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SOME FIXED POINT THEOREMS IN METRIC SPACES BY ALTERING DISTANCES

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Abstract. A generalization is obtained for some of the fixed point theorems of Khan, Swaleh and Sessa, Pathak and Rekha Sharma, and Sastry and Babu for a self-map on a metric space, which involve the idea of alteration of distances between points.

Keywords: fixed point, alteration of distances

MSC 2000: 47H10, 54H25

The famous Banach contraction principle has been generalized by several authors in several ways. A comprehensive literature on the generalizations of the same for self-maps on a metric space can be found in Rhoades [4] and Tasković [9]. Khan, Swaleh and Sessa [1] obtained generalizations of the same for a self-map on a metric space by altering distances between points through the use of certain control functions. Sastry and Babu [5], [6] and [7] continued the study in this direction. It was further pursued by Naidu [2]. In an attempt to unify Theorem 2 of Khan, Swaleh and Sessa [1] and that of Pathak and Rekha Sharma [3], Sastry and Babu obtained a partial generalization (Theorem 2.1 of [5]). Here our aim is to unify all the three results.

Throughout this paper, unless otherwise stated, (X,d) is a metric space, f is a self-map on X, $\mathbb{N}$ is the set of all positive integers, $\mathbb{R}^+$ is the set of all nonnegative real numbers, $\varphi\colon\ \mathbb{R}^+\to\mathbb{R}^+$ is a monotonically increasing function with $\varphi(t+)<t$ $\forall t\in(0,\infty)$, $\theta\colon\ \mathbb{R}^+\to[0,1]$ is a monotonically decreasing function with $\theta(t)<1$ $\forall t\in(0,\infty)$, $\zeta\colon\ \mathbb{R}^+\to[\frac{1}{2},1)$ is continuous at zero, ϱ is a nonnegative real valued function on $X\times X$ with the following two properties:

(i) $\{\varrho(x_n,y_n)\}_{n=1}^{\infty}$ is convergent whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in X such that $\{d(x_n,y_n)\}_{n=1}^{\infty}$ is convergent,

(ii) for any sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty} in X$, the sequence $\{\varrho(x_n, y_n)\}_{n=1}^{\infty}$ converges to zero iff the sequence $\{d(x_n, y_n)\}_{n=1}^{\infty}$ converges to zero;

K is a nonnegative real number, and for $x, y \in X$ we have

$$\begin{aligned}
\alpha(x,y) &= (\max\{\varrho(x,y), \varrho(x,fx), \varrho(y,fy)\}) + K[\varrho(x,fy)\varrho(fx,y)]^{1/2},\\
\beta(x,y) &= (\max\{\varrho(x,y), [\varrho(x,fy)\varrho(fx,y)]^{1/2}\}) + \max\{\varrho(x,fx), \varrho(y,fy)\},\\
\beta_0(x,y) &= (\max\{\varrho(x,y), [\varrho(x,fy)\varrho(fx,y)]^{1/2}\})\\
&\quad + (\min\{\max\{\varrho(x,fx), \varrho(y,fy)\}, \zeta(d(x,y))[\varrho(x,fx) + \varrho(y,fy)]\}),\\
\gamma(x,y) &= \min\{\alpha(x,y), \beta(x,y)\} \text{ and}\\
\gamma_0(x,y) &= \min\{\alpha(x,y), \beta_0(x,y)\}.
\end{aligned}$$

From property (i) of ϱ we note that ϱ is symmetric and that $\{\varrho(x_n, y_n)\}_{n=1}^{\infty}$ converges to $\varrho(x,y)$ whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in X such that $\{d(x_n, y_n)\}_{n=1}^{\infty}$ converges to $d(x,y)$. From property (ii) of ϱ we note that $\varrho(x,y) = 0$ iff $x = y$.

Theorem 1. *Suppose that*

$$\varrho(fx, fy) \leqslant \max\{\varphi(\gamma(x,y)), \theta(d(x,y))\gamma(x,y)\} \tag{1}$$

for all $x, y \in X$. Then f has at most one fixed point in X and for any $x \in X$, $\{f^n x\}$ is Cauchy.

Proof. From inequality (1) we have

$$\varrho(fx, fy) \leqslant \max\{\varphi(\alpha(x,y)), \theta(d(x,y))\alpha(x,y)\}$$

for all $x, y \in X$. Hence

$$\begin{aligned}
\varrho(fx, f^2x) \leqslant\ &\max\{\varphi(\max\{\varrho(x,fx), \varrho(fx,f^2x)\}),\\
&\theta(d(x,fx))\max\{\varrho(x,fx), \varrho(fx,f^2x)\}\}.
\end{aligned} \tag{2}$$

Suppose that $fx \neq x$. Then $\theta(d(x,fx)) < 1$ and $\varrho(x,fx) > 0$. Hence from inequality (2) and the fact that $\varphi(t) \leqslant \varphi(t+) < t\ \forall t \in (0,\infty)$ it follows that

$$\varrho(fx, f^2x) \leqslant \max\{\varphi(\varrho(x,fx)), \theta(d(x,fx))\varrho(x,fx)\}. \tag{3}$$

We note that inequality (3) remains valid even if $fx = x$. Replacing x with $f^{n-1}x$ in inequality (3) we obtain

$$\varrho(f^n x, f^{n+1}x) \leqslant \max\{\varphi(\varrho(f^{n-1}x, f^n x)), \theta(d(f^{n-1}x, f^n x))\varrho(f^{n-1}x, f^n x)\} \tag{4}$$

for all $n \in \mathbb{N}$. Since $\varphi(t) \leqslant t$ and $\theta(t) \leqslant 1$ $\forall t \in \mathbb{R}^+$, from inequality (4) we have

$$\varrho(f^n x, f^{n+1}x) \leqslant \varrho(f^{n-1}x, f^n x)$$

for all $n \in \mathbb{N}$. Consequently, $\{\varrho(f^n x, f^{n+1}x)\}_{n=0}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers. Hence it converges to a nonnegative real number s. First, suppose that $s > 0$. Then from property (ii) of ϱ it follows that $\{d(f^n x, f^{n+1}x)\}_{n=0}^{\infty}$ is a sequence of positive real numbers bounded below by a positive real number δ. Since θ is a monotonically decreasing function on $\mathbb{R}^+$, it follows that $\theta(d(f^{n-1}x, f^n x)) \leqslant \theta(\delta)$ $\forall n \in \mathbb{N}$. Hence from inequality (4) we have

$$\varrho(f^n x, f^{n+1}x) \leqslant \max\{\varphi(\varrho(f^{n-1}x, f^n x)), \theta(\delta)\varrho(f^{n-1}x, f^n x)\}$$

for all $n \in \mathbb{N}$. Taking limit superiors on both sides of the above inequality as $n \to +\infty$, we obtain

$$s \leqslant \max\{\varphi(s+), \theta(\delta)s\}.$$

Since $\varphi(t+) < t$ $\forall t \in (0, \infty)$, $s > 0$ and $\theta(\delta) < 1$, from the above inequality we have $s < s$, which is absurd. Hence $s = 0$. Hence property (ii) of ϱ yields that $\{d(f^n x, f^{n+1}x)\}_{n=0}^{\infty}$ converges to zero.

Now, suppose that $\{f^n x\}$ is not Cauchy. Then there exists a positive real number ε such that for given $N \in \mathbb{N}$ $\exists$ $m, n \in \mathbb{N}$ such that $m > n > N$ and $d(f^n x, f^m x) \geqslant \varepsilon$. Since $\{d(f^n x, f^{n+1}x)\}_{n=0}^{\infty}$ converges to zero, it follows that there exist strictly increasing sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ of positive integers such that $1 < n_k < m_k$, $d(f^{n_k}x, f^{m_k-1}x) < \varepsilon$ and $d(f^{n_k}x, f^{m_k}x) \geqslant \varepsilon$ $\forall k \in \mathbb{N}$. Using the triangle inequality and the fact that $\{d(f^n x, f^{n+1}x)\}_{n=0}^{\infty}$ converges to zero it can be shown that $\{d(f^{n_k}x, f^{m_k}x)\}_{k=1}^{\infty}$, $\{d(f^{n_k}x, f^{m_k-1}x)\}_{k=1}^{\infty}$, $\{d(f^{n_k-1}x, f^{m_k}x)\}_{k=1}^{\infty}$ and $\{d(f^{n_k-1}x, f^{m_k-1}x)\}_{k=1}^{\infty}$ all converge to ε. Hence from property (i) of ϱ it follows that the sequences $\{\varrho(f^{n_k}x, f^{m_k}x)\}_{k=1}^{\infty}$, $\{\varrho(f^{n_k}x, f^{m_k-1}x)\}_{k=1}^{\infty}$, $\{\varrho(f^{n_k-1}x, f^{m_k}x)\}_{k=1}^{\infty}$ and $\{\varrho(f^{n_k-1}x, f^{m_k-1}x)\}_{k=1}^{\infty}$ all converge to the same limit b for some nonnegative real number b. Since $\varepsilon > 0$, from property (ii) of ϱ it follows that $b > 0$. We note that $\{\beta(f^{n_k-1}x, f^{m_k-1}x)\}_{k=1}^{\infty}$ converges to b. Since φ is monotonically increasing on $\mathbb{R}^+$, we have $\limsup\limits_{k\to+\infty} \varphi(\beta(f^{n_k-1}x, f^{m_k-1}x)) \leqslant \varphi(b+)$. Since θ is monotonically decreasing on $\mathbb{R}^+$ and $\{d(f^{n_k-1}x, f^{m_k-1}x)\}_{k=1}^{\infty}$ converges to ε, $\limsup\limits_{k\to+\infty} \theta(d(f^{n_k-1}x, f^{m_k-1}x)) \leqslant \theta(\varepsilon-)$. Since θ is monotonically decreasing on $\mathbb{R}^+$ and $\theta(t) < 1$ $\forall t \in (0, \infty)$, it follows that $\theta(t-) < 1$ $\forall t \in (0, \infty)$. From inequality (1) we have

$$\varrho(fx, fy) \leqslant \max\{\varphi(\beta(x, y)), \theta(d(x, y))\beta(x, y)\} \tag{5}$$

for all $x, y \in X$. Taking $f^{n_k-1}x$ and $f^{m_k-1}x$ istead of x and y in the above inequality and then taking limit superiors on both sides as $k \to +\infty$ we obtain

$$b \leqslant \max\{\varphi(b+), \theta(\varepsilon-)b\}.$$

Since $\varphi(t+) < t$ and $\theta(t-) < 1$ $\forall t \in (0, \infty)$, $b > 0$ and $\varepsilon > 0$, from the above inequality we obtain $b < b$ which is a contradiction. Hence $\{f^n x\}$ is Cauchy.

If x, y are fixed points of f in X, then $\beta(x, y) = \varrho(x, y)$ and hence from inequality (5) we obtain

$$\varrho(x, y) \leqslant \max\{\varphi(\varrho(x, y), \theta(d(x, y))\varrho(x, y)\}.$$

Since $\varphi(t) < t$ and $\theta(t) < 1$ $\forall t \in (0, \infty)$, from the above inequality we have $\varrho(x, y) = 0$. Hence $x = y$. Hence f has at most one fixed point in X. □

Remark 1. Theorem 1 remains valid if inequality (1) is replaced with inequalities (3) and (5).

Theorem 2. *Suppose that*

$$\varrho(fx, fy) \leqslant \max\{\varphi(\gamma(x, y)), \theta(d(x, y))\gamma_0(x, y)\} \tag{6}$$

for all $x, y \in X$. Then for any $x \in X$, $\{f^n x\}$ is Cauchy. For any $x_0 \in X$, the limit of $\{f^n x_0\}$, if it exists, is the unique fixed point of f.

Proof. Since the validity of inequality (6) implies that of inequality (1), it follows from Theorem 1 that f has at most one fixed point in X and that for any $x \in X$, $\{f^n x\}$ is Cauchy. Let $x_0 \in X$. Suppose that $\{f^n x_0\}$ converges to an element z of X. Since ζ is continuous at zero, $\{\zeta(d(f^n x_0, z))\}$ converges to $\zeta(0)$. From the properties of ϱ we note that the sequences $\{\varrho(f^n x_0, fz)\}$, $\{\varrho(f^{n+1}x_0, fz)\}$ converge to $\varrho(z, fz)$ and that the sequences $\{\varrho(f^n x_0, z)\}$, $\{\varrho(f^{n+1}x_0, z)\}$ and $\{\varrho(f^n x_0, f^{n+1}x_0)\}$ converge to zero. Hence $\{\beta(f^n x_0, z)\}$ converges to $\varrho(z, fz)$ and $\{\beta_0(f^n x_0, z)\}$ converges to $\zeta(0)\varrho(z, fz)$. From inequality (6) we have

$$\varrho(fx, fy) \leqslant \max\{\varphi(\beta(x, y)), \theta(d(x, y))\beta_0(x, y)\} \tag{7}$$

for all $x, y \in X$. Taking $x = f^n x_0$ and $y = z$ in inequality (7) and then taking limit superiors on both sides as $n \to +\infty$ we obtain

$$\varrho(z, fz) \leqslant \max\{\varphi(\varrho(z, fz)+), \theta(0)\zeta(0)\varrho(z, fz)\}.$$

Since $\varphi(t+) < t$ $\forall t \in (0, \infty)$, $\theta(0) \leqslant 1$ and $\zeta(0) < 1$, from the above inequality we have $\varrho(z, fz) = 0$. Hence $fz = z$. □

Remark 2. Theorem 2 remains valid if inequality (6) is replaced with inequalities (3) and (7).

From Theorem 2 we have the following

Corollary 1. *Suppose that*

$$\varrho(fx, fy) \leqslant \theta(d(x, y))\gamma_0(x, y)$$

for all $x, y \in X$. *Then for any* $x \in X$, $\{f^n x\}$ *is Cauchy. For any* $x_0 \in X$, *the limit of* $\{f^n x_0\}$, *if it exists, is the unique fixed point of* f.

From Corollary 1 we have

Corollary 2. *Suppose that*

$$\varrho(fx, fy) \leqslant \theta(d(x, y)) \max\bigl\{\varrho(x, y), \tfrac{1}{2}[\varrho(x, fx) + \varrho(y, fy)], [\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}}\bigr\}$$

for all $x, y \in X$. *Then for any* $x \in X$, $\{f^n x\}$ *is Cauchy. For any* $x_0 \in X$, *the limit of* $\{f^n x_0\}$, *if it exists, is the unique fixed point of* f.

Remark 3. In Corollary 2 the conclusion about the existence of a fixed point fails if the expression $\frac{1}{2}[\varrho(x, fx) + \varrho(y, fy)]$ in its governing inequality is replaced with $\max\{\varrho(x, fx), \varrho(y, fy)\}$. Example 1 shows that this is so even when (X, d) is a finite metric space and $\varrho = d$. In particular, the hypothesis of Theorem 1 does not ensure the existence of a fixed point for f.

Example 1 (Example 4 of [8]). Let $X = [0, 1]$ with the usual metric. Define $f\colon X \to X$ as $f(x) = x/2$ if $0 < x \leqslant 1$ and $f(0) = 1$. Define $\theta\colon \mathbb{R}^+ \to [0, 1]$ as $\theta(t) = 1 - t/2$ if $0 \leqslant t \leqslant 1$ and $\theta(t) = \frac{1}{2}$ if $1 < t < +\infty$. Then θ is a monotonically decreasing continuous function on $\mathbb{R}^+$, $\theta(t) < 1$ $\forall t \in (0, \infty)$ and

$$|fx - fy| \leqslant \theta|x - y| \max\{|x - y|, |x - fx|, |y - fy|\}$$

for all $x, y \in X$. Nonetheless, f has no fixed point in X.

Corollary 3 (Theorem 2 of [1]). *Suppose that* (X, d) *is complete,* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is a monotonically increasing continuous function with* $\psi(t) = 0$ *iff* $t = 0$, a, b, c *are monotonically decreasing functions from* $(0, \infty)$ *into* $[0, 1)$ *with* $a(t) + b(t) + c(t) < 1$ $\forall t \in (0, \infty)$, *and*

$$\begin{aligned}\psi(d(fx, fy)) \leqslant a(d(x, y))\psi(d(x, y)) + \tfrac{1}{2}b(d(x, y))[\psi(d(x, fx)) + \psi(d(y, fy))]\\ + c(d(x, y)) \min\{\psi(d(x, fy)), \psi(d(fx, y))\}\end{aligned}$$

for all distinct $x, y \in X$. *Then* f *has a unique fixed point in* X.

Proof. Let $\varrho = \psi \circ d$. Define $\theta\colon \mathbb{R}^+ \to [0,1]$ as $\theta(t) = a(t) + b(t) + c(t)$ if $t \neq 0$ and $\theta(0) = 1$. Then ϱ is a nonnegative real valued function on $X \times X$ having properties (i) and (ii), θ is a monotonically decreasing function on $\mathbb{R}^+$ with $\theta(t) < 1$ $\forall t \in (0,\infty)$ and

$$\begin{aligned}\varrho(fx,fy) &\leqslant \theta(d(x,y))\max\bigl\{\varrho(x,y), \tfrac{1}{2}[\varrho(x,fx)+\varrho(y,fy)], \min\{\varrho(x,fy),\varrho(fx,y)\}\bigr\}\\ &\leqslant \theta(d(x,y))\max\bigl\{\varrho(x,y), \tfrac{1}{2}[\varrho(x,fx)+\varrho(y,fy)], [\varrho(x,fy)\varrho(fx,y)]^{\frac{1}{2}}\bigr\}\end{aligned}$$

for all $x,y \in X$. Hence Corollary 3 follows from Corollary 2. □

Corollary 4 (Theorem 2 of [3]). *Suppose that (X,d) is complete, $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically increasing continuous function with $\psi(t) = 0$ iff $t = 0$, a, b are monotonically decreasing functions from $(0,\infty)$ into $[0,1)$ with $a(t) + b(t) < \frac{1}{2}$ $\forall t \in (0,\infty)$, c is a constant in $[0,1]$ such that $a(t)(1+c) < 1$ $\forall t \in (0,\infty)$, and*

$$\begin{aligned}\psi(d(fx,fy)) \leqslant a(d(x,y))[\psi(d(x,y)) + c[\psi(d(x,fy))\psi(d(fx,y))]^{\frac{1}{2}}]\\ + b(d(x,y))[\psi(d(x,fx)) + \psi(d(y,fy))]\end{aligned}$$

for all distinct $x,y \in X$. Then f has a unique fixed point in X.

Proof. Let $\varrho = \psi \circ d$. Define $\theta\colon \mathbb{R}^+ \to [0,1]$ as $\theta(t) = 2[a(t)+b(t)]$ if $t \neq 0$ and $\theta(0) = 1$. Then ϱ is a nonnegative real valued function on $X \times X$ having properties (i) and (ii), θ is a monotonically decreasing function on $\mathbb{R}^+$ with $\theta(t) < 1$ $\forall t \in (0,\infty)$ and

$$\begin{aligned}\varrho(fx,fy) &\leqslant \theta(d(x,y))\max\bigl\{\tfrac{1}{2}[\varrho(x,y)+c[\varrho(x,fy)\varrho(fx,y)]^{\frac{1}{2}}], \tfrac{1}{2}[\varrho(x,fx)+\varrho(y,fy)]\bigr\}\\ &\leqslant \theta(d(x,y))\max\bigl\{\varrho(x,y), c[\varrho(x,fy)\varrho(fx,y)]^{\frac{1}{2}}, \tfrac{1}{2}[\varrho(x,fx)+\varrho(y,fy)]\bigr\}\\ &\leqslant \theta(d(x,y))\max\bigl\{\varrho(x,y), \tfrac{1}{2}[\varrho(x,fx)+\varrho(y,fy)], [\varrho(x,fy)\varrho(fx,y)]^{\frac{1}{2}}\bigr\}\end{aligned}$$

for all $x,y \in X$. Hence Corollary 4 follows from Corollary 2. □

Remark 4. As observed by Sastry and Babu [5], in Theorem 2 of Pathak and Rekha Sharma [3] the condition '$a(t)(1+c) < 1$ $\forall t \in (0,\infty)$' is redundant in view of the hypothesis on the functions a and b, and the condition '$c \leqslant 1$'.

From Theorem 2 we have

Corollary 5. *Suppose that*

$$\varrho(fx,fy) \leqslant \varphi(\gamma(x,y))$$

for all $x, y \in X$. Then for any $x \in X$, $\{f^n x\}$ is Cauchy. For any $x_0 \in X$, the limit of $\{f^n x_0\}$, if it exists, is the unique fixed point of f.

Corollary 6 (Theorem 2.1 of [5]). *Suppose that (X, d) is complete, $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically increasing continuous function with $\psi(t) = 0$ iff $t = 0$, a, b, c are nonnegative constants with $a + b < 1$ and $a + c < 1$, and*

$$\begin{aligned}\psi(d(fx, fy)) \leqslant a\psi(d(x, y)) + \tfrac{1}{2}b[\psi(d(x, fx)) + \psi(d(y, fy))] \\ + c[\psi(d(x, fy))\psi(d(fx, y))]^{\frac{1}{2}}\end{aligned}$$

for all $x, y \in X$. Then f has a unique fixed point in X.

P r o o f. Let $\varrho = \psi \circ d$. Define $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = \mu t$, where $\mu = \max\{a + b, a + c\}$. Then ϱ is a nonnegative real valued function on $X \times X$ having properties (i) and (ii), φ is a monotonically increasing function on $\mathbb{R}^+$ with $\varphi(t+) < t$ $\forall t \in (0, \infty)$ and

$$\begin{aligned}
&\varrho(fx, fy)\\
&\quad\leqslant a\varrho(x, y) + \tfrac{1}{2}b[\varrho(x, fx) + \varrho(y, fy)] + c[\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}}\\
&\quad\leqslant \min\{(a + b)(\max\{\varrho(x, y), \varrho(x, fx), \varrho(y, fy)\}) + c[\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}},\\
&\qquad (a + c)(\max\{\varrho(x, y), [\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}}\}) + \tfrac{1}{2}b[\varrho(x, fx) + \varrho(y, fy)]\}\\
&\quad\leqslant \min\{(\max\{a + b, c\})[(\max\{\varrho(x, y), \varrho(x, fx), \varrho(y, fy)\}) + [\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}}],\\
&\qquad (\max\{a + c, b\})[(\max\{\varrho(x, y), [\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}}\}) + \tfrac{1}{2}[\varrho(x, fx) + \varrho(y, fy)]]\}\\
&\quad\leqslant \min\{(\max\{a + b, c\})\alpha(x, y), (\max\{a + c, b\})\beta_0(x, y)\}\\
&\quad\leqslant \min\{\mu\alpha(x, y), \mu\beta_0(x, y)\}\\
&\quad= \mu\gamma_0(x, y) \leqslant \mu\gamma(x, y) = \varphi(\gamma(x, y))
\end{aligned}$$

for all $x, y \in X$, where we have taken $K = 1$ in the definition of $\alpha(x, y)$. Hence Corollary 6 follows from Corollary 5. □

Corollary 7. *Suppose that a, b, c are nonnegative monotonically decreasing functions on $(0, \infty)$ with $a(t) + b(t) < 1$ and $a(t) + c(t) < 1$ $\forall t \in (0, \infty)$, and*

$$\begin{aligned}\varrho(fx, fy) \leqslant a(d(x, y))\varrho(x, y) + \tfrac{1}{2}b(d(x, y))[\varrho(x, fx) + \varrho(y, fy)] \\ + c\,(d(x, y))[\varrho(x, fy)\varrho(fx, y)]^{\frac{1}{2}}\end{aligned}$$

for all distinct $x, y \in X$. Then for any $x \in X$, $\{f^n x\}$ is Cauchy. For any $x_0 \in X$, the limit of $\{f^n x_0\}$, if it exists, is the unique fixed point of f.

Proof. Define $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ as $\theta(t) = \max\{a(t)+b(t), a(t)+c(t)\}$ if $t \neq 0$ and $\theta(0) = 1$. Then θ is a monotonically decreasing function on $\mathbb{R}^+$ with $\theta(t) < 1$ $\forall t \in (0,\infty)$. Proceeding as in the proof of Corollary 6 it can be shown that

$$\varrho(fx, fy) \leqslant \theta(d(x,y))\gamma_0(x,y)$$

for all $x, y \in X$, with $K = 1$ in the definition of $\alpha(x,y)$. Hence Corollary 7 follows from Corollary 1. □

Remark. Corollary 7 is also a generalization of Corollaries 3, 4 and 6. Corollary 7 shows that in Theorem 2 of Pathak and Rekha Sharma [3] the condition '$a(t) + b(t) < \frac{1}{2}$ $\forall t \in (0,\infty)$' can be replaced by the weaker conditions '$2a(t) < 1$ and $a(t) + 2b(t) < 1$ $\forall t \in (0,\infty)$'.

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