b$. □

We use this machinery to study idempotents in $\hat{T}$. *For the remainder of this section we simplify notation slightly by writing* 1 *instead of* $i(1)$ *since this will cause no confusion.* We begin by applying Propositions 3.1, 3.2, and 3.6 to $X = \hat{T}$.

3.7. Proposition. *The following hold in* $\hat{T}$.

(1) r *is recurrent in* $\hat{T}$ *if there is some* $p \neq 1$ *such that* $pr = r$.
(2) r *is uniformly recurrent in* $\hat{T}$ *if and only if for all* q *there is some* p *such that* $pqr = r$, *i.e., if and only if* r *lies in some minimal (closed) left ideal of* $\hat{T}$.
(3) r_1 *and* r_2 *are proximal if and only if there is some* p *for which* $pr_1 = pr_2$.
(4) r *generates a minimal closed subsemigroup of* $\hat{T}$ *if and only if* r *is idempotent, i.e., if and only if* $r^2 = r$.

Proposition 3.7 (4) is Namakura's Theorem [12]. The proofs of the next two results are straightforward.

3.8. Proposition. *Each of the following statements about an element* $r \neq 1$ *of* $\hat{T}$ *implies the next.*

(1) r *is uniformly recurrent and proximal to* 1.
(2) r *is idempotent.*
(3) r *is recurrent and proximal to* 1.

3.9. Proposition. *Let* X *be a compact flow and let* $x \in X$. *If* p *is (uniformly) recurrent in* $\hat{T}$ *then* px *is (uniformly) recurrent in* X. *If* r_1 *and* r_2 *are proximal in* $\hat{T}$ *then* r_1x *and* r_2x *are proximal in* X.

It is useful to order the idempotents of $\hat{T}$. A natural first step is to preorder them according to the containment of the left ideals they generate.

3.10. Lemma. *The following are equivalent for idempotents* $p, q \in \hat{T}$.

(1) $\hat{T}q \subseteq \hat{T}p$.
(2) *For all* r *there is an* s *such that* $rq = sp$.
(3) $qp = q$.

P r o o f. The equivalence of (1) and (2) is clear. If (3) holds and r is given, then s can be taken to be rq in (2). If (2) holds then in particular there is some s for which $q = q^2 = sp$. But then

$$qp = (sp)p = s(p^2) = sp = q.$$

$\square$

An actual partial order refines the preorder of Lemma 3.10.

Definition. The idempotent elements of $\hat{T}$ are partially ordered by

$$p \leqslant q \iff qp = pq = p.$$

3.11. Lemma. *Suppose p and q are related as in Lemma 3.10. Set $q_1 \equiv pq$. Then q_1 is an idempotent and $q_1 \leqslant p$.*

Proof. We calculate

$$\begin{aligned} q_1^2 &= (pq)^2 = p(qp)q = pq^2 = pq = q_1, \\ pq_1 &= p(pq) = p^2q = pq = q_1, \\ q_1p &= (pq)p = p(qp) = pq = q_1. \end{aligned}$$

$\square$

The next result combines the foregoing ideas. Its proof is left as an exercise.

3.12. Proposition. *The following are equivalent for $p \in \hat{T}$.*
(1) *p is uniformly recurrent and proximal to 1.*
(2) *p is idempotent and belongs to some minimal left ideal of $\hat{T}$.*
(3) *p is a minimal idempotent.*

We now turn our attention to the issue of when $\hat{T}$ has a nontrivial idempotent, i.e., an idempotent $p \neq 1$.

3.13. Proposition. *An element $q \in \hat{T}$ has a left inverse if and only if it can be moved arbitrarily close to 1, i.e., if and only if for every $b \in \mathcal{N}_1$ there is some $t \in T$ such that $tq \in b$.*

Proof. If $pq = 1$ then the facts that $p \mapsto pq$ is continuous and $i(T)$ is dense in $\hat{T}$ imply that q can be moved arbitrarily close to 1. If q can be moved arbitrarily close to 1 then $a_b \equiv \{r\colon rq \in b\}$ is a nonempty open set for each $b \in \mathcal{N}_1$, and since the family of all a_b's clearly has the finite intersection property, there is some p which lies in $\bigcap\limits_{b \in \mathcal{N}_1} \mathrm{cl}(a_b)$. It is easy to see that pq must be 1. $\square$

3.14. Theorem. *The following are equivalent for T.*

(1) $\hat{T}$ *has no proper left ideals.*

(2) *Every element of* $\hat{T}$ *has a left inverse.*

(3) $\hat{T}$ *is a group.*

(4) *Every element of* $\hat{T}$ *can be moved arbitrarily close to* 1, *i.e., for every* $q \in \hat{T}$ *and every neighborhood* b *of the identity in* $\hat{T}$ *there is some* $t \in T$ *such that* $tq \in b$.

(5) $\hat{T}$ *has no nonidentity idempotents.*

(6) *No nonidentity recurrent point of* $\hat{T}$ *is proximal to* 1.

(7) *For every neighborhood* b *of the identity in* $\hat{T}$ *there is a finite subset* $T_0 \subseteq T$ *such that for every* $p \in \hat{T}$ *there is some* $t_0 \in T_0$ *with* $t_0 p \in b$, *i.e.,*

$$\bigcup_{T_0} t_0^{-1} b = \hat{T}.$$

Proof. The equivalence of (1) and (2) is clear, and the equivalence of (2) and (3) is an exercise in the elementary theory of monoids. If (2) holds and $q \in \hat{T}$ then there is some p for which $pq = 1$. For given $b \in \mathcal{N}_1$, the continuity of the map $r \mapsto rq$ at $r = p$ and the density of $i(T)$ in $\hat{T}$ furnish $t \in T$ such that $tq \in b$, i.e., (4) holds. If (4) holds and $q \in \hat{T}$ then $V_b \equiv \{r \in \hat{T} \colon\ rq \in b\}$ is a nonempty open set for each $b \in \mathcal{N}_1$, and since the family of all V_b's clearly has the finite intersection property, there is some p which lies in $\bigcap_{b \in \mathcal{N}_1} \mathrm{cl}(V_b)$. It is easy to see that pq must be 1. This shows that (2) holds.

The implication from (2) to (5) is likewise easy: if p is an idempotent with left inverse q then

$$p = 1p = (qp)p = q(pp) = qp = 1.$$

To show that (5) implies (2) observe that if p has no left inverse then $\hat{T}p$ is a closed subsemigroup omitting 1. This must contain a minimal closed subsemigroup by Zorn's Lemma, and the latter must be of the form $\{q\}$ for some idempotent $q \neq 1$ by Theorem 3.7 (4). The equivalence of (5) and (6) is a consequence of Theorem 3.12, for $\hat{T}$ has a nontrivial idempotent if and only if it has a nontrivial minimal idempotent. The equivalence of (4) and (7) is a consequence of the compactness of $\hat{T}$. Indeed, for $t \in T$ put $U_t = \{p \in \hat{T} \colon\ tp \in b\}$. Each U_t is open in $\hat{T}$ and Proposition 3.13 insures that $\{U_t \colon\ t \in T\}$ covers $\hat{T}$. □

We are interested in conditions on T which are equivalent to the existence (or nonexistence) of idempotents in $\hat{T}$. The following corollary comes close to providing such an equivalence.

Definition. An open set a in T will be called *large* if there is some finite subset $T_0 \subseteq T$ for which $\bigcup_{T_0} t_0^{-1} a = T$. An open set which is not large is called *small*.

3.15. Corollary. *Any topological monoid T in which every neighborhood of the identity is large satisfies Theorem 3.14. Conversely, if T satisfies Theorem 3.14, and if for every neighborhood a of the identity in T there is a neighborhood b of the identity in $\hat{T}$ with $i^{-1}(b) \subseteq a$, then every neighborhood of the identity in T is large. In particular, if T is compactifiable then it satisfies Theorem 3.14 if and only if every neighborhood of the identity is large.*

P r o o f. Suppose that every neighborhood of the identity in T is large. We aim to show that Theorem 3.14 (6) holds. For that purpose consider a neighborhood b of 1 in $\hat{T}$, and then find a neighborhood c of 1 in $\hat{T}$ such that $\mathrm{cl}(c) \subseteq b$. Let $a = i^{-1}(c)$. Since a is large there is some finite $T_0 \subseteq T$ such that $\bigcup_{T_0} t_0^{-1} a = T$. We claim that $\bigcup_{T_0} t_0^{-1} b = \hat{T}$. If not, there is a $p \in \hat{T}$ such that $t_0 p \notin b$ for all $t_0 \in T_0$. Then there is a neighborhood U of p such that $t_0 q \notin \mathrm{cl}(c)$ for all $t_0 \in T_0$ and $q \in U$. Now if $t \in i^{-1}(U)$ then it follows that $i(t_0 t) = t_0(i(t)) \notin \mathrm{cl}(c)$ for all $t_0 \in T_0$, which implies that $t_0 t \notin i^{-1}(c) = a$ for all $t_0 \in T_0$, a contradiction.

Now assume that T satisfies Theorem 3.14, and that for every neighborhood a of the identity in T there is a neighborhood b of the identity in $\hat{T}$ with $i^{-1}(b) \subseteq a$. Let a be open in T with $1 \in a$. Pick b in $\hat{T}$ containing 1 with $i^{-1}(b) \subseteq a$ and pick a finite T_0 satisfying property (6) of Theorem 3.14 for the set b. It is routine to check that $\bigcup_{T_0} t_0^{-1} a = T$. □

Not every neighborhood of the identity in T need be large even if T satisfies Theorem 3.14. Indeed, let T be the monoid discussed in Example 2.4. Then $\hat{T}$ is a singleton so it trivially satisfies Theorem 3.14, yet T has small neighborhoods of the identity. One such is $a \equiv \{t\colon\ t(1) = 1\}$. For if we define the functions $s_n \in T$ by $s_n(1) = n$ and $s_n(j) = j$ if $j \neq 1$, then for any finite set $T_0 \subseteq T$ there is an m such that $t_0(m) = m$ for all $t_0 \in T_0$. In particular, $t_0 s_m \notin a$ for all $t_0 \in T_0$.

The Auslander-Ellis Theorem holds for arbitrary topological monoids in the following form.

3.16. Theorem. *If $\hat{T}$ contains a nontrivial idempotent, then every point of every compact flow has a uniformly recurrent point proximal to it. If T is compactifiable, and every point of every compact flow has a uniformly recurrent point proximal to it, then $\hat{T}$ contains a nontrivial idempotent.*

4. Hindman's theorem

We apply the machinery developed in the previous section to produce a generalization of Hindman's theorem. In order to state our version of Hindman's theorem, Theorem 4.5, we must introduce some notation and technical machinery. Let T be a topological monoid and let $\{c_n\}$ be a sequence of open sets in T. Then for indices $i_1 < i_2 < \ldots < i_k$ we write

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \equiv \{s_1 s_2 \ldots s_k \colon\ s_j \in c_{i_j},\ \ 1 \leqslant j \leqslant k\}.$$

It is important to understand that the factors in these products occur in the order that the sets from which they are chosen appear in the sequence, without repetition, i.e., that $j < l$ implies $i_j < i_l$.

Definition. Let a and a' be open sets in a flow X. We say that a' is *well below* a, and write $a' \lhd a$, if there is some $g \in C^T(X)$ such that g is 0 on a' and 1 off a.

Note that if $a' \lhd a$ by virtue of some $g \in C^T(X)$, then by replacing g by $(g \vee 0) \wedge 1$ if necessary, we may assume that $0 \leqslant g \leqslant 1$. The well below relation may be recast slightly using standard facts from general topology [6]: a' lies well below a if and only if there are disjoint zero sets Z and Z' of functions in $C^T(X)$ such that $a' \subseteq Z'$ and $a \cup Z = X$. In particular, if X is metric and $C^T(X) = C^*(X)$ then every closed set is the zero set of some function in $C^T(X)$, and in this case $a' \lhd a$ if and only if $\mathrm{cl}(a') \subseteq a$.

Throughout the remainder of this section we use upper case letters towards the end of the alphabet $(U, V, W, \ldots)$ to designate open sets in $\hat{X}$, and lower case letters towards the beginning of the alphabet $(a, b, c, \ldots)$ to designate open sets in X.

4.1. Lemma. *Let X be a flow with flow compactification $i\colon X \to \hat{X}$ as in Proposition 2.2. If $U \lhd V$ in $\hat{X}$ then $i^{-1}(U) \lhd i^{-1}(V)$ in X.*

Proof. Suppose $\hat{f} \in C(\hat{X})$ is 0 on U and 1 off V. Then $f \equiv \hat{f} i$ lies in $C^T(X)$ and is 0 on $i^{-1}(U)$ and 1 off $i^{-1}(V)$. □

Definition. Let X be a flow with flow compactification $i\colon X \to \hat{X}$ as in Proposition 2.2. For any subset $S \subseteq X$ we define

$$\begin{aligned} i_*(S) &\equiv \bigcup \{U \subseteq \hat{X} \colon\ U \text{ open and } i^{-1}(U) \subseteq S\} \\ &= \hat{X} \setminus \mathrm{cl}(i(X \setminus S)). \end{aligned}$$

Observe that for $S_1, S_2 \subseteq X$, $i_*(S_1 \cap S_2) = i_*(S_1) \cap i_*(S_2)$, $i^{-1} i_*(S_1) \subseteq \mathrm{int}(S_1)$, and $i_* i^{-1} i_*(S_1) = i_*(S_1) = i_*(\mathrm{int}(S_1))$.

4.2. Lemma. $i^{-1}i_*(a) = \bigcup\{a'\colon\ a' \lhd a\}$.

Proof. Let us take $a' \lhd a$. Then for some $f \in C^T(X)$, $f(a') = 0$ and $f(X \backslash a) = 1$. Extend f to $\hat{f}$ on $C^T(\hat{X}) = C(\hat{X})$. Then $U \equiv \hat{f}^{-1}(-\infty, .5)$ is open in $\hat{X}$ and $a' \subseteq i^{-1}(U) = f^{-1}(-\infty, .5) \subseteq a$. This implies $i^{-1}i_*(a) \supseteq \bigcup\{a'\colon\ a' \lhd a\}$.

For the reverse inclusion consider $t \in i^{-1}i_*(a)$. Put $p = i(t) \in i_*(a)$ and take $p \in U \subseteq \hat{X}$ with U open and $\mathrm{cl}(U) \subseteq i_*(a)$. Find $\hat{f} \in C(\hat{X})$ such that $\hat{f}(U) = 0$ and $\hat{f}(\hat{X} \backslash i_*(a)) = 1$. Set $f = \hat{f} \circ i \in C^T(X)$ and define $a' = i^{-1}(U)$. We have $f(t) = \hat{f}(i(t)) = \hat{f}(p) = 0$, so $t \in a'$. Now f is 0 on a', and if $s \notin a$ then since $i^{-1}i_*(a) \subseteq a$ we also have $s \notin i^{-1}i_*(a)$, so $f(s) = \hat{f}(i(s)) = 1$. This f shows that $a' \lhd a$. □

4.3. Lemma. *If $a' \lhd a$ then $\mathrm{cl}(i_*(a')) \cup \mathrm{cl}(i(a')) \subseteq i_*(a)$.*

Proof. Find $f \in C^T(X)$ such that f is 0 on a' and 1 off a. Extend f to $\hat{f} \in C(\hat{X})$ and put $U \equiv \{y \in \hat{X}\colon\ \hat{f}(y) < 1\}$. Then U is open in $\hat{X}$ and since $i^{-1}(U) = \{x \in X\colon\ f(x) < 1\} \subseteq a$, $U \subseteq i_*(a)$. Now if there were some point $y \in i_*(a')$ for which $\hat{f}(y) > 0$ then there would be some neighborhood V of y such that $i^{-1}(V) \subseteq a$ and $\hat{f}(z) > 0$ for all $z \in V$. But such a neighborhood must contain points of $i(X)$, and since $i^{-1}i_*(a') \subseteq a$, any such point must be of the form $i(x)$ for some $x \in a'$, a violation of the assumption that $\hat{f}(i(x)) = f(x) = 0$. We conclude that $\hat{f}$ is 0 on $i_*(a')$, and since $\hat{f}$ is also 0 on $i(a')$ because f is 0 on a', it follows that $\hat{f}$ is 0 on $\mathrm{cl}(i_*(a')) \cup \mathrm{cl}(i(a'))$. Therefore $\mathrm{cl}(i_*(a')) \cup \mathrm{cl}(i(a')) \subseteq U \subseteq i_*(a)$. □

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be open covers of a flow X. We say that $\mathcal{C}$ *T-refines* $\mathcal{D}$, and write $\mathcal{C} \lhd \mathcal{D}$, provided that for every $c \in \mathcal{C}$ there is some $d \in \mathcal{D}$ such that $c \lhd d$. A cover $\mathcal{C}$ of X is *fat* if each $c \in \mathcal{C}$ is open and $i_*(\mathcal{C}) \equiv \{i_*(c)\colon\ c \in \mathcal{C}\}$ covers $\hat{X}$.

4.4. Lemma. *Every open cover of a flow which is T-refined by a finite open cover is fat. Conversely, every fat cover is T-refined by a finite fat cover. In particular, every fat cover has a finite fat subcover.*

Proof. Suppose that $\mathcal{C} \lhd \mathcal{D}$ for open covers $\mathcal{C}$ and $\mathcal{D}$ of a flow X such that $\mathcal{C}$ is finite. Then $\{\mathrm{cl}(i(c))\colon\ c \in \mathcal{C}\}$ covers $\hat{X}$ because $i(X)$ is dense in X. Since for each $c \in \mathcal{C}$ there is some $d \in \mathcal{D}$ such that $c \lhd d$, we get $\mathrm{cl}(i(c)) \subseteq i_*(d)$ by Lemma 4.3. Therefore $\mathcal{D}$ is fat. On the other hand, let $\mathcal{D}$ be a fat cover of X. Because every open subset of $\hat{X}$ is the union of those open subsets well below it, it follows that the family of open subsets $U \subseteq \hat{X}$, for which $U \lhd i_*(d)$ for some $d \in \mathcal{D}$, covers $\hat{X}$. Let $\mathcal{E}$ be a finite subcover of this family. For each $d \in \mathcal{D}$ set

$$c_d \equiv \bigcup\{i^{-1}(U)\colon\ U \in \mathcal{E} \text{ and } U \lhd i_*(d)\}.$$

Now for any $d \in \mathcal{D}$ we have $\bigcup\{U \in \mathcal{E}\colon\ U \lhd i_*(d)\} \lhd i_*(d)$ because $\mathcal{E}$ is finite, from which we get $c_d \lhd d$ by Lemma 4.1. That is, $\mathcal{C} \equiv \{c_d\colon\ d \in \mathcal{D}\} \lhd \mathcal{D}$. And $\mathcal{C}$ is fat because for each $d \in \mathcal{D}$ we have

$$i_*(c_d) \supseteq \bigcup\{U \in \mathcal{E}\colon\ U \lhd i_*(d)\}.$$

$\square$

Now we can state our generalization of Hindman's Theorem.

4.5. Theorem. *Let $\{d_n\}$ be a sequence of open sets in T such that $\bigcap_{\mathbb{N}} i_*(d_n)$ contains a nonisolated idempotent. Then for any finite open cover $\mathcal{C}$ of T there is some $c \in C$ such that for every $a \rhd c$ there is a sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of c such that*

$$\prod_{1\leqslant j\leqslant k} c_{i_j} \subseteq a \cap d_n$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$. If $\mathcal{C}$ is fat then there is some $c \in \mathcal{C}$ and some sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of c such that

$$\prod_{1\leqslant j\leqslant k} c_{i_j} \subseteq c \cap d_n$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$.

Here is an example to show that the passage from c to a in Theorem 4.5 is necessary.

Example. Take $T = \mathbb{R}$, and cover it with two open sets, c_1 and c_2, as follows. Let $\{r_i\colon\ i \in \mathbb{Z}\} \subseteq \mathbb{R}$ satisfy

$$\begin{gathered} r_i < r_j \quad \text{for } i < j, \\ \lim_{i\to\infty} r_i = \infty, \quad \lim_{i\to-\infty} r_i = -\infty, \\ \lim_{i\to\infty}(r_{i+1} - r_i) = \lim_{i\to-\infty}(r_{i+1} - r_i) = 0. \end{gathered}$$

Set

$$c_1 \equiv \bigcup_{\mathbb{Z}}(r_{4i}, r_{4i+3}), \quad c_2 \equiv \bigcup_{\mathbb{Z}}(r_{4i+2}, r_{4i+5}).$$

Then a little reflection shows that neither c_1 nor c_2 can contain a sequence $\{c_n\}$ of the type guaranteed by Theorem 4.5. But any $a \rhd c_i$ has bounded complement, and so easily contains such a sequence.

The proof of Theorem 4.5 follows the Galvin-Glaser proof of the classical Hindman Theorem [5], [7], [10]. The proof rests on the following lemma.

4.7. Lemma. *Let p be a nonisolated idempotent of $\hat{T}$ such that $p \in \mathrm{cl}(i(c)) \cap i_*(a)$ for open $a, c \subseteq T$. Then there exist nonempty disjoint open sets $a_1, c_1 \subseteq T$ with the following properties.*

(1) $a_1 \cup c_1 \subseteq a$ *and* $c_1 \subseteq c$.

(2) $p \in i_*(a_1)$.

(3) $c_1 a_1 \subseteq a$.

Proof. For fixed p the map $q \longmapsto qp$ is continuous. Since p is an idempotent there exists a neighborhood U of p such that $qp \in i_*(a)$ for all $q \in U$. Without loss of generality $U \subseteq i_*(a)$. Since $i(c) \cap U$ is infinite there is some $t_0 \in c$ such that $p \neq i(t_0) \in U$. Let V and W be disjoint neighborhoods of $i(t_0)$ and p, respectively, with $V \cup W \subseteq U$.

Observe that $t_0 p = i(t_0)p \in i_*(a)$ since $i(t_0) \in U$. Since $\hat{T}$ is a flow there are neighborhoods b of t_0 in T and W' of p in $\hat{T}$ such that $sq \in i_*(a)$ for all $s \in b$ and $q \in W'$. Without loss of generality $W' \subseteq W$. Put

$$a_1 \equiv a \cap i^{-1}(W'), \quad c_1 \equiv b \cap c \cap i^{-1}(V).$$

Clearly $a_1 \subseteq a$ and $c_1 \subseteq c$, and

$$c_1 \subseteq i^{-1}(V) \subseteq i^{-1}(U) \subseteq i^{-1} i_*(a) \subseteq a.$$

And $i_*(a_1) = i_* i^{-1}(W') \cap i_*(a)$ is a neighborhood of p because it contains $W' \cap i_*(a)$. Now consider $s \in c_1$ and $t \in a_1$. Since $s \in b$ and $i(t) \in W'$, it follows that

$$i(st) = i(s)i(t) = si(t) \in i_*(a).$$

Therefore $st \in i^{-1} i_*(a) \subseteq a$, i.e., $c_1 a_1 \subseteq a$. □

Proof of Theorem 4.5. Let $\mathcal{C}$ be a finite open cover of T and let p be a nonisolated idempotent in $\bigcap_{\mathbb{N}} i_*(d_n)$. First observe that there is some $c \in \mathcal{C}$ such that $p \in \mathrm{cl}(i(c))$, and if $\mathcal{C}$ is fat then, since $i_*(\mathcal{C})$ covers $\hat{X}$, we get that $p \in i_*(c)$. To prove that this c satisfies the theorem, fix $a \rhd c$ and label a as a_0; in the fat case take a_0 to be c itself. Then $p \in i_*(a)$ by Lemma 4.3. We now use Lemma 4.7 to find nonempty disjoint open sets a_1 and c_1 in T such that $a_1 \cup c_1 \cup c_1 a_1 \subseteq a_0 \cap d_1$, $c_1 \subseteq c$, and $p \in i_*(a_1)$. Proceeding inductively, we generate sequences $\{a_n\}$ and $\{c_n\}$ of nonempty open subsets of T with the following properties for every $n \in \mathbb{N}$.

(1) $a_n \cap c_n = \emptyset$, $a_n \cup c_n \subseteq a_{n-1} \cap \bigcap_{1 \leqslant j \leqslant n} d_j$, and $c_n \subseteq c$.
(2) $p \in i_*(a_n)$.
(3) $c_n a_n \subseteq a_{n-1} \cap \bigcap_{1 \leqslant j \leqslant n} d_j$.

We claim that for indices $0 \leqslant m < i_1 < i_2 < \ldots < i_k$ we have

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq a_m \cap d_{m+1}.$$

We prove the claim by induction on k. The claim is valid for $k = 1$ because

$$c_{i_1} \subseteq a_{i_1 - 1} \cap \bigcap_{1 \leqslant j \leqslant i_1} d_j \subseteq a_m \cap d_{m+1}.$$

Now assume the claim for $k - 1$ and consider indices $0 \leqslant m < i_1 < i_2 < \ldots < i_k$. Then

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} = c_{i_1} \prod_{2 \leqslant j \leqslant k} c_{i_j} \subseteq c_{i_1} a_{i_1} \subseteq a_{i_1 - 1} \cap \bigcap_{1 \leqslant j \leqslant i_1} d_j \subseteq a_m \cap d_{m+1}.$$

If we replace m by $n - 1$ we get

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq a_{n-1} \cap d_n \subseteq a_0 \cap d_n = a \cap d_n$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$. $\square$

The following corollaries are intended to give an idea of the scope of Theorem 4.5. The authors are sure that these results only scratch the surface.

4.8. Corollary. *Let T be a compactifiable topological monoid with no nonidentity idempotent, e.g., a topological group. Suppose T has a small neighborhood of the identity, and let $\{D_n\}$ be a sequence of compact subsets of T. Then for any finite open cover $\mathcal{C}$ of T there is some $c \in \mathcal{C}$ such that for all $a \rhd c$ there is a sequence $\{c_n\}$ of nonempty pairwise disjoint open subsets of c such that*

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq a \cap \left(T \setminus \bigcup_{1 \leqslant l \leqslant n} D_l \right)$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$. If $\mathcal{C}$ is fat then there is some $c \in \mathcal{C}$ and some sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of c such that

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq c \cap d_n$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$.

P r o o f. The presence of a small neighborhood of the identity implies that T has a nonidentity idempotent p by Corollary 3.5, and p cannot be isolated because it does not lie in T. For each $n \in \mathbb{N}$ let d_n be $T \setminus D_n$; note that $i_*(d_n)$ is $\hat{T} \setminus i(D_n)$ and so contains p. Apply Theorem 4.5. $\square$

4.9. Corollary. *Let T be a compactifiable topological monoid with no nonidentity idempotent, e.g., a topological group. Suppose T has a small neighborhood of the identity, and let $\{t_n\}$ be a sequence of elements of T. Then for any finite open cover $\mathcal{C}$ of T there is some $c \in \mathcal{C}$ such that for all $a \rhd c$ there is a sequence $\{c_n\}$ of nonempty pairwise disjoint open subsets of c such that*

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq a \cap (T \setminus \{t_l\colon\ 1 \leqslant l \leqslant n\})$$

for all indices $n < i_1 < i_2 < \ldots < i_k$. If $\mathcal{C}$ is fat then there is some $c \in \mathcal{C}$ and some sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of c such that

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq c \cap (T \setminus \{t_l\colon\ 1 \leqslant l \leqslant n\})$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$.

4.10. Corollary. *Let T be a locally compact monoid such that $\hat{T} \setminus i(T)$ is a nonempty subsemigroup of $\hat{T}$. Let $\{D_n\}$ be a sequence of compact subsets of T. Then for any finite open cover $\mathcal{C}$ of T there is some $c \in \mathcal{C}$ such that for all $a \rhd c$ there is a sequence $\{c_n\}$ of nonempty pairwise disjoint open subsets of c such that*

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq a \cap \Bigl(T \setminus \bigcup_{1 \leqslant l \leqslant n} D_l\Bigr)$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$. If $\mathcal{C}$ is fat then there is some $c \in \mathcal{C}$ and some sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of c such that

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq c \cap \Bigl(T \setminus \bigcup_{1 \leqslant l \leqslant n} D_l\Bigr)$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$.

P r o o f. Since T is locally compact, every Tychonov flow is compactifiable, including T itself. Since T is open in $\hat{T}$, $\hat{T} \setminus i(T)$ is a nonempty closed subsemigroup of $\hat{T}$ and therefore contains an idempotent p by Namakura's Theorem [12]. Furthermore, p is not isolated since it does not lie in T. Take $d_n \equiv T \setminus D_n$ and apply Theorem 4.5. $\square$

In the discrete case all bounded real-valued functions on T lie in $C^T(T)$, so that $a \rhd c$ if and only if $a \supseteq c$. Thus by taking T to be $\mathbb{N}$ in the following corollary we get Hindman's classical result.

4.11. Corollary. *Let T be a discrete monoid such that $\hat{T} \setminus i(T)$ is a nonempty subsemigroup of $\hat{T}$. Let $\{t_n\}$ be a sequence of elements of T. If $\mathcal{C}$ is any finite partition of T, then there is a $c \in \mathcal{C}$ which contains a nonrepeating sequence $\{s_n\}$ such that*

$$\prod_{1 \leqslant j \leqslant k} s_{i_j} \subseteq c \cap (T \setminus \{t_l\colon\ 1 \leqslant l \leqslant n\})$$

for all indices $n < i_1 < i_2 < \ldots < i_k$.

The point of the next result is that a fat cover of a flow begets a fat cover of T, to which Hindman's Theorem applies.

4.12. Corollary. *Let $\{d_n\}$ be a sequence of open sets in T such that $\bigcap\limits_{\mathbb{N}} i_*(d_n)$ contains a nonisolated idempotent. Let X be a flow with distinguished points $\{x_j\colon 1 \leqslant j \leqslant n\}$. Then for any fat cover $\mathcal{C}$ of X there is some subset $\{c_j\colon\ 1 \leqslant j \leqslant n\} \subseteq \mathcal{C}$ and some sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of T such that*

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq d_n \cap \{t\colon\ tx_j \in c_j,\ 1 \leqslant j \leqslant n\}$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$.

Proof. For any finite open cover $\mathcal{C}$ of X let $\Theta(\mathcal{C})$ designate the set of all maps $\theta\colon\ \{1, 2, \ldots, n\} \to \mathcal{C}$, and for each $\theta \in \Theta(\mathcal{C})$ let

$$d(\theta) \equiv \{t \in T\colon\ tx_j \in \theta(j),\ 1 \leqslant j \leqslant n\}$$

and let $\mathcal{D}(\mathcal{C}) \equiv \{d(\theta)\colon\ \theta \in \Theta(\mathcal{C})\}$. Then $\mathcal{D}(\mathcal{C})$ is evidently a finite open cover of T.

We first claim that $\mathcal{D}(\mathcal{C}') \lhd \mathcal{D}(\mathcal{C})$ whenever $\mathcal{C}' \lhd \mathcal{C}$. To verify this claim consider $\theta' \in \mathcal{D}(\mathcal{C}')$ and use the fact that $\mathcal{C}' \lhd \mathcal{C}$ to find $\theta \in \mathcal{D}(\mathcal{C})$ such that $\theta'(j) \lhd \theta(j)$ for $1 \leqslant j \leqslant n$. For each j find $f_j \in C^T(X)$ such that f_j is 0 on $\theta'(j)$ and 1 off $\theta(j)$, and such that $0 \leqslant f_j \leqslant 1$. For each j define $g_j\colon\ T \to \mathbb{R}$ by the rule

$$g_j(t) \equiv f_j(tx_j).$$

It is clear that g is continuous, that $0 \leqslant g \leqslant 1$, and that g is 0 on $\{t\colon\ tx_j \in \theta'(j)\}$ and 1 off $\{t\colon\ tx_j \in \theta(j)\}$. To show that g is T-uniformly continuous, fix $t_0 \in T$ and

$\varepsilon > 0$. Since f_j is T-uniformly continuous there is some neighborhood a_j of t_0 in T such that

$$|f_j(t_0'x) - f_j(t_0x)| < \varepsilon$$

for all $t_0' \in a_j$. But then

$$|g_j(t_0'r) - g_j(t_0r)| = |f_j(t_0'rx_j) - f_j(t_0rx_j)| < \varepsilon$$

for all $t_0' \in a_j$ and all $r \in T$. This shows that $g_j \in C^T(X)$ for each j. Finally, set

$$g \equiv 1 - \prod_{1 \leqslant j \leqslant n} (1 - g_j).$$

Then g is an element of $C^T(T)$ which is 0 on $d(\theta')$ and 1 off $d(\theta)$. This proves the claim.

Now fix a fat cover $\mathcal{C}$ of X. By Lemma 4.4 we may assume that $\mathcal{C}$ is finite. In conjunction with that lemma, the claim shows that $\mathcal{D}(\mathcal{C})$ is a fat cover of T. Apply Theorem 4.5 to $\mathcal{D}(\mathcal{C})$. □

Now we can resolve the issues raised by Example 4.6. The reason that no sequence satisfying Hindman's Theorem can be found is that the given covering $\{c_1, c_2\}$ is not fat.

4.13. Corollary. *Consider $\mathbb{R}$ to be a flow acting upon itself by left addition. Then for any fat cover $\mathcal{C}$ of $\mathbb{R}$ there is some $c \in \mathcal{C}$ and some sequence $\{c_n\}$ of pairwise disjoint nonempty open subsets of $\mathbb{R}$ such that*

$$\sum_{1 \leqslant j \leqslant k} c_{i_j} \subseteq c \cap (R \setminus [-n, n]).$$

for all indices $1 \leqslant n \leqslant i_1 < i_2 < \ldots < i_k$.

P r o o f. $\mathbb{R}$ has small neighborhoods of the identity, and so $\hat{\mathbb{R}} \setminus \mathbb{R}$ contains an idempotent which is not isolated because it does not lie in $\mathbb{R}$. Furthermore, $i_*(\mathbb{R} \setminus [-n, n])$ is a neighborhood of any such idempotent for any $n \in \mathbb{N}$. To get the result simply apply Corollary 4.12 with base point 0. □

Let X be a Tychonoff space and let T be a monoid of continuous functions from X into X. Let $i\colon X \to Y$ be a compactification of X such that the action of T on X extends to an action on Y. For each $g \in C(Y)$ and $\varepsilon > 0$ define

$$U(g, \varepsilon) \equiv \{(t_1, t_2)\colon \forall x \in X \ \ (|g(t_1x) - g(t_2x)| < \varepsilon)\}.$$

These neighborhoods of the diagonal in T generate a uniformity, whose topology we term the *Y-topology.* With respect to the Y-topology, one easily sees that T is a Hausdorff topological monoid and i is a flow map. Thus Y is a flow compactification of X, which is therefore certainly compactifiable. In fact, the Y-topology is the coarsest with respect to which the functions of $C(Y)$ are T-uniformly continuous. Consequently, for open sets $a, c \subseteq X$ we have $c \lhd a$ whenever there is some $g \in C(Y)$ such that g is 0 on c and 1 off a. In this case we shall say that *a is Y-above c.*

The above considerations always apply when i is taken to be the embedding of X in its Stone-Čech compactification, for the action of T on X certainly extends to an action on βX. In this case $C^T(X) = C(Y) = C^*(X)$, so that for open sets $a, c \subseteq X$, $c \lhd a$ if and only if c and $X \setminus a$ are completely separated.

4.14. Corollary. *Let X and Y be as above, and assume that T has a small Y-neighboorhood of the identity. Then for every fat cover C of T by Y-open sets there is some $c \in C$, some Y-neighborhood d of 1, and some sequence $\{c_n\}$ of pairwise disjoint nonempty Y-open subsets of c such that*

$$\prod_{1 \leqslant j \leqslant k} c_{i_j} \subseteq c \setminus d.$$

5. Small projective flows

The considerations in this section take place in the category of compact flows, and we assume that context for the rest of this paper. A flow is *elementary* if it has a point with a dense orbit, and *minimal* if every point has a dense orbit. A flow X is *projective* if for every flow map $f\colon X \to Y$ and every flow surjection $h\colon Z \to Y$ there is some flow map $g\colon X \to Z$ such that $hg = f$.

Our construction of $\hat{T}$ allows us to sharpen slightly some results from [1]. The reader should exercise caution, however, when consulting that article, for it assumes that the topology on T is based at 1, an assumption not in force here. And the definition of an idempotent point there differs from its definition here. (In fact, Proposition 5.1 shows that the two definitions agree.) The basic result is Proposition 5.1, which sharpens [1], 12.2.

5.1. Proposition. *The following are equivalent for a compact flow X.*

(1) *X is a flow retract of $\hat{T}$.*
(2) *X is elementary and projective.*
(3) *X is flow homeomorphic to $\hat{T}p$ for some idempotent $p \in \hat{T}$.*

P r o o f. The implication from (1) to (2) follows from general considerations. $\hat{T}$ is projective because it is the free compact flow over the one-point space [1], 11.2. Here is a sketch of the argument that $\hat{T}$ is projective. Let Z and Y be compact flows, $f\colon \hat{T} \to Y$ a flow map and $h\colon Z \to Y$ a flow surjection. Let $F = f \circ i\colon T \to Y$ and $y_0 = F(1)$. Since h is a surjection, we may pick $z_0 \in Z$ so that $h(z_0) = y_0$ and define $G\colon T \to Z$ by $G(t) = tz_0$. It is routine to verify that both F and G are flow maps and $hG = F$. Now by Corollary 2.3, G extends to a flow map $g\colon \hat{T} \to Z$ and because $h \circ g \circ i = f \circ i$, it follows that $hg = f$. Now any retract of $\hat{T}$ is also projective [8], 12.14, and is, of course, elementary.

The implication from (3) to (1) is likewise easy. Let $j\colon \hat{T}p \to \hat{T}$ be the flow injection which takes each member of the $\hat{T}p$ to itself, let $\tilde{p}\colon T \to \hat{T}p$ designate flow map given by $\tilde{p}(t) = tp$, and let $\hat{p}\colon \hat{T} \to \hat{T}p$ be the flow surjection which is the continuation of $\tilde{p}$ given by Corollary 2.3. Then any member of $\hat{T}p$ is of the form qp for some $q \in \hat{T}$, for which we have

$$\hat{p}j(qp) = \hat{p}(qp) = (qp)p = qp^2 = qp.$$

That is to say that $\hat{p}j$ is the identity on $\hat{T}p$, meaning that $\hat{T}p$ is a retract of $\hat{T}$.

Assume (2) to prove (3). Since X is elementary there is some point $x \in X$ with a dense orbit. Let $\tilde{x}\colon T \to X$ designate the flow map $\tilde{x}(t) = tx$ and let $\hat{x}$ be the continuation of $\tilde{x}$ to $\hat{T}$ (Corollary 2.3 again). Since X is projective and $\hat{x}\colon \hat{T} \to X$ is a flow surjection, there is a flow injection $j\colon X \to \hat{T}$ such that $\hat{x}j$ is the identity on X. Set $p \equiv j(x)$. Then

$$px = \hat{x}(p) = \hat{x}(j(x)) = x.$$

Consequently by Proposition 2.7,

$$p = j(x) = j(px) = pj(x) = p^2,$$

so p is an idempotent. It remains to show that $j(X) = \hat{T}p$. But $j(X) \supseteq \hat{T}p$ follows from the fact that $j(qx) = qj(x) = qp$ for any $q \in \hat{T}$. And since $\hat{T}x \equiv \{qx\colon\ q \in \hat{T}\}$ is the image of the compact set $\hat{T}$ under the continuous function $\hat{x}$, it is closed. Because $\hat{T}x$ contains the dense orbit of x, we have $\hat{T}x = X$. From this it follows easily that $j(X) \subseteq \hat{T}p$. $\square$

We remark that it is quite possible for $\hat{T}p$ to be projective without p being idempotent; a finite example can be found in [1].

5.2. Corollary. *A compact flow is minimal and projective if and only if it is flow homeomorphic to $\hat{T}p$ for some minimal idempotent p.*

P r o o f. A minimal flow X is certainly elementary, so that if it is also projective then it is flow homeomorphic to $\hat{T}p$ for some idempotent p by Proposition 5.1. But the minimality of X as a flow also implies that $\hat{T}p$ is minimal among left ideals, hence p is minimal by Theorem 3.12. On the other hand, the same theorem implies that if p is minimal then the left ideal $\hat{T}p$ it generates is minimal among left ideals, hence $\hat{T}q = \hat{T}p$ for all $q \in \hat{T}p$, meaning every point has a dense orbit, i.e., $\hat{T}p$ is a minimal flow. □

With Proposition 5.1 and Corollary 5.2 in hand, one may improve some results in [1] by replacing T by $\hat{T}$, thus removing the hypothesis that T is compact. This strengthens conditions 4 and 5 of Theorem 12.5, along with conditions 5 and 6 of Theorem 12.10. And all the results of Section 13, save the last one, benefit from this modification.

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