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EQUIVALENCE BIMODULE BETWEEN NON-COMMUTATIVE TORI

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Abstract. The non-commutative torus $C^*(\mathbb{Z}^n, \omega)$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres isomorphic to $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ for a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$. D. Poguntke [9] proved that A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_{kl}(\mathbb{C})$ for a simple non-commutative torus A_φ and an integer kl. It is well-known that a stable isomorphism of two separable C^*-algebras is equivalent to the existence of equivalence bimodule between them. We construct an A_ω-$C(\widehat{S_\omega}) \otimes A_\varphi$-equivalence bimodule.

Keywords: Morita equivalent, twisted group C^*-algebra, crossed product

MSC 2000: 46L05, 46L87, 55R15

1. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate with them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. The twisted group C^*-algebra $C^*(\mathbb{Z}^n, \omega)$ is called a *non-commutative torus of rank n* and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group.* A multiplier ω on an abelian group G is called *totally skew* if the symmetry group S_ω is trivial. A non-commutative torus A_ω is said to be a *completely irrational non-commutative torus* if ω is totally skew (see [1], [7], [8]). Baggett and Kleppner [1] showed that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra.

It was shown in [1], [7] that even when ω is not totally skew on a locally compact abelian group G, the restriction of ω-representations from G to S_ω induces a canonical

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homeomorphism of $\mathrm{Prim}(C^*(G,\omega))$ with $\widehat{S_\omega}$, where $\mathrm{Prim}(C^*(G,\omega))$ is the primitive ideal space of the twisted group C^*-algebra $C^*(G,\omega)$, and that there is a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$ such that ω is similar to the pull-back of ω_1. Furthermore, it is known (see [1], [7], [9]) that $C^*(G,\omega)$ may be realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $\widehat{S_\omega} = \mathrm{Prim}(C^*(G,\omega))$ with fibres $C^*(G,\omega)/x$ for $x \in \mathrm{Prim}(C^*(G,\omega))$ and all $C^*(G,\omega)/x$ turn out to form the simple twisted group C^*-algebra $C^*(G/S_\omega,\omega_1)$. So $A_\omega \cong C^*(\mathbb{Z}^n,\omega)$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$.

D. Poguntke proved in [8] that any primitive quotient of the group C^*-algebra $C^*(G)$ of a locally compact two step nilpotent group G is isomorphic to the tensor product of a completely irrational non-commutative torus A_φ with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable (possibly finite-dimensional) Hilbert space $\mathcal{H}$. Since $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ is the primitive quotient of $C^*(\mathbb{Z}^n/S_\omega(\omega_1))$, where $\mathbb{Z}^n/S_\omega(\omega_1)$ is the extension group of $\mathbb{Z}^n/S_\omega$ by $\mathbb{T}$ defined by ω_1, $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ is isomorphic to $A_\varphi \otimes M_{kl}(\mathbb{C})$ for an integer kl.

It was shown in [9] that A_ω is stably isomorphic to $C(\widehat{S_\omega})\otimes C^*(\mathbb{Z}^n/S_\omega,\omega_1)$. In [3], the authors showed that two separable C^*-algebras A and B are stably isomorphic if and only if they are strongly Morita equivalent, i.e., there exists an A-B-equivalence bimodule defined in [10]. Thus the non-commutative torus A_ω is strongly Morita equivalent to $C(\widehat{S_\omega})\otimes C^*(\mathbb{Z}^n/S_\omega,\omega_1)$, which in turn is strongly Morita equivalent to $C(\widehat{S_\omega})\otimes A_\varphi$. This implies that there exists an A_ω-$C(\widehat{S_\omega})\otimes A_\varphi$-equivalence bimodule.

M. Brabanter [2] constructed an $A_{m/k}$-$C(\mathbb{T}^2)$-equivalence bimodule. Modifying his construction, we are going to construct an A_ω-$C(\widehat{S_\omega})\otimes A_\varphi$-equivalence bimodule.

2. Equivalence bimodule between non-commutative tori

The following result of Poguntke clarifies the structure of the fibres of the canonical bundle associated with a non-commutative torus A_ω.

1. Theorem [8, Theorem 1]**.** *Let G be a compactly generated locally compact abelian group and ω_1 a totally skew multiplier on G. Let K be the maximal compact subgroup of E and E_ϱ the stabilizer of an irreducible unitary representation ϱ of K restricting on $\mathbb{T}^1$ to the identity. Then*

$$C^*(G,\omega_1) \cong C^*(E_\varrho/K,m)\otimes \mathcal{K}(\mathcal{L}^2(E/E_\varrho))\otimes M_{\dim(\varrho)}(\mathbb{C}),$$

where m is the associated Mackey obstruction.

This theorem is applied to understand the structure of $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$. The noncommutative torus A_ω is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres isomorphic to the simple twisted group C^*-algebra $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ of a finitely generated discrete abelian group $\mathbb{Z}^n/S_\omega$ defined by a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$, where ω is similar to the pull-back of ω_1. Then $\mathbb{Z}^n/S_\omega \cong F \oplus T$, where F is a maximal torsion-free subgroup of $\mathbb{Z}^n/S_\omega$ and T is the maximal torsion subgroup of $\mathbb{Z}^n/S_\omega$. Let $G = \mathbb{Z}^n/S_\omega$, $E = (\mathbb{Z}^n/S_\omega)(\omega_1)$, and let E_ϱ be the stabilizer of an irreducible unitary representation ϱ of the extension $K := T(\omega_1|_T)$, which restricts to the identity on $\mathbb{T}^1$. Here we denote by $\omega_1|_T$ the restriction of ω_1 to T. The Mackey method says that $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F \oplus T, \omega_1)$ is isomorphic to the primitive quotient of $C^*(E)$ lying over ϱ. Then by Theorem 1,

$$C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(E_\varrho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\varrho)) \otimes M_{\dim(\varrho)}(\mathbb{C}).$$

Now by definition, E_ϱ is of index $|S_{\omega_1|_T}|$ in E. So

$$[E : E_\varrho] = \# \text{ of irreducible } \omega_1|_T\text{-representations of } T = |S_{\omega_1|_T}|$$

and $\dim(\varrho) = \sqrt{|T|/|S_{\omega_1|_T}|}$, and E_ϱ/K is a subgroup of a finite index $[E : E_\varrho]$ in E/K. Let F_ϱ be the isomorphic image of E_ϱ/K under the natural map of E/K to F. Then $\{x \in F \mid h_{\omega_1}(x)(y) = 1, \ \forall y \in S_{\omega_1|_T}\}$ is exactly F_ϱ, and F_ϱ is a subgroup of a finite index $[E : E_\varrho]$ in F. Let $J_F = F/F_\varrho, J = J_F \oplus S_{\omega_1|_T}$ and $T_t = T/S_{\omega_1|_T}$. Then $|J_F| = |S_{\omega_1|_T}|$. Since F_ϱ is a subgroup of F, we can consider $J_F \oplus S_{\omega_1|_T}$ as a subgroup of $(F \oplus T)/F_\varrho$. So $(\mathbb{Z}^n/S_\omega)/F_\varrho$ is isomorphic to $J_F \oplus T$ and $((\mathbb{Z}^n/S_\omega)/F_\varrho)/J$ is isomorphic to T_t.

Next, we show that $C^*(E_\varrho/K, m)$ is isomorphic to $C^*(F_\varrho, \omega_1|_{F_\varrho})$. By Theorem 1, $C^*(F_\varrho, \omega_1|_{F_\varrho})$ is isomorphic to $C^*(F_\varrho(\omega_1|_{F_\varrho})/\mathbb{T}^1, m_1)$, where m_1 is the associated Mackey obstruction. Let ω_2 be a totally skew multiplier on T_t whose pull-back to T is similar to $\omega_1|_T$. It is enough to show that the Mackey obstruction m_2, in the isomorphism

$$\begin{aligned} C^*(F_\varrho \oplus T_t, \omega_1|_{F_\varrho} \oplus \omega_2) &\cong C^*((F_\varrho \oplus T_t)(\omega_1|_{F_\varrho} \oplus \omega_2)/T_t(\omega_2), m_2) \otimes C^*(T_t, \omega_2) \\ &\cong C^*(F_\varrho, \omega_1|_{F_\varrho}) \otimes C^*(T_t, \omega_2), \end{aligned}$$

is essentially the same as m_1. However, for $h \in F_\varrho$, the unitary operators E'_h given in [5, XII.1.17] are the same for F_ϱ and for $F_\varrho \oplus T_t$ up to a scalar. They give the same Mackey obstructions. So

$$\begin{aligned} C^*((F_\varrho \oplus T_t)(\omega_1|_{F_\varrho} \oplus \omega_2)/T_t(\omega_2), m_2) &\cong C^*(F_\varrho(\omega_1|_{F_\varrho})/\mathbb{T}^1, m_1) \\ &\cong C^*(F_\varrho, \omega_1|_{F_\varrho}), \end{aligned}$$

and $C^*(E_\varrho/K, m)$ is isomorphic to $C^*(F_\varrho, \omega_1|_{F_\varrho})$. See [5, Section XII] for details.

2. Corollary. $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F_\varrho, \omega_1|_{F_\varrho}) \otimes M_{[E:E_\varrho]}(\mathbb{C}) \otimes M_{\dim(\varrho)}(\mathbb{C})$.

P r o o f. By Theorem 1,

$$\begin{aligned} C^*(\mathbb{Z}^n/S_\omega, \omega_1) &\cong C^*(E_\varrho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\varrho)) \otimes M_{\dim(\varrho)}(\mathbb{C}) \\ &\cong C^*(F_\varrho, \omega_1|_{F_\varrho}) \otimes M_{[E:E_\varrho]}(\mathbb{C}) \otimes M_{\dim(\varrho)}(\mathbb{C}). \end{aligned}$$

Here $M_{[E:E_\varrho]}(\mathbb{C}) \cong M_{|J_F|}(\mathbb{C})$ and $M_{\dim(\varrho)}(\mathbb{C}) \cong M_{\sqrt{|T_t|}}(\mathbb{C})$.

Therefore, $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F_\varrho, \omega_1|_{F_\varrho}) \otimes M_{[E:E_\varrho]}(\mathbb{C}) \otimes M_{\dim(\varrho)}(\mathbb{C})$. □

Note that $C^*(F_\varrho, \omega_1|_{F_\varrho})$ is a completely irrational non-commutative torus. So A_ω is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $A_\varphi \otimes M_{kl}(\mathbb{C})$, where $A_\varphi \cong C^*(F_\varrho, \omega_1|_{F_\varrho})$ and $M_{kl}(\mathbb{C}) \cong M_{[E:E_\varrho]}(\mathbb{C}) \otimes M_{\dim(\varrho)}(\mathbb{C})$.

M. Brabanter [2, Proposition 1] showed that the rational rotation algebra $A_{m/k}$ is isomorphic to the C^*-algebra of matrices $(f_{ij})_{i,j=1}^k$ of functions f_{ij} with

$$\begin{aligned} &f_{ij} \in C^*(k\mathbb{Z} \times k\mathbb{Z}) && \text{if } i,j \in \{1,2,\ldots,k-1\} \quad \text{or } (i,j) = (k,k), \\ &f_{ik} \in \Omega && \text{if } i \in \{1,2,\ldots,k-1\}, \\ &f_{ki} \in \Omega^* && \text{if } i \in \{1,2,\ldots,k-1\}, \end{aligned}$$

where Ω and Ω^* are the $C^*(k\mathbb{Z} \times k\mathbb{Z})$-modules defined as

$$\begin{aligned} \Omega &= \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f(z,1) = z^s f(z,0), \ \forall z \in \widehat{k\mathbb{Z}}\}, \\ \Omega^* &= \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f^* \in \Omega\} \end{aligned}$$

for an integer s such that $sm = 1 \pmod{k}$.

The non-commutative torus A_ω of rank n is obtained by an iteration of $n-1$ crossed products by actions of $\mathbb{Z}$, the first action on $C(\mathbb{T}^1)$ (see [6]). When A_ω has a primitive ideal space $\widehat{S_\omega} \cong \mathbb{T}^1$ and fibres $A_\varphi \otimes M_k(\mathbb{C})$, then by a change of basis, A_ω can be obtained by an iteration of $n-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{m/k}$, where the actions of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ are trivial, since $M_k(\mathbb{C})$ is factored out of the fibre $A_\varphi \otimes M_k(\mathbb{C})$ of A_ω. When A_ω has a primitive ideal space $\widehat{S_\omega} \cong \mathbb{T}^3$ with fibres $M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$, then by a change of basis, A_ω can be obtained by a crossed product by an action of $\mathbb{Z}$ on a rational rotation algebra $A_{m/k}$, where the action of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ is trivial, since the existence of the above crossed product representation for A_ω implies the existence of such an action, and the crossed product by the action of $\mathbb{Z}$ on $A_{m/k}$ is a kl-homogeneous C^*-algebra over $\mathbb{T}^3$, and so the crossed product is isomorphic to A_ω by the Disney and Raeburn result [4, Proposition 3.10]. Combining the previous two comments yields that when A_ω is not simple, then by a change of basis, A_ω can be obtained by an iteration of $n-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{m/k}$, where the actions of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ are trivial.

3. Theorem. *A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\varphi$.*

Proof. Let A_ω be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $A_\varphi \otimes M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$. Then A_ω may be realized as the crossed product $A_{m/k} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ are trivial. So A_ω has a matrix representation induced from the matrix representation of the rational rotation subalgebra $A_{m/k}$, i.e., $A_{m/k}$ has a $C^*(k\mathbb{Z} \times k\mathbb{Z})$-module structure and A_ω must be given by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ with $A_{r(\omega)} := C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$. Thus A_ω is isomorphic to the C^*-algebra of matrices $(g_{ij})_{i,j=1}^k$ of g_{ij} with

$$\begin{aligned} g_{ij} &\in A_{r(\omega)} && \text{if } i,j \in \{1,2,\ldots,k-1\} \text{ or } (i,j)=(k,k), \\ g_{ik} &\in \widetilde{\Omega} && \text{if } i \in \{1,2,\ldots,k-1\}, \\ g_{kj} &\in \widetilde{\Omega}^* && \text{if } j \in \{1,2,\ldots,k-1\}, \end{aligned}$$

where $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ are $A_{r(\omega)}$-modules defined as

$$\widetilde{\Omega} = A_{r(\omega)} \cdot \Omega \quad \& \quad \widetilde{\Omega}^* = A_{r(\omega)} \cdot \Omega^*,$$

where Ω and Ω^* are given above.

Let X be the complex vector space $\left(\oplus_1^{k-1}\widetilde{\Omega}\right) \oplus A_{r(\omega)}$. We will consider the elements of X as $(k,1)$ matrices where the first $(k-1)$ entries are in $\widetilde{\Omega}$ and the last entry is in $A_{r(\omega)}$. If $x \in X$, denote by x^* the $(1,k)$ matrix resulting from x by transposition and involution so that $x^* \in \left(\oplus_1^{k-1}\widetilde{\Omega}^*\right) \oplus A_{r(\omega)}$. The space X is a left A_ω-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = (g_{ij})_{i,j=1}^k \in A_\omega$ and $x \in X$. If $g \in A_{r(\omega)}$ and $x \in X$, then $x \cdot [g]$ defines a right $A_{r(\omega)}$-module structure on X. Now we define an A_ω-valued and an $A_{r(\omega)}$-valued inner products $\langle \cdot,\cdot \rangle_{A_\omega}$ and $\langle \cdot,\cdot \rangle_{A_{r(\omega)}}$ on X by

$$\langle x,y \rangle_{A_\omega} = x \cdot y^* \quad \& \quad \langle x,y \rangle_{A_{r(\omega)}} = x^* \cdot y$$

if $x, y \in X$ and we have matrix multiplication on the right. Equipped with this structure, by the same reasoning as in the proof given in [2, Theorem 3], X becomes an A_ω-$A_{r(\omega)}$-equivalence bimodule. So A_ω is strongly Morita equivalent to $A_{r(\omega)}$, which is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $A_\varphi \otimes M_l(\mathbb{C})$. One can proceed in this way finitely many times to obtain that A_ω is strongly Morita equivalent to $C^*(S_\omega \times P, \omega|_{S_\omega \times P}) \cong C^*(S_\omega) \otimes C^*(P, \omega|_P)$, where P is a torsion-free subgroup of $\mathbb{Z}^n$, which is isomorphic to F_ϱ, $\omega|_{S_\omega \times P}$ which is similar to the pull-back of $\omega_1|_{F_\varrho}$, and $C^*(P, \omega|_P) \cong C^*(F_\varrho, \omega_1|_{F_\varrho}) \cong A_\varphi$.

Therefore, A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\varphi$. □

We have obtained that A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\varphi$, which is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\varphi \otimes M_{kl}(\mathbb{C}) \cong C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$. So A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$.

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