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A CHARACTERIZATION OF 1-, 2-, 3-, 4-HOMOMORPHISMS OF ORDERED SETS

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Abstract. We characterize totally ordered sets within the class of all ordered sets containing at least four-element chains. We use a simple relationship between their isotone transformations and the so called 1-endomorphism which is introduced in the paper. Later we describe 1-, 2-, 3-, 4-homomorphisms of ordered sets in the language of super strong mappings.

Keywords: ordered sets, morphisms

MSC 2000: 06A10, 06A99

0. Introduction

In [4] new concepts of 2-, 3-, 4-endomorphisms of ordered sets were introduced. They appeared to be an efficient tool for the determination of chains in the class of all ordered sets satisfying a certain condition (the existence of a three-element chain). In this contribution we introduce a 1-endomorphism and demonstrate its conjunction with the above mentioned results. We declare that the requirement of a four-element chain is essential.

Let $(P, \leqslant)$ be an ordered set, $\emptyset \neq X \subseteq P$. The symbol $E_f(X)$ denotes $f^-(f(X))$ where $f^-(X)$ is the preimage of X under a mapping f, i.e. $f^-(X) = \{y \mid f(y) = x$ for some $x \in X\}$. By $[X)_\leqslant = \{y \in P\colon\ y \geqslant x$ for some $x \in X\}$ we denote the upper end of an ordered set $(P, \leqslant)$ generated by a subset X. Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and let $f\colon\ P \longrightarrow Q$ be a mapping. The mapping f is isotone if for any pair of elements $a, b \in P$ such that $a \leqslant b$ we have $f(a) \leqslant f(b)$. The mapping f is a *strong homomorphism* if $f(z) \geqslant f(x)$ implies $f(z) = f(u)$, $f(x) = f(a)$ for some

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$a, u \in P$ such that $u \geqslant a$. An isotone mapping of an ordered set into itself is called an *endomorphism*. The set of all endomorphisms of $(P, \leqslant)$ endowed with a composition forms a monoid which is denoted by $\mathrm{End}(H, \leqslant)$.

Remark. There exists also another concept of a strong homomorphism. A mapping $f\colon P \longrightarrow Q$ between ordered sets $(P, \leqslant)$, $(Q, \leqslant)$ is called a strong homomorphism if for any pair of elements $x \in P, y \in Q$ we have $f(x) \leqslant y$ if and only if there exists an element $x' \in P$ such that $x \leqslant x'$ and $f(x') = y$ (*L. L. Esakia*: Heyting algebras I. Duality theory. Mecniereba, 1985, Tbilisi).

Definition 1 ([4]). Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets. A mapping $f\colon P \to Q$ is called

(1) a 1-*homomorphism* if it satisfies the condition

$$f^{-}([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant}) \quad \text{for any } x \in P,$$

(2) a 2-*homomorphism* if it satisfies the condition

$$f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant})) \quad \text{for any } x \in P,$$

(3) a 3-*homomorphism* if it satisfies the condition

$$f^{-}([f(x))_{\leqslant}) = [f^{-}(f(x)))_{\leqslant} \quad \text{for any } x \in P,$$

(4) a 4-*homomorphism* if both the conditions for 2- and 3-homomorphisms are satisfied

$$[f^{-}(f(x)))_{\leqslant} = f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant})) \quad \text{for any } x \in P.$$

1. 1-endomorphisms

Proposition 1. *Let $(X, \leqslant)$ be an ordered set containing at least a four-element chain. Then for any ordered pair (x, y) of $\leqslant$-incomparable elements $x, y \in X$ there exists an isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ such that*

$$f(x) < f(y) \quad \text{and} \quad \{x\} = E_f(x),\ \{y\} = E_f(y).$$

Proof. Suppose $(X, \leqslant)$ contains at least a four-element chain C. Consider $C_0 \subseteq C$ such that $C_0 = \{a, b, c, d\}$, $a < b < c < d$, and $x, y \in X$ are incomparable elements. Now let X^{xy}, X_{xy} be subsets of X such that

$$X^{xy} = \{z\colon\ z > x \ \text{ or } \ z > y\} = [\{x, y\})_{\leqslant} \setminus \{x, y\},$$
$$X_{xy} = \{z\colon\ z < x \ \text{ or } \ z < y\} = (\{x, y\}]_{\leqslant} \setminus \{x, y\},$$
$$Y = X \setminus (X^{xy} \cup X_{xy} \cup \{x, y\}).$$

Let $f(x) = b$ and $f(y) = c$, which means $f(x) < f(y)$. Furthermore let $f(t) = a$ for any $t \in X_{xy}$, $f(s) = d$ for any $s \in X^{xy}$ and $f(r) = d$ for any $r \in Y$ (cf. Fig. 1). Now $f(u) = f(v)$ for any pair $(u, v) \in X^{xy} \times X^{xy}$, $(u, v) \in X_{xy} \times X_{xy}$, $(u, v) \in Y \times Y$, and $f(u) < f(v)$ for any pair $(u, v) \in X_{xy} \times X^{xy}$, which implies f is isotone, because $p \leqslant q$ implies $f(p) \leqslant f(q)$ for any $p, q \in X$ and $\{x\} = f^{-}(b) = E_f(x)$, $\{y\} = f^{-}(c) = E_f(y)$. Thus the proposition holds. $\square$

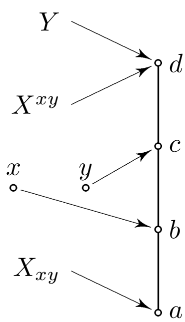

Figure 1

Lemma 1. *Let $f\colon X_1 \longrightarrow X_2$ be a mapping of an ordered set $(X_1, \leqslant)$ into another one $(X_2, \leqslant)$. The following conditions are equivalent:*

(1) *f is isotone,*

(2) *$E_f([E_f(x))_{\leqslant}) \subseteq f^{-}([f(x))_{\leqslant})$ for any $x \in X_1$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X_1$ be an arbitrary element and in addition suppose $z \in E_f([E_f(x))_{\leqslant})$, which means $f(z) \in f([E_f(x))_{\leqslant})$. Then there exists $q \in [E_f(x))_{\leqslant}$ such that $f(z) = f(q)$. It follows that there exists $r \in E_f(x)$, i.e. $f(r) = f(x)$ such that $r \leqslant q$. Since $f(r) \leqslant f(q)$ we have $f(x) \leqslant f(z)$, which implies $f(z) \in [f(x))_{\leqslant}$ and consequently $z \in f^{-}([f(x))_{\leqslant})$. We have $E_f([E_f(x))_{\leqslant}) \subseteq f^{-}([f(x))_{\leqslant})$.

(2) $\Rightarrow$ (1): Let x, y be elements from X_1 such that $x \leqslant y$. Since $x \in E_f(x)$ we have $y \in [E_f(x))_{\leqslant}$ and further $y \in E_f([E_f(x))_{\leqslant})$. By the assumption $y \in f^{-}([f(x))_{\leqslant})$, which implies $f(y) \in [f(x))_{\leqslant}$ and thus $f(x) \leqslant f(y)$. Finally, the mapping f is isotone. $\square$

Proposition 2. *Let $(X, \leqslant)$ be an ordered set containing at least a four-element chain. Then $(X, \leqslant)$ is a chain if and only if any isotone selfmap f of the poset $(X, \leqslant)$ satisfies the following condition:*

$$(*) \qquad E_f([E_f(x))_{\leqslant}) = f^{-}([f(x))_{\leqslant}) \quad \textit{for any } x \in X.$$

P r o o f. $\Rightarrow$: Let $(X, \leqslant)$ be a chain and $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ an isotone mapping. Let $x \in X$ be an arbitrary element and suppose $z \in f^{-}([f(x))_{\leqslant})$, which means $f(z) \in [f(x))_{\leqslant}$, i.e. $f(x) \leqslant f(z)$. If $f(x) = f(z)$ then $z \in E_f(x)$ and as

$$E_f(x) \subseteq [E_f(x))_{\leqslant} \subseteq E_f([E_f(x))_{\leqslant}),$$

we have $z \in E_f([E_f(x))_{\leqslant})$. If $f(x) < f(z)$ then $x \leqslant z$ (since the mapping f is isotone and $(X, \leqslant)$ is a chain). Further, from $[E_f(x))_{\leqslant} = \{t\colon \exists u \in X\colon f(u) = f(x), u \leqslant t\}$ we obtain $z \in [E_f(x))_{\leqslant}$, which implies $f(z) \in f([E_f(x))_{\leqslant})$ and consequently $z \in E_f([E_f(x))_{\leqslant})$. We have $f^{-}([f(x))_{\leqslant}) \subseteq E_f([E_f(x))_{\leqslant})$. Since $E_f([E_f(x))_{\leqslant}) \subseteq f^{-}([f(x))_{\leqslant})$ (Lemma 1) we have finally $f^{-}([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant})$.

$\Leftarrow$: Let $(X, \leqslant)$ be a poset containing at least a four-element chain, let $x, y \in X$ be incomparable ($x \parallel y$) and suppose $f^{-}([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant})$ for any isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$. Let f_0 be a mapping from Proposition 1, i.e. $f_0(x) < f_0(y)$ and $\{x\} = E_{f_0}(x)$, $\{y\} = E_{f_0}(y)$. Since $f_0(x) < f_0(y)$, then $f_0(y) \in [f_0(x))_{\leqslant}$, which implies $y \in f_0^{-}([f_0(x))_{\leqslant})$. Now $y \in E_{f_0}([E_{f_0}(x))_{\leqslant})$ by the assumption $(*)$. We get $y \in E_{f_0}([\{x\})_{\leqslant})$, which implies $f_0(y) \in f_0([\{x\})_{\leqslant})$. Then there exists $z \in [\{x\})_{\leqslant}$ such that $f_0(z) = f_0(y)$. We get

$$z \in E_{f_0}(z) = E_{f_0}(y) = \{y\},$$

which implies $z = y$ and therefore $y \in [\{x\})_{\leqslant}$, which means $x \leqslant y$. This is a contradiction to the assumption of incomparability of x and y. Thus $(X, \leqslant)$ is a chain. □

Remark. It can be easily proved that the condition $(*)$ can be replaced by the dual one:

$$f^{-}((f(x)]_{\leqslant}) = E_f((E_f(x)]_{\leqslant}) \quad \text{for any } x \in X.$$

In the proof it is useful again to consider such an isotone mapping that $f(x) < f(y)$ and $\{x\} = E_f(x)$, $\{y\} = E_f(y)$ whose existence was stated in Proposition 1.

Theorem 1. *Let $(X, \leqslant)$ be an ordered set containing at least a four-element chain. Then the following conditions are equivalent:*

(1) $(X, \leqslant)$ *is a totally ordered set,*
(2) $\mathrm{End}(X, \leqslant) \subseteq 1\text{-}\mathrm{End}(X, \leqslant)$,
(3) $\mathrm{End}(X, \leqslant) = 1\text{-}\mathrm{End}(X, \leqslant)$.

Proof. (1) $\Rightarrow$ (2): It follows from Proposition 2.

(2) $\Rightarrow$ (3): Let $f \in 1\text{-}\mathrm{End}(X, \leqslant)$ be an arbitrary mapping and suppose $x, y \in X$, $x \leqslant y$ are arbitrary elements. Since $x \leqslant y$ and $x \in E_f(x)$ hence $y \in [E_f(x))_{\leqslant}$ and further $y \in E_f([E_f(x))_{\leqslant})$. Now $y \in f^-([f(x))_{\leqslant})$ by the assumption of 1-endomorphism. This implies $f(y) \in [f(x))_{\leqslant}$ and we get $f(x) \leqslant f(y)$, thus the mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ is isotone. Finally, $\mathrm{End}(X, \leqslant) \supseteq 1\text{-}\mathrm{End}(X, \leqslant)$, which implies $\mathrm{End}(X, \leqslant) = 1\text{-}\mathrm{End}(X, \leqslant)$.

(3) $\Rightarrow$ (1): It follows from Proposition 2. $\square$

Proposition 3. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and $f\colon P \longrightarrow Q$ a mapping. Then the following conditions are equivalent:*
(1) *f is a 1-homomorphism,*
(2) a) *f is isotone,*
b) *for any $z, x \in P$ the inequality $f(z) \geqslant f(x)$ implies $f(z) = f(u)$, $f(x) = f(a)$ for some $a, u \in P$ such that $u \geqslant a$,*
i.e. f is an isotone strong homomorphism.

Proof. (1) $\Rightarrow$ (2): b) Suppose (1) is satisfied and $f(z) \geqslant f(x)$ for some $x, z \in P$. We have $f(z) \in [f(x))_{\leqslant}$ thus $z \in f^-([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant})$. Now $f(z) \in f([E_f(x))_{\leqslant})$, which means that there exists $u \in P$ such that $f(z) = f(u)$ and $u \in [E_f(x))_{\leqslant}$, therefore there exists $a \in P$ such that $a \leqslant u$ and $a \in f^-(f(x))$, i.e. $f(a) = f(x)$. The condition a) follows from Lemma 1.

(2) $\Rightarrow$ (1): Assume (2) and $z \in f^-([f(x))_{\leqslant})$, i.e. $f(z) \in [f(x))_{\leqslant}$, which is $f(z) \geqslant f(x)$. By (2) we have $f(z) = f(u)$, $f(a) = f(x)$ for some $a, u \in P$ such that $u \geqslant a$, which means $f(u) \geqslant f(a)$. Consequently $u \in [E_f(a))_{\leqslant} = [E_f(x))_{\leqslant}$ and $f(z) = f(u) \in f([E_f(x))_{\leqslant}$, i.e. $z \in E_f([E_f(x))_{\leqslant})$. The converse inclusion $E_f([E_f(x))_{\leqslant}) \subseteq f^-([f(x))_{\leqslant})$ follows from (2) a) by Lemma 1. $\square$

2. Super-strong mappings

Proposition 4. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and $f\colon P \longrightarrow Q$ a mapping. Then the following conditions are equivalent:*
(1) *f is a 2-homomorphism,*
(2) a) *f is isotone,*
b) *for any $z, x \in P$ the inequality $f(z) \geqslant f(x)$ implies $f(z) = f(u)$ for some $u \geqslant x$, $u \in P$.*

P r o o f. (1) $\Rightarrow$ (2): b) Suppose (1) is satisfied and $f(z) \geqslant f(x)$ for some $x, z \in P$. Then $f(z) \in [f(x))_{\leqslant}$, which means $z \in f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant}))$, i.e. $f(z) \in f([x)_{\leqslant})$. Thus there exists $u \in [x)_{\leqslant}$, i.e. $u \geqslant x$ such that $f(u) = f(z)$.

a) Suppose $x, y \in P$, $x \leqslant y$ are arbitrary elements. Then $y \in [x)_{\leqslant}$, which implies $f(y) \in f([x)_{\leqslant})$ and $y \in f^{-}(f(y)) \subseteq f^{-}(f([x)_{\leqslant})) = f^{-}([f(x))_{\leqslant})$, thus $f(y) \in [f(x))_{\leqslant}$, which means $f(x) \leqslant f(y)$.

(2) $\Rightarrow$ (1): Suppose (2) holds and $z \in f^{-}([(f(x))_{\leqslant})$, which is $f(z) \in [(f(x))_{\leqslant}$, i.e. $f(z) \geqslant f(x)$. Applying (2) we have $f(z) = f(u)$ for some $u \geqslant x$ and consequently $u \in [x)_{\leqslant}$, which implies $f(u) \in f([x)_{\leqslant}))$. Finally $f(z) \in f([x)_{\leqslant})$ and $z \in f^{-}(f([x)_{\leqslant}))$. The converse inclusion follows from $f([x)_{\leqslant}) \subseteq [(f(x))_{\leqslant}$, which holds for any isotone mapping f (cf. [4], Lemma 2). $\square$

Proposition 5. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and $f\colon P \longrightarrow Q$ a mapping. Then the following conditions are equivalent:*

(1) *f is a 3-homomorphism,*

(2) a) *f is isotone,*

b) *for any $y, x \in P$ the inequality $f(y) \geqslant f(x)$ implies $y \geqslant z$ for some $z \in P$ such that $f(z) = f(x)$.*

P r o o f. (1) $\Rightarrow$ (2): b) Suppose (1) and $f(y) \geqslant f(x)$ for some $x, y \in P$. Clearly $f(y) \in [(f(x))_{\leqslant}$ and thus $y \in f^{-}([(f(x))_{\leqslant}) = [f^{-}(f(x)))_{\leqslant}$, hence there exists $z \in f^{-}(f(x))$, i.e. $f(z) = f(x)$ such that $y \geqslant z$.

a) Suppose $x, y \in P$, $x \leqslant y$. Since $x \in f^{-}(f(x))$ we have $y \in [x)_{\leqslant} \subseteq [f^{-}(f(x)))_{\leqslant} = f^{-}([f(x))_{\leqslant})$, hence $f(y) \in [f(x))_{\leqslant}$. Consequently $f(x) \leqslant f(y)$.

(2) $\Rightarrow$ (1): Suppose (2) and let $y \in f^{-}([(f(x))_{\leqslant})$, which means $f(y) \in [(f(x))_{\leqslant}$, i.e. $f(y) \geqslant f(x)$. We have $y \geqslant z$ for some $z \in P$ such that $f(z) = f(x)$ by (2) and consequently $y \geqslant z \in f^{-}(f(z)) = f^{-}(f(x))$ and $y \in [f^{-}(f(x)))_{\leqslant}$. The converse inclusion follows from $[f^{-}(f(x)))_{\leqslant} \subseteq f^{-}([(f(x))_{\leqslant})$, which holds for any isotone mapping f (cf. [4], Lemma 2). $\square$

A mapping satisfying the condition (2) b) of Proposition 4 or 5 is called *u-super strong* or *l-super strong*, respectively. If it satisfies both the conditions, it is called a *super strong* mapping.

There is a natural question whether 2-, 3-endomorphisms are closed under composition. The answer is negative, which means that 2, 3-End$(P, \leqslant)$ is not a subgroupoid of End$(P, \leqslant)$. Let $P = \{a, b, c\}$ and $a \leqslant b$, $a \parallel c \parallel b$ (cf. Fig. 2). The mappings $f, g\colon (P, \leqslant) \to (P, \leqslant)$ $(f, g\colon (P, \geqslant) \to (P, \geqslant))$ are 2-endomorphisms (3-endomorphisms) but for $h = g \circ f$ we have $h \notin$ 2-End$(P, \leqslant)$ $(h \notin$ 3-End$(P, \geqslant))$.

Now we can extend in a certain sense Theorem 1 from [4] to the case of 4-endomorphisms.

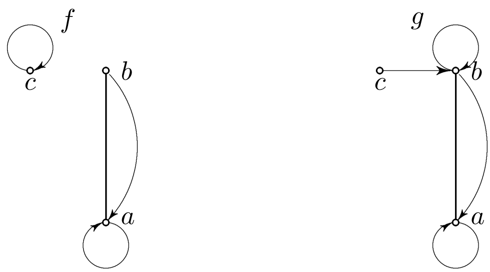

Figure 2

Theorem 2. *Let $(P, \leqslant)$ be a totally ordered set. Then*

$$\operatorname{End}(P, \leqslant) = 4\text{-}\operatorname{End}(P, \leqslant).$$

P r o o f. The inclusion $\operatorname{End}(P, \leqslant) \supseteq 4\text{-}\operatorname{End}(P, \leqslant)$ has been proved in [4], Lemma 3.

Suppose $f(z) \geqslant f(x)$. Since $(P, \leqslant)$ is a chain we have either $z \geqslant x$, i.e. condition (2) a) from Proposition 4 is satisfied, or $z < x$, which implies $f(z) \leqslant f(x)$ and consequently $f(z) = f(x)$, i.e. for $u = x$ f is also a 2-homomorphism. Similarly we can prove condition (2) a) from Proposition 5. □

There is a natural question how to construct 2-, 3-, 4-homomorphisms.

Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$. Further, let us define two relations $\lhd$, $\blacktriangleleft$ on P/θ in the following way:

$$[x]_\theta \blacktriangleleft [y]_\theta \quad \text{iff for any } q \in [x]_\theta \text{ there exists } p \in [y]_\theta \text{ such that } q \leqslant p,$$

$$[x]_\theta \lhd [y]_\theta \quad \text{iff for any } p \in [y]_\theta \text{ there exists } q \in [x]_\theta \text{ such that } q \leqslant p.$$

It is easy to see that they are both reflexive and transitive but not antisymmetric in general.

Lemma 2. *If the equivalence blocks of P/θ are convex then $\lhd \cap \blacktriangleleft$ is an order relation on P/θ.*

P r o o f. It has been proved in [2].

Corollary 1. *Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$ such that $\blacktriangleleft$ is an order relation on P/θ. Then the canonical mapping $\psi\colon P \to P/\theta$, $x \mapsto [x]_\theta$ is a 2-homomorphism.*

P r o o f. It is enough to verify the validity of conditions (2) a), b) from Proposition 4. The definition of the relation $\blacktriangleleft$ yields

(i) $[x]_\theta \blacktriangleleft [y]_\theta$ implies $[y]_\theta = [z]_\theta$ for some $x \leqslant z$,

(ii) $z \leqslant y$ implies $[z]_\theta \blacktriangleleft [y]_\theta$

and the corollary holds. □

Corollary 2. *Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$ such that $\lhd$ is an order relation on P/θ. Then the canonical mapping $\psi\colon\ P \to P/\theta$, $x \mapsto [x]_\theta$ is a 3-homomorphism.*

P r o o f. The definition of the relation $\lhd$ yields

(i) $[x]_\theta \lhd [y]_\theta$ implies $z \leqslant y$ for some $z \in [x]_\theta$,

(ii) $z \leqslant y$ implies $[z]_\theta \lhd [y]_\theta$

and the corollary holds. □

Corollary 3. *Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$ such that the equivalence blocks are convex. Let us order P/θ by $\lhd \cap \blacktriangleleft$. Then the canonical mapping $\psi\colon\ P \to P/\theta$, $x \mapsto [x]_\theta$ is a 4-homomorphism.*

P r o o f. It follows immediately from Corollary 1 and Corollary 2. □

Theorem 3. *Let $(P, \leqslant)$ be a poset. Then the following conditions are equivalent:*

(1) a) *$(P, \leqslant)$ is an antichain or*

b) *there exists an element $a \in P$ such that $(P, \leqslant) = X \oplus \{a\}$ where $X \neq \emptyset$ is an antichain or*

c) *$(P, \leqslant)$ is at least a three element chain,*

(2) $\operatorname{End}(P, \leqslant) \subseteq 2\text{-}\operatorname{End}(P, \leqslant)$,

(3) $\operatorname{End}(P, \leqslant) = 2\text{-}\operatorname{End}(P, \leqslant)$.

P r o o f. Conditions (2) and (3) are equivalent due to [4] Lemma 3 (this also follows from Proposition 4). It is enough to demonstrate the equivalence of (1) and (2). It has been recently proved in [4] that if P has at least a three-element chain it has to be a chain, i.e. (1) c) holds. Thus we can study only the cases where $(P, \leqslant)$ is of length one, i.e. it contains two-element chains only.

(1) $\Rightarrow$ (2): This follows immediately from Proposition 4.

(2) $\Rightarrow$ (1): Suppose that any isotone mapping is a 2-homomorphism, i.e. condition (2) b) from Proposition 4 is satisfied. This is clear if $(P, \leqslant)$ is an antichain or $(P, \leqslant)$ is a two-element chain. Suppose $(P, \leqslant)$ contains at least one two-element chain $b < a$ and incomparable elements. Then for any pair of incomparable elements $x, y \in P$ we can construct an isotone mapping f such that $f(x) > f(y)$, $f(z) = a$ for any $z \in X^{xy}$, $f(z) = b$ for any $z \in X_{xy}$ (X^{xy}, X_{xy} were defined in the proof of Proposition 1) and $f(z) = a$ otherwise. The mapping f has to be a 2-homomorphism, i.e there exists an element $z > y$ such that $f(z) = f(x)$. If $x \parallel z$ then we can again construct a similar mapping but for elements x and z. This leads to the existence of a three-element chain and consequently $(P, \leqslant)$ is a chain. Thus $x \leqslant z$, which means that P is up directed and must be of the form $X \oplus \{a\}$ for $a \in P$, X an antichain. □

Theorem 4. *Let* $(P, \leqslant)$ *be a poset. Then the following conditions are equivalent:*

(1) a) $(P, \leqslant)$ *is an antichain or*

b) *there exists an element* $a \in P$ *such that* $(P, \leqslant) = \{a\} \oplus X$ *where* $X \neq \emptyset$ *is an antichain or*

c) $(P, \leqslant)$ *is at least a three element chain,*

(2) $\mathrm{End}(P, \leqslant) \subseteq 3\text{-}\mathrm{End}(P, \leqslant)$,

(3) $\mathrm{End}(P, \leqslant) = 3\text{-}\mathrm{End}(P, \leqslant)$.

P r o o f. Dually to the proof of the previous Theorem 3. □

Theorem 5. *Let* $(P, \leqslant)$ *be a poset. Then the following conditions are equivalent:*

(1) a) $(P, \leqslant)$ *is an antichain or*

b) $(P, \leqslant)$ *is at least a three element chain,*

(2) $\mathrm{End}(P, \leqslant) \subseteq 4\text{-}\mathrm{End}(P, \leqslant)$,

(3) $\mathrm{End}(P, \leqslant) = 4\text{-}\mathrm{End}(P, \leqslant)$.

P r o o f. It follows from Theorem 3 and Theorem 4. □

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