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# NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS WITH MULTIVALUED NONLINEARITIES

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*Abstract.* In this paper we study nonlinear elliptic boundary value problems with monotone and nonmonotone multivalued nonlinearities. First we consider the case of monotone nonlinearities. In the first result we assume that the multivalued nonlinearity is defined on all $\mathbb{R}$. Assuming the existence of an upper and of a lower solution, we prove the existence of a solution between them. Also for a special version of the problem, we prove the existence of extremal solutions in the order interval formed by the upper and lower solutions. Then we drop the requirement that the monotone nonlinearity is defined on all of $\mathbb{R}$. This case is important because it covers variational inequalities. Using the theory of operators of monotone type we show that the problem has a solution. Finally, in the last part we consider an eigenvalue problem with a nonmonotone multivalued nonlinearity. Using the critical point theory for nonsmooth locally Lipschitz functionals we prove the existence of at least two nontrivial solutions (multiplicity theorem).

*Keywords*: upper solution, lower solution, order interval, truncation function, pseudomonotone operator, coercive operator, extremal solution, Yosida approximation, nonsmooth Palais-Smale condition, critical point, eigenvalue problem

*MSC 2000*: 35J20, 35J60, 35R70

## 1. Introduction

In this paper we employ the method of upper and lower solutions, the theory of nonlinear operators of monotone type and the critical point theory for nonsmooth functionals in order to solve certain nonlinear elliptic boundary value problems, involving discontinuous nonlinearities of both monotone and nonmonotone type.

Most of the works so far have treated semilinear problems. Only Deuel-Hess [12] deal with a fully nonlinear equation, but their forcing term on the right hand side is a Carathéodory function. Deuel-Hess use the method of upper and lower solutions in order to show that the problem has a solution located in the order interval formed by the upper and lower solutions. More recently, Dancer-Sweers [11] have considered a semilinear elliptic problem with a Carathéodory forcing term which is independent of the gradient of the solution and they proved the existence of extremal solutions in the order interval (i.e the existence of a maximal and of a minimal solution there). Semilinear elliptic problems with discontinuities have been studied by Chang [8] and Costa-Goncalves [10], who used the critical point theory for nondifferentiable functionals, by Ambrosetti-Turner [4] and Ambrosetti-Badiale [5], who used the dual variational principle of Clarke [9] and by Stuart [23] and Carl-Heikkila [7], who used monotonicity techniques. In Carl-Heikkila [7], we encounter differential inclusions but they assume that the monotone term $\beta(\cdot)$ corresponding to the discontinuous nonlinearity is defined everywhere (i.e. $\operatorname{dom}\beta = \mathbb{R}$), while here we have a result where $\operatorname{dom}\beta \neq \mathbb{R}$, a case of special importance since it incorporates variational inequalities. We also consider the case where the term $\beta(\cdot)$ is nonmonotone, which corresponds to problems in mechanics in which the constitutive laws are nonmonotone and multivalued and so are described by the subdifferential of nonsmooth and nonconvex potential functions (hemivariational inequalities).

## 2. Preliminaries

Let X be a reflexive Banach space and $X^*$ its topological dual. In what follows by $(\cdot,\cdot)$ we denote the duality brackets of the pair $(X, X^*)$. A map $A\colon X \mapsto 2^{X^*}$ is said to be "monotone", if for all $[x_1, x_1^*], [x_2, x_2^*] \in \operatorname{Gr} A$, we have $(x_2^*, x_1^*, x_2 - x_1) \geqslant 0$. The set $D = \{x \in X\colon A(x) \neq \emptyset\}$ is called the "domain of $A$". We say that $A(\cdot)$ is maximal monotone, if its graph is maximal with respect to inclusion among the graphs of all monotone maps from $X$ into $X^*$. It follows from this definition that $A(\cdot)$ is maximal monotone if and only if $(v^* - x^*, v - x) \geqslant 0$ for all $[x, x^*] \in \operatorname{Gr} A$ implies $[v, v^*] \in \operatorname{Gr} A$. For a maximal monotone map $A(\cdot)$, for every $x \in D$, $A(x)$ is nonempty, closed and convex. Moreover, $\operatorname{Gr} A \subseteq X \times X^*$ is demiclosed, i.e. if $x_n \to x$ in $X$ and $x_n^* \xrightarrow{w} x^*$ in $X^*$ or if $x_n \xrightarrow{w} x$ in $X$ and $x_n^* \to x^*$ in $X^*$, then $[x, x^*] \in \operatorname{Gr} A$. A single-valued $A\colon X \mapsto X^*$ with the domain all of $X$ is said to be hemicontinuous if for all $x, y, z \in X$, the map $\lambda \mapsto (A(x+\lambda y), z)$ is continuous from $[0,1]$ into $\mathbb{R}$ (i.e. for all $x, y \in X$, the map $\lambda \mapsto A(x+\lambda y)$ is continuous from $[0,1]$ into $X^*$ furnished with the weak topology). A monotone hemicontinuous operator is maximal monotone. A map $A\colon X \mapsto 2^{X^*}$ is said to be "pseudomonotone", if for all $x \in X$, $A(x)$ is nonempty, closed and convex, for every sequence $\{[x_n, x_n^*]\}_{n\geqslant 1} \subseteq \operatorname{Gr} A$ such that $x_n \xrightarrow{w} x$ in $X$, $x_n^* \xrightarrow{w} x^*$ in $X^*$ and $\limsup (x_n^*, x_n - x) \leqslant 0$, we have that for each $y \in X$, there corresponds a $y^*(y) \in A(x)$ such that $(y^*(y), x - y) \leqslant \liminf (x^*, x_n - y)$, and finally $A$ is upper semicontinuous (as a set-valued map) from every finite dimensional subspace of $X$ into $X^*$ endowed with the weak topology. Note that this requirement is automatically satisfied if $A(\cdot)$ is bounded, i.e. maps bounded sets into bounded sets. A map $A\colon X \mapsto 2^{X^*}$ with nonempty, closed and convex values, is said to be generalized pseudomonotone if for any sequence $\{[x_n, x_n^*]\}_{n\geqslant 1} \subseteq \operatorname{Gr} A$ such that $x_n \xrightarrow{w} x$ in $X$, $x_n^* \xrightarrow{w} x^*$ in $X^*$ and $\limsup(x_n^*, x_n - x) \leqslant 0$, we have $[x, x^*] \in \operatorname{Gr} A$ and $(x_n^*, x_n) \to (x^*, x)$ (generalized pseudomonotonicity). The sum of two pseudomonotone maps is pseudomonotone and a maximal monotone map with domain $D = X$ is pseudomonotone. A pseudomonotone map which is also coercive (i.e. $\inf[(x^*, x)\colon\ x^* \in A(x)]/\|x\| \to \infty$) is surjective.

A function $\varphi\colon\ X \to \widehat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be proper, if it is not identically $+\infty$, i.e. $\operatorname{dom}\varphi = \{x \in X\colon\ \varphi(x) < +\infty\}$ (the effective domain of $\varphi$) is nonempty. By $\Gamma_0(X)$ we denote the space of all proper, convex and lower semicontinuous functions. Given a proper, convex function $\varphi(\cdot)$, its subdifferential $\partial\varphi\colon\ X \mapsto 2^{X^*}$ is defined by

$$\partial\varphi(x) = \{x^* \in X^*\colon\ (x^*, y - x) \leqslant \varphi(y) - \varphi(x) \quad \text{for all } \ y \in \operatorname{dom}\varphi\}.$$

If $\varphi \in \Gamma_0(X)$, then $\partial\varphi(\cdot)$ is maximal monotone (in fact cyclically maximal monotone). Finally, recall that $\varphi \in \Gamma_0(X)$ is locally Lipschitz in the interior of its effective domain.

Next let $\varphi\colon\ X \to \mathbb{R}$ be locally Lipschitz. For such a function we can define the generalized directional derivative of $\varphi$ at $x \in X$ in the direction $h \in X$ as follows:

$$\varphi^0(x;h) = \limsup_{x' \to x, \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $\varphi^0(x;\cdot)$ is sublinear and continuous and so by the Hahn-Banach theorem we can define a nonempty, weakly compact and convex set,

$$\partial\varphi(x) = \{x^* \in X^*\colon\ (x^*, h) \leqslant \varphi^0(x;h) \quad \text{for all } \ h \in X\}.$$

The set $\partial\varphi(x)$ is called the (generalized) subdifferential of $\varphi$ at $x$ (see Clarke [9]). If $\varphi$ is also convex, then this subdifferential coincides with the subdifferential of $\varphi$ in the sense of convex analysis defined earlier. Moreover, in this case $\varphi^0(x;h) = \lim\limits_{\lambda\downarrow 0} \big(\varphi(x + \lambda h) - \varphi(x)\big)/\lambda = \varphi'(x;h)$ (the directional derivative of $\varphi$ at $x$ in the direction $h$). A function $\varphi$ for which $\varphi^0(x;\cdot) = \varphi'(x;\cdot)$ is said to be regular at $x$. Finally, recall that if $x$ is a local extremum of $\varphi$, then $0 \in \partial\varphi(x)$. More generally, a point $x \in X$ for which we have $0 \in \partial\varphi(x)$, is said to be a critical point of $\varphi$. For further details on operators of monotone type and subdifferentials we refer to Hu-Papageorgiou [16] and Zeidler [25].

## 3. Existence results with monotone nonlinearities

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a $C^1$-boundary $\Gamma$. In what follows we denote by $A_1(\cdot)$ the nonlinear, second order differential operator in divergence form defined by $A_1(x)(\cdot) = -\sum_{k=1}^{N} D_k a_k(\cdot, x(\cdot), Dx(\cdot))$. In this section we study the following boundary value problem:

$$
(1) \qquad \left\{ \begin{array}{l} A_1(x)(z) + a_0(z, x(z), Dx(z)) + \beta(z, x(z)) \ni g(x(z)) \text{ in } Z, \\ x|_\Gamma = 0. \end{array} \right\}
$$

First, using the method of upper and lower solutions, we establish the existence of (weak) solutions for problem (1), when $\operatorname{dom}\beta = \mathbb{R}$. Let us start by introducing the hypotheses on the coefficient functions $a_k(z,x,y)$, $k \in \{1,2,\ldots,N\}$ and on the multifunction $\beta(r)$.

$H(\alpha_k)$: $a_k\colon Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, k \in \{1,2,\ldots,N\}$, are functions such that

(i) for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, $z \to a_k(z,x,y)$ is measurable;

(ii) for almost all $z \in Z$, $(x,y) \to a_k(z,x,y)$ is continuous;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, we have

$$
|a_k(z,x,y)| \leqslant \gamma(z) + c(|x|^{p-1} + \|y\|^{p-1})
$$
$$
\text{with } \gamma \in L^q(Z), \ \ c > 0, \ \ 1 < p < \infty \ \text{ and } \ \frac{1}{p} + \frac{1}{q} = 1;
$$

(iv) for almost all $z \in Z$, all $x \in \mathbb{R}^N$ and all $y, y' \in \mathbb{R}$, $y \neq y'$, we have

$$
\sum_{k=1}^{N} (a_k(z,x,y) - a_k(z,x,y'))(y_k - y'_k) > 0;
$$

(v) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, we have

$$
\sum_{k=1}^{N} a_k(z,x,y) y_k \geqslant c_1 \|y\|^p - \gamma_1(z)
$$

with $c_1 > 0$, $\gamma_1 \in L^1(Z)$.

**Remark.** By virtue of these hypotheses, we can define semilinear form

$$
\widehat{a}\colon \ W_0^{1,p}(Z) \times W_0^{1,p}(Z) \to \mathbb{R}
$$

by setting

$$
\widehat{a}(x,v) = \int_Z \sum_{k=1}^{n} a_k(z, x(z), Dx(z)) D_k v(z) \, \mathrm{d}z.
$$

$H(\beta)$: $\beta\colon Z \times \mathbb{R} \to 2^{\mathbb{R}}$ is a graph measurable multifunction such that for all $z \in Z$, $\beta(z,\cdot)$ is maximal monotone, $\operatorname{dom}\beta(z,\cdot) = \mathbb{R}$, $0 \in \beta(z,0)$ and $|\beta(z,x)| = \max[|v|\colon v \in \beta(z,x)] \leqslant k(z) + \eta|x|^{p-1}$ a.e on Z with $k \in L^q(Z)$, $\eta > 0$.

**Remark.** It is well-known (see for example [16], example III.4.28 (a), p. 348 and theorem III.5.6, p. 362) that for all $z \in Z$, $\beta(z,x) = \partial j(z,x)$ with $j(z,x)$ a jointly measurable function such that $j(z,\cdot)$ is convex and continuous (in fact, locally Lipschitz). If $\beta^0(z,x) = \operatorname{proj}(0;\beta(z,x))$ (= the unique element of $\beta(z,x)$ with the smallest absolute value), then $x \to \beta^0(z,x)$ is nondecreasing and for every $(z,x) \in Z \times \mathbb{R}$, we have $\beta(z,x) = [\beta^0(z,x^-), \beta^0(z,x^+)]$. Moreover, $j(z,x) = j(z,0) + \int_0^x \beta^0(z,s)\,\mathrm{d}s$. Since $j(z,\cdot)$ is unique up to an additive constant, we can always have $j(z,0) = 0$. Since by hypothesis $0 \in \beta(z,0)$, we infer that for all $z \in Z$ and all $x \in \mathbb{R}$, $j(z,x) \geqslant 0$. In what follows $\beta_-(z,x) = \beta^0(z,x^-)$ and $\beta_+(z,x) = \beta^0(z,x^+)$. So $\beta(z,x) = [\beta_-(z,x), \beta_+(z,x)]$. Evidently we have $|\beta_-(z,x)|, |\beta_+(z,x)| \leqslant k(z) + \eta|x|^{p-1}$ for almost all $z \in Z$ and all $x \in \mathbb{R}$.

To introduce the hypotheses on the rest of the data of (1), we need the following definitions.

**Definition.** A function $\varphi \in W^{1,p}(Z)$ is said to be an "upper solution" of (1), if there exists $x_1^* \in L^q(Z)$ such that $x_1^*(z) \in \beta(z,\varphi(z))$ a.e. on Z and

$$\widehat{a}(\varphi, v) + \int_Z a_0(z,\varphi,D\varphi)v(z)\,\mathrm{d}z + \int_Z x_1^*(z)v(z)\,\mathrm{d}z \geqslant \int_Z g(\varphi(z))v(z)\,\mathrm{d}z$$

for all $v \in W_0^{1,p}(Z) \cap L^p(Z)_+$ and $\varphi|_\Gamma \geqslant 0$.

**Definition.** A function $\psi \in W^{1,p}(Z)$ is said to be a "lower solution" of (1), if there exists $x_0^* \in L^q(Z)$ such that $x_0^*(z) \in \beta(z,\psi(z))$ a.e. on Z and

$$\widehat{a}(\psi, v) + \int_Z a_0(z,\psi,D\psi)v(z)\,\mathrm{d}z + \int_Z x_0^*(z)v(z)\,\mathrm{d}z \leqslant \int_Z g(\psi(z))v(z)\,\mathrm{d}z$$

for all $v \in W_0^{1,p}(Z) \cap L^p(Z)_+$ and $\psi|_\Gamma \leqslant 0$.

We can continue with the hypotheses on the data of (1):

$\mathrm{H}_0$: There exist an upper solution $\varphi \in W^{1,p}(Z)$ and a lower solution $\psi \in W^{1,p}(Z)$ such that $\psi(z) \leqslant 0 \leqslant \varphi(z)$ a.e. on $Z$ and for all $y \in L^p(Z)$ such that $\psi(z) \leqslant y(z) \leqslant \varphi(z)$ a.e. on $Z$ we have $g(y(\cdot)) \in L^q(Z)$. Moreover, $g(\cdot)$ is nondecreasing.

$\mathrm{H}(\alpha_0)$: $a_0\colon Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that
(i) for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, $z \to a_0(z,x,y)$ is measurable;
(ii) for almost all $z \in Z$, $(x,y) \to a_0(z,x,y)$ is continuous;

(iii) for almost all $z \in Z$ and all $x \in [\psi(z), \varphi(z)]$, we have

$$|a_0(z,x,y)| \leqslant \gamma_2(z) + c_2\|y\|^{p-1}$$

with $\gamma_2 \in L^q(Z)$, $c_2 > 0$.

**Definition.** By a "(weak) solution" of (1) we mean a function $x \in W_0^{1,p}(Z)$ such that there exists $f \in L^q(Z)$ with $f(z) \in \beta(z, x(z))$ a.e. on Z and

$$\widehat{a}(x,v) + \int_Z a_0(z,x,Dx)v(z)dz + \int_Z f(z)v(z)\,\mathrm{d}z = \int_Z g(x(z))v(z)\,\mathrm{d}z$$

for all $v \in W_0^{1,p}(Z)$.

Let $K = [\psi, \varphi] = \{y \in W^{1,p}(Z)\colon\ \psi(z) \leqslant y(z) \leqslant \varphi(z)\ \text{ a.e. on } Z\}$. Our approach will involve truncation and penalization techniques. So we introduce the following two functions:

$\tau\colon W^{1,p}(Z) \to W^{1,p}(Z)$ (the truncation function) defined by

$$\tau(x)(z) = \begin{cases} \varphi(z) & \text{if } \varphi(z) \leqslant x(z), \\ x(z) & \text{if } \psi(z) \leqslant x(z) \leqslant \varphi(z), \\ \psi(z) & \text{if } x(z) \leqslant \psi(z) \end{cases}$$

and $u\colon Z \times \mathbb{R} \to \mathbb{R}$ (the penalty function) defined by

$$u(z,x) = \begin{cases} (x - \varphi(z))^{p-1} & \text{if } \varphi(z) \leqslant x, \\ 0 & \text{if } \psi(z) \leqslant x \leqslant \varphi(z), \\ -(\psi(z) - x)^{p-1} & \text{if } x(z) \leqslant \psi(z). \end{cases}$$

It is easy to check that the following lemma is true (see also Deuel-Hess [12]):

**Lemma 1.**

(a) *The truncation function map $\tau\colon W^{1,p}(Z) \to W^{1,p}(Z)$ is bounded and continuous;*

(b) *the penalty function $u(z,x)$ is a Carathéodory function such that*

$$\int_Z u(z,x(z))x(z)\,\mathrm{d}z \geqslant c_3\|x\|_p^p - c_4$$

*for all $x \in L^p(Z)$ and some $c_3, c_4 > 0$.*

To solve (1), we first investigate the following auxiliary problem, with $y \in K$:

$$\text{(2)} \qquad \left\{ \begin{array}{l} A_2(x)(z) + a_0(z, \tau(x)(z), D\tau(x)(z)) + \beta(z, x(z)) \\ \qquad\qquad + \varrho u(z, x(z)) \ni g(y(z)) \ \text{ on } \ Z, \\ x|_\Gamma = 0, \ \ \varrho > 0. \end{array} \right\}$$

Here $A_2(x)$ is a nonlinear, second order differential operator in divergence form, defined by

$$A_2(x)(z) = -\sum_{k=1}^{N} D_k a_k(z, \tau(x), Dx).$$

In the next proposition we establish the nonemptiness of the solution set $S(y) \subseteq W_0^{1,p}(Z)$ of (2) for all $y \in K$.

**Proposition 2.** *If hypotheses* $\mathrm{H}(a_k)$, $\mathrm{H}(\beta)$, $\mathrm{H}_0$, $\mathrm{H}(a_0)$ *hold and* $y \in K$, *then the solution set* $S(y) \subseteq W_0^{1,p}(Z)$ *of* (2) *is nonempty for* $\varrho > 0$ *large.*

Proof. Let $\theta\colon W_0^{1,p}(Z) \times W_0^{1,p}(Z) \to \mathbb{R}$ be a semilinear Dirichlet form defined by

$$\theta(x, y) = \int_Z \sum_{k=1}^{N} a_k(z, \tau(x), Dx) D_k y(z) \, \mathrm{d}z.$$

By virtue of hypotheses $H(a_k)$, this Dirichlet form defines a nonlinear operator $\widehat{A}_1\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ by $\langle \widehat{A}_1(x), y\rangle = \theta(x, y)$ (here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets of the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$). Also let $\widehat{a}_0\colon W_0^{1,p}(Z) \to L^q(Z)$ be defined by $\widehat{a}_0(x)(z) = a_0(z, \tau(x)(z), D\tau(x)(z))$. This is continuous and bounded (see hypothesis $\mathrm{H}(a_0)$).

**Claim 1.** *The operator* $\widehat{A}_2 = \widehat{A}_1 + \widehat{a}_0\colon W^{1,p}(Z) \to W^{-1,q}(Z)$ *is pseudomonotone.*

To this end, let $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ as $n \to \infty$ and assume that $\limsup\langle \widehat{A}_2(x_n), x_n - x\rangle \leqslant 0$. Then $\limsup\langle \widehat{A}_1(x_0) + \widehat{a}_0(x_n), x_n - x\rangle \leqslant 0$. From the Sobolev embedding theorem, we have $x_n \to x$ in $L^p(Z)$ and so $\langle \widehat{a}_0(x_n), x_n - x\rangle = (\widehat{a}_0(x_n), x_n - x)_{pq} \to 0$ (by $(\cdot,\cdot)_{pq}$ we denote the duality brackets of $(L^p(Z), L^q(Z))$. Therefore we obtain $\limsup\langle \widehat{A}_1(x_n), x_n - x\rangle \leqslant 0$.

We have

$$\begin{aligned}
\langle \widehat{A}_1(x_n), x_n - x\rangle &= \int_Z \sum_{k=1}^N a_k(z, \tau(x_n), Dx_n)(D_k x_n - D_k x)\,\mathrm{d}z \\
&= \int_Z \sum_{k=1}^N (a_k(z, \tau(x_n), Dx_n) \\
&\quad - a_k(z, \tau(x_n), Dx)(D_k x_n - D_k x)\,\mathrm{d}z \\
&\quad + \int_Z \sum_{k=1}^N a_k(z, \tau(x_n), Dx)(D_k x_n - D_k x)\,\mathrm{d}z \\
&\geqslant \int_Z \sum_{k=1}^N a_k(z, \tau(x_n), Dx)(D_k x_n - D_k x)\,\mathrm{d}z \\
&\quad \text{(hypothesis H}(a_k)\text{(iv))}.
\end{aligned}$$

Since $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, we have $x_n \to x$ in $L^p(Z)$ and then directly from the definition of the truncation map $\tau$ we have $\tau(x_n) \to \tau(x)$ in $L^p(Z)$. Therefore

$$\int_Z \sum_{k=1}^N a_k(z, \tau(x_n), Dx)(D_k x_n - D_k x)\,\mathrm{d}z \to 0 \ \text{ as } \ n \to \infty.$$

On the other hand, we already know that $\limsup \langle A_1(x_n), x_n - x\rangle \leqslant 0$. Hence $\langle A_1(x_n), x_n - x\rangle \to 0$ as $n \to \infty$. This implies that

$$\int_Z \sum_{k=1}^N (a_k(z, \tau(x_n), Dx_n) - a_k(z, \tau(x_n), Dx))(D_k x_n - D_k x)\,\mathrm{d}z \to 0 \quad \text{as } \ n \to \infty.$$

Then invoking Lemma 6 of Landes [17], we infer that $D_k x_n(z) \to D_k x(z)$ a.e. on $Z$ for all $k \in \{1, 2, \ldots, N\}$. So using Lemma 3.2 of Leray-Lions [18], we have that $\widehat{A}_1(x_n) \xrightarrow{w} \widehat{A}_1(x)$ in $W^{-1,q}(Z)$. We have already established earlier that $\langle \widehat{A}_1(x_n), x_n - x\rangle \to 0$. Since $\langle \widehat{A}_1(x_n), x\rangle \to \langle \widehat{A}_1(x), x\rangle$, we obtain that $\langle \widehat{A}_1(x_n), x_n\rangle \to \langle \widehat{A}_1(x), x\rangle$. Also $\langle \widehat{a}_0(x_n), x_n\rangle = (\widehat{a}_0(x_n), x_n)_{pq}$. But again by Lemma 3.2 Leray-Lions [18], we have that $\widehat{a}_0(x_n) \xrightarrow{w} \widehat{a}_0(x)$ in $L^q(Z)$. Since $x_n \to x$ in $L^p(Z)$ (by the Sobolev imbedding theorem), we have that $\langle \widehat{a}_0(x_n), x_n\rangle = (\widehat{a}_0(x_n), x_n)_{pq} \to (\widehat{a}_0(x), x)_{pq} = \langle \widehat{a}_0(x), x\rangle$. Therefore finally we have $\widehat{A_2}(x_n) \xrightarrow{w} \widehat{A}_2(x)$ in $W^{-1,q}(Z)$ and $\langle \widehat{A}_2(x_n), x_n\rangle \to \langle \widehat{A}_2(x), x\rangle$ which proves that $\widehat{A}_2$ is generalized pseudomonotone. But $\widehat{A}_2$ is everywhere defined, single-valued and bounded. So from Proposition III.6.11, p. 366 of Hu-Papageorgiou [16], it follows that $\widehat{A}_2$ is pseudomonotone. This proves the claim.

Next, let $U\colon W_0^{1,p}(Z) \to L^q(Z)$ be defined by $U(x)(z) = u(z, x(z))$. From the compact embedding of $W_0^{1,p}(Z)$ in $L^p(Z)$ and Lemma 1 we infer that $U(\cdot)$ is completely continuous (i.e. sequentially continuous from $W_0^{1,p}(Z)$ with the weak topology into $L^q(Z)$ with the strong topology). Therefore $\widehat{A} = \widehat{A}_2 + \varrho U\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is pseudomonotone.

From Lebourg's subdifferential mean value theorem (see Clarke [9], Theorem 2.3.7, p. 41), we have that for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|j(z,x)| \leqslant k(z)|x| + \eta|x|^p$. Thus if we define $\widehat{G}\colon L^p(Z) \to \mathbb{R}$ by $\widehat{G}(x) = \int_Z j(z, x(z))\,\mathrm{d}z$, we have that $\widehat{G}(\cdot)$ is continuous (in fact locally Lipschitz) and convex. Let $G = \widehat{G}|_{W_0^{1,p}(Z)}$. Then from Lemma 2.1 of Chang [8], we have that $\partial G(x) = \partial\widehat{G}(x) \subseteq L^q(Z)$ for all $x \in W_0^{1,p}(Z)$.

Then the auxiliary boundary value problem is equivalent to the abstract operator inclusion

$$\widehat{A}(x) + \partial G(x) \ni \widehat{g}(y)$$

with $\widehat{g}(y)(\cdot) = g(y(\cdot)) \in L^q(Z)$ (see hypothesis $\mathrm{H}_0$).

**Claim 2.** *$x \to \widehat{A}(x) + \partial G(x)$ is coercive from $W_0^{1,p}(Z)$ into $W^{-1,q}(Z)$ for $\varrho > 0$ large.*

To this end, we note that

$$\langle \widehat{A}(x), x\rangle = \langle \widehat{A}_1(x) + \widehat{a}_0(x) + \varrho U(x), x\rangle.$$

From hypothesis $\mathrm{H}(a_k)$(v) we have

$$\langle \widehat{A}_1(x), x\rangle \geqslant c_1\|Dx\|_p^p - \|\gamma_1\|_1 \geqslant c_5\|x\|_{1,p} - c_6, \quad \text{with } c_5, c_6 > 0. \tag{3}$$

Also from hypothesis $\mathrm{H}(a_0)$ (iii) we have

$$\langle \widehat{a}_0(x), x\rangle \geqslant -c_7\|x\|_p\|x\|_{1,p}^{p-1} - c_8\|x\|_p \ \text{ for some } c_7, c_8 > 0. \tag{4}$$

From Young's inequality with $\varepsilon > 0$ we obtain

$$\|x\|_p\|x\|_{1,p}^{p-1} \leqslant \frac{1}{\varepsilon^p p}\|x\|_p^p + \frac{\varepsilon^q}{q}\|x\|_{1,p}^p$$

and so using (4) we have

$$\langle \widehat{a}_0(x), x\rangle \geqslant -c_7\frac{1}{\varepsilon^p p}\|x\|_p^p - c_7\frac{\varepsilon^q}{q}\|x\|_{1,p}^p - c_8\|x\|_p. \tag{5}$$

Finally, from Lemma 1 we have

$$\langle \varrho U(x), x\rangle \geqslant c_9\varrho\|x\|_p^p - c_{10} \ \text{ for some } c_9, c_{10} > 0. \tag{6}$$

From (3), (5) and (6) it follows that

$$\text{(7)} \qquad \langle \widehat{A}(x), x\rangle \geqslant \Big(c_5 - c_7\frac{\varepsilon^q}{q}\Big)\|x\|_{1,p}^p + \Big(c_9\varrho - c_7\frac{1}{\varepsilon^p p}\Big)\|x\|_p^p - c_8\|x\|_p - c_6.$$

Choose $\varepsilon > 0$ such that $c_5 > c_7\frac{\varepsilon^q}{q}$. Then with $\varepsilon > 0$ fixed in this way choose $\varrho > 0$ such that $c_2\varrho > c_7\frac{1}{\varepsilon^p p}$. From (7) it follows that $\widehat{A}$ is coercive.

Moreover, since by hypothesis $\mathrm{H}(\beta)$ we have $0 \in \beta(z, 0)$, it follows that $0 \in \partial G(0)$ and so $\langle x^*, x\rangle \geqslant 0$ for all $x^* \in \partial G(x)$. Thus $\widehat{A} + \partial G$ is coercive ant this proves the claim.

Finally, because $\partial G(\cdot)$ is maximal monotone and $\operatorname{dom}\partial G = X$, we have that $\partial G(\cdot)$ is pseudomonotone. So $\widehat{A}+\partial G$ is pseudomonotone (Claim 1) and coercive (Claim 2). Apply Corollary III.6.30, p. 372, of Hu-Papageorgiou [16] to conclude that $\widehat{A} + \partial G$ is surjective. So there exists $x \in W_0^{1,p}(Z)$ such that $\widehat{A}(x) + \partial G(x) \ni \widehat{g}(y)$. □

Having this auxiliary result, we can now prove the first existence theorem concerning our original problem (1).

**Theorem 3.** *If hypotheses* $\mathrm{H}(a_k)$, $\mathrm{H}(a_0)$, $\mathrm{H}_0$ *and* $\mathrm{H}(\beta)$ *hold, then problem* (1) *has a nonempty solution set.*

P r o o f. We consider the solution multifunction $S\colon K \to 2^{W_0^{1,p}(Z)}$ for the auxiliary problem (2), i.e. for every $y \in K, S(y) \subseteq W_0^{1,p}(Z)$ is the solution set of (2). From Proposition 2 we know that $S(\cdot)$ has nonempty values.

**Claim 1.** $S(K) \subseteq K$.

Let $y \in K$ and let $x \in S(y)$. We have

$$\langle \widehat{A}_2(x), v\rangle + \langle x^*, v\rangle + \varrho\langle U(x), v\rangle = \langle \widehat{g}(y), v\rangle$$

for some $x^* \in \partial G(x)$ and all $v \in W_0^{1,p}(Z)$. Since $\psi \in W^{1,p}(Z)$ is a lower solution, by definition we have

$$\widehat{a}(\psi, v) + \int_Z a_0(z, \psi, D\psi)v(z) + \langle x_1^*, v\rangle \leqslant \langle \widehat{g}(\psi), v\rangle$$

for all $v \in W_0^{1,p}(Z) \cap L^p(Z)_+$ and for some $x_1^* \in L^q(Z)$ with $x_1^*(z) \in \beta(z, \psi(z))$ a.e. on $Z$.

Let $v = (\psi - x)^+ \in W_0^{1,p}(Z) \cap L^p(Z)_+$ (see for example Gilbarg-Trudinger [13], Lemma 7.6, p. 145). From the definition of the convex subdifferential we have

$$\langle x^*, (\psi - x)^+\rangle \leqslant G(x + (\psi - x)^+) - G(x),$$

and

$$\langle x_1^*, (\psi - x)^+\rangle \geqslant G(\psi) - G(\psi - (\psi - x)^+).$$

Using these two inequalities, we obtain

$$\begin{aligned}(8)\quad -\langle \widehat{A}_2(x), (\psi - x)^+\rangle - G(x + (\psi - x)^+) + G(x) - \varrho\langle U(x), (\psi - x)^+\rangle \\ \leqslant -\langle \widehat{g}(y), (\psi - x)^+\rangle\end{aligned}$$

and

$$\begin{aligned}(9)\quad \widehat{a}(\psi, (\psi - x)^+) + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+ \,\mathrm{d}z + G(\psi) - G(\psi - (\psi - x)^+) \\ \leqslant \langle \widehat{g}(\psi), (\psi - x)^+\rangle.\end{aligned}$$

Note that $G(x) + G(\psi) - G(x + (\psi - x)^+) - G(\psi - (\psi - x)^+) = 0$. So adding (8) and (9) we obtain

$$\begin{aligned}(10)\quad \widehat{a}(\psi, (\psi - x)^+) + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+ \,\mathrm{d}z - \langle \widehat{A}_2(x), (\psi - x)^+\rangle \\ - \varrho\langle U(x), (\psi - x)^+\rangle \leqslant \langle \widehat{g}(\psi) - \widehat{g}(y), (\psi - x)^+\rangle.\end{aligned}$$

First we estimate the quantity

$$\widehat{a}(\psi, (\psi - x)^+) + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+ - \langle \widehat{A}_2(x), (\psi - x)^+\rangle.$$

We have

$$\begin{aligned}&\widehat{a}(\psi, (\psi - x)^+) - \langle \widehat{A}_2(x), (\psi - x)^+\rangle + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+ \,\mathrm{d}z \\ &\quad = \int_Z \sum_{k=1}^N (a_k(z, \psi, D\psi) - a_k(z, \tau(x), Dx)) D_k(\psi - x)^+(z) \,\mathrm{d}z \\ &\qquad + \int_Z (a_0(z, \psi, D\psi) - a_0(z, \tau(x), D\tau(x))(\psi - x)^+ \,\mathrm{d}z.\end{aligned}$$

Since
$$D_k(\psi-x)^+(z)=\begin{cases} D_k(\psi-x)(z) & \text{if } x(z)<\psi(z),\\ 0 & \text{if } x(z)\geqslant\psi(z)\end{cases}$$
(see Gilbarg-Trudinger [13]), we have
$$\int_Z\sum_{k=1}^N(a_k(z,\psi,D\psi)-a_k(z,\tau(x),Dx))D_k(\psi-x)^+(z)\,\mathrm{d}z$$
$$=\int_{\{\psi>x\}}\sum_{k=1}^N(a_k(z,\psi,D\psi)-a_k(z,\psi,Dx))D_k(\psi-x)(z)\,\mathrm{d}z\geqslant 0$$
(see hypothesis $\mathrm{H}(a_k)$ (iv)).

Also because $D\tau(x)(z)=\begin{cases} D\varphi(z) & \text{if } \varphi(z)<x(z),\\ Dx(z) & \text{if } \varphi(z)\leqslant x(z)\leqslant\varphi(z),\\ D\psi(z) & \text{if } x(z)<\psi(z),\end{cases}$ we have
$$\int_Z(a_0(z,\psi,D\psi)-a_0(z,\tau(x),D\tau(x))(\psi-x)^+(z)\,\mathrm{d}z$$
$$=\int_{\{\psi>x\}}(a_0(z,\psi,D\psi)-a_0(z,\psi,D\psi))(\psi-x)(z)\,\mathrm{d}z=0.$$
Therefore finally we can write that

$$\text{(11)}\qquad \widehat{a}(\psi,(\psi-x)^+)+\int_Z a_0(z,\psi,D\psi)(\psi-x)^+\,\mathrm{d}z-\langle\widehat{A}_2,(\psi-x)^+\rangle\geqslant 0.$$

Because $g(\cdot)$ is nondecreasing (see hypothesis $\mathrm{H}_0$) and $y\in K$, we have

$$\text{(12)}\qquad \langle\widehat{g}(\psi)-\widehat{g}(y),(\psi-x)^+\rangle=\int_Z(g(\psi(z))-g(y(z)))(\psi-x)^+(z)\,\mathrm{d}z\leqslant 0.$$

Using (11) and (12) in (10), we obtain
$$\varrho\langle U(x),(\psi-x)^+\rangle\geqslant 0$$
$$\Longrightarrow\varrho\int_Z-(\psi-x)^{p-1}(z)(\psi-x)^+(z)\,\mathrm{d}z=-\varrho\int_Z[(\psi-x)^+(z)]^p\,\mathrm{d}z\geqslant 0$$
$$\Longrightarrow\|(\psi-x)^+\|_p=0\ \text{ i.e. }\ \psi\leqslant x.$$

Similarly we show that $x\leqslant\varphi$, hence $x\in K$. This proves the claim.

**Claim 2.** *If $y_1\leqslant x_1\in S(y_1)$ and $y_1\leqslant y_2\in K$, then there exists $x_2\in S(y_2)$ such that $x_1\leqslant x_2$.*

Since $x_1 \in S(y_1) \subseteq K$, we have for some $f_1 \in L^q(Z)$ with $f_1(z) \in \beta(z, x_1(z))$ a.e. on Z the equality

$$\widehat{a}(x_1, v) + \int_Z a_0(z, x_1, Dx_1)v(z)\,\mathrm{d}z + \int_Z f_1(z)v(z)\,\mathrm{d}z = \int_Z g(y_1(z))v(z)\,\mathrm{d}z$$

for all $v \in W_0^{1,p}(Z)$, which implies

$$\widehat{a}(x_1, v) + \int_Z a_0(z, x_1, Dx_1)v(z)\,\mathrm{d}z + \int_Z f_1(z)v(z)\,\mathrm{d}z \leqslant \int_Z g(y_2(z))v(z)\,\mathrm{d}z$$

for all $v \in W_0^{1,p}(Z) \cap L^p(Z)_+$, since $g(\cdot)$ is nondecreasing and $y_1 \leqslant y_2$. Thus $x_1 \in W_0^{1,p}(Z)$ is a lower solution of the problem

$$\left\{\begin{array}{l} A_1(x)(z) + a_0(z, x, Dx) + \beta(z, x(z)) \ni g(y_2(z)), \\ x|_\Gamma = 0. \end{array}\right\} \tag{13}$$

An argument similar to that of Claim 1 gives us a solution $x_2 \in W_0^{1,p}(Z)$ of (13) such that $x_1 \leqslant x_2 \leqslant \varphi$. Note that $\varphi \in W^{1,p}(Z)$ remains an upper solution of (13) since $y_2 \in K$ and so $g(y_2(z)) \leqslant g(\varphi(z))$ a.e. on Z. This proves the claim.

**Claim 3.** *For every $y \in K$, $S(y) \subseteq W_0^{1,p}(Z)$ is weakly closed.*

To this end, let $x_n \in S(y)$, $n \geqslant 1$, and assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$. By definition we have

$$\widehat{A}(x_n) + x_n^* = \widehat{g}(y), \quad n \geqslant 1, \quad \text{with } \; x_n^* \in \partial G(x_n)$$

which implies

$$\langle \widehat{A}(x_n), x_n - x\rangle = (\widehat{g}(y), x_n - x)_{pq} - \langle x_n^*, x_n - x\rangle.$$

From the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have that $x_n \to x$ in $L^p(Z)$ and so $(\widehat{g}(y), x_n - x)_{pq} \to 0$. Also $\{x_n^*\}_{n\geqslant 1} \subseteq L^q(Z)$ is bounded (see the proof of Proposition 2) and so $\langle x_n^*, x_n - x\rangle = (x_n^*, x_n - x)_{pq} \to 0$. Therefore

$$\lim \langle \widehat{A}(x_n), x_n - x\rangle = 0 \Longrightarrow \widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x) \;\text{ in }\; W^{-1,q}(Z)$$

(since $\widehat{A}$ is bounded, pseudomonotone).

Also we may assume that $x_n^* \xrightarrow{w} x^*$ in $L^q(Z)$. Since $[x_n, x_n^*] \in \operatorname{Gr} \partial G = \operatorname{Gr} \partial \widehat{G} \cap (W_0^{1,p}(Z) \times L^q(Z))$ (see the proof of Proposition 2 and Chang [8], Lemma 2.1) and $\operatorname{Gr} \partial \widehat{G}$ is demiclosed, we conclude that $x^* \in \partial G(x)$. Thus finally we have

$$\widehat{A}(x) + x^* = \widehat{g}(y), \quad \text{with} \quad x^* \in \partial G(x),$$

which implies $x \in S(y)$, which proves the claim.

Claims 1, 2 and 3 and the fact the $W^{1,p}(Z)$ is separable, permit the application of Proposition 2.4 of Heikkila-Hu [15], which yields $x \in S(x)$ (a fixed point of $S(\cdot)$). Evidently this is a weak solution of problem (1). □

**Remark.** In fact, with a little additional effort we can show that the result is still valid, if we assume that there exists $M \geqslant 0$ such that $x \to g(x) + Mx$ is nondecreasing. However, to simplify our presentation we have decided to proceed with the stronger hypothesis that $g(\cdot)$ is nondecreasing. Moreover, it is clear from our proof that if $a\colon Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is defined by $a(z,x,y) = (a_k(z,x,y))_{k=1}^N$ and $x \in W_0^{1,p}(Z)$ is a solution of (1), then $-\operatorname{div} a(z,x,Dx) \in L^q(Z)$ and

$$\begin{cases} -\operatorname{div} a(z,x(z),Dx(z)) + a_0(z,x(z),Dx(z)) + f(z) = g(x(z)) \text{ a.e. on } Z, \\ x|_\Gamma = 0 \end{cases}$$

with $f \in L^q(Z)$, $f(z) \in \beta(z,x(z))$ a.e. on $Z$ (i.e. $x$ is a strong solution).

For a particular version of problem (1) we can show the existence of extremal solutions in the order interval $K$, i.e. of solutions $x_l$, $x_u$ in $K$ such that for every solution $x \in K$ we have $x_l \leqslant x \leqslant x_u$.

So let $A_3 x(z) = -\sum\limits_{k=1}^N D_k a_k(z,Dx)$ (second order nonlinear differential operator in divergence form) and consider the boundary value problem

$$\left\{ \begin{aligned} & A_3(x)(z) + a_0(z,x(z)) + \beta(z,x(z)) \ni g(x(z)) \text{ on } Z, \\ & x|_\Gamma = 0. \end{aligned} \right\} \tag{14}$$

The hypotheses on the functions $a_k$ and $a_0$ are the following:

H$(\alpha_k)'$: $a_k\colon Z \times \mathbb{R}^N \to \mathbb{R}$, $k \in \{1,2,\dots,N\}$, are functions such that
(i) for all $y \in \mathbb{R}^N$, $z \to a_k(z,y)$ is measurable;
(ii) for almost all $z \in Z$, $y \to a_k(z,y)$ is continuous;
(iii) for almost all $z \in Z$ and all $y \in \mathbb{R}^N$, we have

$$|a_k(z,y)| \leqslant \gamma(z) + c\|y\|^{p-1}$$

with $\gamma \in L^q(Z)$, $c > 0$, $1 < p < \infty$ and $1/p + 1/q = 1$;

(iv) for almost all $z \in Z$ and all $y, y' \in \mathbb{R}^N$, $y \neq y'$, we have

$$\sum_{k=1}^{N} (a_k(z,y) - a_k(z,y'))(y_k - y'_k) > 0;$$

(v) for almost all $z \in Z$ and all $y \in \mathbb{R}^N$, we have

$$\sum_{k=1}^{N} a_k(z,y) y_k \geqslant c_1 \|y\|^p - \gamma_1(z)$$

with $c_1 > 0$, $\gamma_1 \in L^1(Z)$.

$\mathrm{H}(\alpha_0)'$: $a_0\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to a_0(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to a_0(z,x)$ is continuous, nondecreasing;

(iii) for almost all $z \in Z$ and all $x \in [\psi(z), \varphi(z)]$, we have $|a_0(z,x)| \leqslant \gamma_2(z)$ with $\gamma_2 \in L^q(Z)$.

Then we can prove the following result

**Proposition 4.** *If hypotheses* $\mathrm{H}(a_k)'$, $\mathrm{H}(a_0)'$, $\mathrm{H}(\beta)$ *and* $\mathrm{H}_0$ *hold, then problem* (14) *has extremal solutions in the order interval* $K$.

P r o o f. Hypotheses $\mathrm{H}(a_k)'$ and $\mathrm{H}(a_0)'$ imply that the map $S\colon K \to K$ is actually single-valued. Also we claim that it is increasing with respect to the induced partial order on $K$. Indeed, let $y_1, y_2 \in K$, $y_1 \leqslant y_2$ and let $x_1 = S(y_1)$, $x_2 = S(y_2)$. We have

$$\widehat{A}(x_1) + x_1^* = \widehat{g}(y_1)$$

and

$$\widehat{A}(x_2) + x_2^* = \widehat{g}(y_2)$$

with $x_i^* \in \partial G(x_i)$, $i = 1, 2$.

Using $(x_1 - x_2)^+ \in W_0^{1,p}(Z) \cap L^p(Z)_+$ as our test function, we have

$$\begin{aligned} \langle \widehat{A}(x_1) - \widehat{A}(x_2), (x_1 - x_2)^+ \rangle &= \langle x_1^* - x_2^*, (x_1 - x_2)^+ \rangle \\ &= \langle \widehat{g}(y_1) - \widehat{g}(y_2), (x_1 - x_2)^+ \rangle. \end{aligned} \tag{15}$$

By virtue of hypotheses $\mathrm{H}(a_k)'$ and $\mathrm{H}(a_0)'$ (ii), we have

$$\langle \widehat{A}(x_1) - \widehat{A}(x_2), (x_1 - x_2)^+ \rangle \geqslant 0 \ \text{ (strictly if } x_1 \neq x_2). \tag{16}$$

Also from the monotonicity of the subdifferential, we have

$$\langle x_1^* - x_2^*, (x_1 - x_2)^+ \rangle = (x_1^* - x_2^*, (x_1 - x_2)^+)_{pq} \geqslant 0. \tag{17}$$

Finally, since by hypothesis $\mathrm{H}_0$, $g(\cdot)$ is nondecreasing, it follows that

$$\langle \widehat{g}(y_1) - \widehat{g}(y_2), (x_1 - x_2)^+ \rangle = (\widehat{g}(y_1) - \widehat{g}(y_2), (x_1 - x_2)^+)_{pq} \leqslant 0. \tag{18}$$

Using (16), (17) and (18) in (15), we infer that $(x_1 - x_2)^+ = 0$, hence $x_1 \leqslant x_2$. This proves the claim. Using Corollary 1.5 of Amann [2], we infer that $S(\cdot)$ has extremal fixed points in $K$. Clearly these are the extremal solutions of (14) in $K$. $\square$

Now we will consider a multivalued nonlinear elliptic problem, with a $\beta(\cdot)$ such that $\operatorname{dom} \beta \neq \mathbb{R}$. This case is important because it covers variational inequalities.

So now we examine the following boundary value problem:

$$\left\{ \begin{array}{l} A_1(x)(z) + a_0(z, x(z)) + \beta(x(z)) \ni g(z) \ \text{ on } \ Z, \\ x|_\Gamma = 0. \end{array} \right\} \tag{19}$$

Our hypotheses on $a_0$ and $\beta$ are the following:

$\mathrm{H}(\alpha_0)''$: $a_0\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that
(i) for all $x \in \mathbb{R}$, $z \to a_0(z, x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to a_0(z, x)$ is continuous, nondecreasing;
(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have $|a_0(z, x)| \leqslant \gamma_2(z) + c_2|x|$ with $\gamma_2 \in L^q(Z), c_2 > 0$.

$\mathrm{H}(\beta)_1$: $\beta\colon \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone map with $0 \in \beta(0)$.

**Theorem 5.** *If hypotheses* $\mathrm{H}(a_k)$, $\mathrm{H}(a_0)''$, $\mathrm{H}(\beta)_1$ *hold and* $g \in L^p(Z)$, *then the solution set of problem* (19) *is nonempty.*

Proof. Recall that $\beta = \partial j$ with $j \in \Gamma_0(\mathbb{R})$. Let $\beta_\varepsilon = \frac{1}{\varepsilon}(1 - (1 + \varepsilon\beta)^{-1})$, $\varepsilon > 0$, be the Yosida approximation of $\beta(\cdot)$ and consider the following approximation of problem (19):

$$\left\{ \begin{array}{l} \widehat{A}(x_1) - a_0(z, x(z)) + \beta_\varepsilon(x(z)) = g(z) \ \text{ on } \ Z, \\ x|_\Gamma = 0. \end{array} \right\} \tag{20}$$

As before let $\widehat{a}\colon W_0^{1,p}(Z) \times W_0^{1,p}(Z)$ be the semilinear form defined by

$$\widehat{a}(x, y) = \int_Z \sum_{k=1}^N a_k(z, x, Dx) D_k y(z) \, \mathrm{d}z$$

and let $\widehat{A}_1\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by

$$\langle \widehat{A}_1(x), y\rangle = \widehat{a}(x,y) \ \text{ for all } \ x, y \in W_0^{1,p}(Z).$$

Also let $\widehat{a}_0\colon L^p(Z) \to L^q(Z)$ be the Nemitsky operator corrresponding to $a_0$, i.e. $\widehat{a}_0(x)(\cdot) = a_0(\cdot, x(\cdot))$ (in fact note that by H$(a)''$ (iii), $\widehat{a}(x) \in L^p(Z) \subseteq L^q(Z)$ since $p \geqslant 2 \geqslant q$).

From Theorem 3.1 of Gossez-Mustonen [14] we know that $\widehat{A}_1$ is pseudomonotone, while exploiting the compact embedding of $W_0^{1,p}(Z)$ in $L^p(Z)$, we can easily see that $\widehat{a}_0|_{W_0^{1,p}}$ is completely continuous. Therefore $\widehat{A}_2 = \widehat{A}_1 + \widehat{a}_0$ is pseudomonotone.

Let $G_\varepsilon\colon W_0^{1,p}(Z) \to \mathbb{R}$ be the integral functional defined by $G_\varepsilon(x) = \int_Z j_\varepsilon(x(z))\,\mathrm{d}z$ with $j_\varepsilon(r)$ being the Moreau-Yosida regularization of $j(\cdot)$ (see for example Hu-Papageorgiou [16], Definition III.4.30, p. 349). We know that $G_\varepsilon(\cdot)$ is Gateaux differentiable and $\partial G_\varepsilon(x) = \partial j_\varepsilon(x(\cdot))$ (see Hu-Papageorgiou [16], Proposition III.4.32, p. 350). Then problem (20) is equivalent to the operator equation

$$\widehat{A}_2(x) + \partial G_\varepsilon(x) = g. \tag{21}$$

Note that $\partial G_\varepsilon$ is maximal monotone, with $\operatorname{dom}\partial G_\varepsilon = W_0^{1,p}(Z)$. Therefore $\partial G_\varepsilon$ is pseudomonotone and hence so is $\widehat{A}_2 + \partial G_\varepsilon$. We will show that $\widehat{A}_2 + \partial G_\varepsilon$ is coercive. Since $0 = G_\varepsilon(0)$ and $\langle \partial G_\varepsilon(x), x\rangle \geqslant 0$, to establish the desired coercivity of $\widehat{A}_2 + \partial G_\varepsilon$ it suffices to show that $\widehat{A}_2$ is coercive. To this end we have

$$\langle \widehat{A}_2, x\rangle \geqslant c_1\|Dx\|_p^p - \|\gamma_1\|_1 + \int_Z a_0(z, x(z))x(z)\,\mathrm{d}a.$$

Since $a_0(z,\cdot)$ is nondecreasing (hypothesis H$(a)''$ (ii)) we have $(a_0(z,x(z)) - a_0(z,0))x(z) \geqslant 0$ a.e on $\mathbb{R}$ and so

$$\begin{aligned}\int_Z a_0(z,x(z))x(z)\,\mathrm{d}z &= \int_Z (a_0(z,x(z)) - a_0(z,0))x(z)\,\mathrm{d}z + \int_Z a_0(z,0)x(z)\,\mathrm{d}z\\ &\geqslant \int_Z a_0(z,0)x(z)\,\mathrm{d}z.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}\langle \widehat{A}_2(x), x\rangle &\geqslant c_1\|Dx\|_p^p - \|\gamma_1\|_1 + \int_Z a_0(z,0)x(z)\,\mathrm{d}z\\ &\geqslant c_1\|Dx\|_p^p - \|\gamma_1\|_1 - \|a_0(\cdot,0)\|_q\|x\|_p,\end{aligned}$$

from which we infer the coercivity of $x \to (\widehat{A}_2 + \partial G_\varepsilon)(x)$. Thus Corollary III.6.30, p. 372, of Hu-Papageorgiou [16] implies that there exists $x_\varepsilon \in W_0^{1,p}(Z)$ which solves (21). Now let $\varepsilon_n \downarrow 0$ and set $x_n = x_{\varepsilon_n} n \geqslant 1$. We will derive some uniform bounds for the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$. To this end, note that

$$\int_Z \sum_{k=1}^N a_k(z, x_n, Dx_n) D_k x_n(z)\,\mathrm{d}z + \int_Z a_0(z, x_n) x_n(z)\,\mathrm{d}z + \int_Z \beta_{\varepsilon_n}(x_n) x_n(z)\,\mathrm{d}z$$
$$= \int_Z g(z) x_n(z)\,\mathrm{d}z$$

implies $c_1 \|Dx_n\|_p^p - \|\gamma_1\|_1 - \|a_0(\cdot, 0)\|_q \|x_n\|_p \leqslant \|g\|_q \|x_n\|_p$ (since $\beta_\varepsilon(x_n(z)) x_n(z) \geqslant 0$ a.e on $Z$).

From this inequality we deduce that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Also note that $\eta_n(r) = |\beta_{\varepsilon_n}(r)|^{p-2} \beta_{\varepsilon_n}(r)$ is locally Lipschitz on $\mathbb{R}$ and $\eta_n(0) = 0$. So from Marcus-Mizel [20] we know that $\eta_n(x_n(\cdot)) \in W_0^{1,p}(Z)$, $n \geqslant 1$. Using this as our test function, we obtain

$$\int_Z \sum_{k=1}^N a_k(z, x_n, Dx_n) D_k \eta_n(x_n)\,\mathrm{d}z + \int_Z a_0(z, x_n) \eta_n(x_n)\,\mathrm{d}z + \int_Z |\beta_{\varepsilon_n}(x_n)|^p\,\mathrm{d}z = \int_Z g(z) \eta_n(x_n(z))\,\mathrm{d}z. \tag{22}$$

Note that $D_k \eta_n(x_n(z)) = (p-1)|\beta_{\varepsilon_n}(x_n(z))|^{p-2} \beta'_{\varepsilon_n}(x_n(z)) D_k x_n(z)$ a.e. on $Z$ (see Marcus-Mizel [20], and recall that $\beta_{\varepsilon_n}(\cdot)$ being Lipschitz is differentiable almost everywhere). Since $\beta_{\varepsilon_n}(\cdot)$ is nondecreasing, $(p-1)|\beta_{\varepsilon_n}(x_n(z))|^{p-2} \beta'_{\varepsilon_n}(x_n)) \geqslant 0$ a.e. on $Z$. Thus using hypothesis H$(a_k)$(v) we have

$$\sum_{k=1}^N a_k(z, x_n, Dx_n) D_k \eta_n(x_n)\,\mathrm{d}z \geqslant -\|\gamma_1\|_1.$$

Moreover, from hypothesis H$(a_0)''$ (iii) we have $a_0(\cdot, x_n(\cdot)) \in L^p(Z)$. In addition, since $\beta_{\varepsilon_n}(\cdot)$ is $\varepsilon_n^{-1}$-Lipschitz and $0 = \beta_{\varepsilon_n}(0)$, we have $|\beta_{\varepsilon_n}(r)| \leqslant \varepsilon_n^{-1}|r|$, which implies that $|\beta_{\varepsilon_n}(x(\cdot))| \in L^q(Z)$. So by Hölder's inequality we have

$$\int_Z a_0(z, x_n(z)) \eta_n(x(z))\,\mathrm{d}z \geqslant -\|\widehat{a}_0(x_n)\|_p \|\beta_{\varepsilon_n}(x_n)\|_p^{p-1}.$$

But since $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded, we have $\sup\limits_{n\geqslant 1} \|\widehat{a}_0(x_n)\|_p \leqslant M_1$ (see hypothesis H$(a_0)''$ (iii)). So we obtain

$$\int_Z a_0(z, x_n(z)) \eta_n(x(z))\,\mathrm{d}z \geqslant -M_1 \|\beta_{\varepsilon_n}(x_n)\|_p^{p-1}.$$

Returning to (22), we can write

$$\|\beta_{\varepsilon_n}(x_n)\|_p^p - M_1\|\beta_{\varepsilon_n}(x_n)\|_p^{p-1} \leqslant \|g\|_p\|\beta_{\varepsilon_n}(x_n)\|_p^{p-1} + \|\gamma_1\|_1$$

which implies that $\{\beta_{\varepsilon_n}(x_n)\}_{n\geqslant 1} \subseteq L^p(Z)$ is bounded, hence it is bounded also in $L^2(Z)$.

Hence by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $\beta_{\varepsilon_n}(x_n) \xrightarrow{w} v^*$ in $L^2(Z)$ as $n \to \infty$.

Also we have

$$\langle \widehat{A}(x_n), x_n - x_n\rangle + \langle \beta_{\varepsilon_n}(x_n), x_n - x\rangle = (g, x_n - x)_{pq}.$$

Exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we obtain

$$\limsup\langle \widehat{A}_2(x_n), x_n - x\rangle \leqslant 0$$
$$\text{consequently } \ \widehat{A}_2(x_n) \xrightarrow{w} \widehat{A}_2(x) \ \text{ in } \ W^{-1,q}(Z)$$

(recall that $\widehat{A}_2$ is pseudomonotone and bounded). Hence in the limit as $n \to \infty$ we have $\widehat{A}_2(x) + v^* = g$ in $W^{-1,q}(Z)$. Let $\widehat{\beta}\colon L^2(Z) \to 2^{L^2(Z)}$ be defined by

$$\widehat{\beta}(x) = \{u \in L^2(Z)\colon\ u(z) \in \beta(x(z)) \ \text{ a.e. on } \ Z\}.$$

We know that $\widehat{\beta}$ is maximal monotone (see Hu-Papageorgiou [16], p. 328). Using Proposition III.2.29, p. 325, of Hu-Papageorgiou [16], we conlcude that $v^* \in \widehat{\beta}(x)$ and so $v^*(z) \in \beta(x(z))$. So $x \in W_0^{1,p}(Z)$ is a solution of (19). □

## 4. Existence results with nonmonotone nonlinearities

In this section we examine a quasilinear elliptic problem with a multivalued nonmonotone nonlinearity. The problem that we study is a hemivariational inequality. Hemivariational inequalities are a new type of variational inequalities, where the convex subdifferential is replaced by the subdifferential in the sense of Clarke [9] of a locally Lipschitz function. Such inequalities are motivated by problems in mechanics, where the lack of convexity does not permit the use of the convex superpotential of Moreau [21]. Concrete applications to problems of mechanics and engineering can be found in the book of Panagiotopoulos [22]. Our formulation incorporates also the case of elliptic boundary value problems with discontinuous nonlinearities. Such problems have been studied (primarily for semilinear systems) by Ambrosetti-Badiale [4], Ambrosetti-Turner [5], Badiale [6], Chang [8] and Stuart [23].

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a $C^1$-boundary $\Gamma$. We start with a few remarks concerning the first eigenvalue of the negative $p$-Laplacian $-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2}Dx)$, $1 < p < \infty$, with Dirichlet boundary conditions. We consider the following nonlinear eigenvalue problem:

$$
\left\{
\begin{array}{l}
-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda |x(z)|^{p-2}x(z) \ \text{ a.e. on } Z, \\
x|_\Gamma = 0.
\end{array}
\right\}
\tag{23}
$$

The least $\lambda \in \mathbb{R}$ for which (20) has a nontrivial solution is called the first eigenvalue of $-(\Delta_p, W_0^{1,p}(Z))$. From Lindqvist [19] we know that $\lambda_1 > 0$ is isolated and simple. Moreover, $\lambda_1 > 0$ is characterized via the Rayleigh quotient, namely

$$
\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} \colon\ x \in W_0^{1,p}(Z),\ \ x \neq 0\right].
$$

This minimum is realized at the normalized first eigenfunction $u_1$, which we know to be positive, i.e. $u_1(z) > 0$ a.e. on $Z$ (note that by nonlinear elliptic regularity theory $u_1 \in C^{1,\beta}_{\mathrm{loc}}(Z)$, $0 < \beta < 1$; see Tolksdorf [24]).

We consider the following nonlinear eigenvalue problem:

$$
\left\{
\begin{array}{l}
-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \lambda \partial j(z, x(z)) \ \text{ a.e. on } Z, \\
x|_\Gamma = 0, 1 < p < \infty,\ \ \lambda > 0.
\end{array}
\right\}
\tag{24}
$$

Our approach to problem (24) will be variational, based on the critical point theory for nonsmooth locally Lipschitz functionals, due to Chang [8]. In this case the classical Palais-Smale condition (PS-condition for short) takes the following form. Let $X$ be a Banach space and $f\colon X \to \mathbb{R}$ a locally Lipschitz function. We say that $f(\cdot)$ satisfies the nonsmooth PS-condition, if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ for which $\{f(x_n)\}_{n\geqslant 1}$ is bounded and $m(x_n) = \min\{\|x^*\|\colon x^* \in \partial f(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. When $f \in C^1(X)$, we know that $\partial f(x_n) = \{f'(x_n)\}$ and so we see that the above definition of the PS-condition coincides with the classical one.

Our hypotheses on the function $j(z, r)$ in problem (24) are the following:

H(j): $j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $x \to j(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $v \in \partial j(z, x)$, we have

$$
|v| \leqslant c_1(1 + |x|^{r-1})
$$

with $c_1 > 0$, $1 \leqslant r < p$;

(iv) $j(\cdot,0) \in L^\infty(Z)$, $\int_Z j(z,0)\,\mathrm{d}z = 0$ and there exists $x_0 \in \mathbb{R}$ such that $j(z,x_0) > 0$ for almost all $z \in Z$;

(v) $\limsup\limits_{x\to 0} pj(z,x)/|x|^p < 0$ uniformly for almost all $z \in Z$.

We will need the following nonsmooth variant of the classical "Mountain Pass theorem". The result is due to Chang [8].

**Theorem 6.** *If $X$ is a reflexive Banach space, $V\colon X \to \mathbb{R}$ is a locally Lipschitz functional which satisfies the* (PS)*-condition and for some $r > 0$ and $y \in X$ with $\|y\| > r$ we have*

$$\max[V(0), V(y)] < \inf[V(x)\colon \|x\| = r]$$

*then there exists a nontrivial critical point $x \in X$ of $V$ (i.e. $0 \in \partial V(x)$) such that the critical value $c = V(x)$ is characterized by the minimax principle*

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant\tau\leqslant 1} V(\gamma(\tau))$$

*where $\Gamma = \{\gamma \in C([0,1],X)\colon \gamma(0) = 0,\ \gamma(1) = y\}$.*

We have the following multiplicity result for problem (1).

**Theorem 7.** *If hypotheses* H(j) *hold, then there exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ problem* (24) *has at least two nontrivial solutions.*

Proof. For $\lambda > 0$, let $V_\lambda\colon W_0^{1,p}(Z) \to \mathbb{R}$ be defined by

$$V_\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \lambda \int_Z j(z,x(z))\,\mathrm{d}z.$$

We know that $V_\lambda$ is locally Lipschitz (see Clarke [9]).

**Claim 1.** *$V_\lambda$ satisfies the nonsmooth* (PS)*-condition.*

Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, be such that $|V_\lambda(x_n)| \leqslant M_1$ for all $n \geqslant 1$ and $m(x_n) \to 0$ as $n \to \infty$. Let $x_n^* \in \partial V_\lambda(x_n)$ be such that $m(x_n) = \|x_n^*\|$ for all $n \geqslant 1$. Its existence follows from the fact that $\partial V_\lambda(x_n)$ is $w$-compact and the norm functional is weakly lower semicontinuous. We have

$$x_n^* = A(x_n) - \lambda v_n^*, \quad n \geqslant 1.$$

Here $A\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx,Dy)_{\mathbb{R}^N}\,\mathrm{d}z$$

for all $x, y \in W_0^{1,p}(Z)$ and $v_n^* \in \partial\psi(x_n)$ where $\psi(x) = \int_Z j(z, x(z))\,\mathrm{d}z$. It is easy to see that $A$ is monotone, demicontinuous, thus maximal monotone.

From the Lebourg mean value theorem (see Clarke [9], Theorem 2.3.7, p. 41), we know that there exists $v^* \in \partial j(z, \eta x)$, $0 < \eta < 1$ such that $j(z, x) - j(z, 0) = v^* x$. Using this together with hypothesis H(j) (iii) and the fact that $j(\cdot, 0) \in L^\infty(Z)$, we can write that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have $|j(z, x)| \leqslant \beta_1 + \beta_2 |x|^r$ with $\beta_1, \beta_2 > 0$. Hence we have

$$\begin{aligned} M_1 \geqslant V_\lambda(x_n) &= \frac{1}{p}\|Dx_n\|_p^p - \lambda \int_Z j(z, x_n(z))\,\mathrm{d}z \\ &\geqslant \frac{1}{p}\|Dx_n\|_p^p - \lambda\beta_1|Z| - \lambda\beta_3\|x_n\|_p^r \text{ for some } \beta_3 > 0. \end{aligned}$$

Here $|Z|$ denotes the Lebesgue measure of the domain $Z \subseteq \mathbb{R}^N$. Using Young's inequality with $\varepsilon > 0$, we have

$$\lambda\beta_3\|x_n\|_p^r \leqslant M_8 + \varepsilon\|x_n\|_p^p$$

for some $M_8 > 0$. Let $\varepsilon < \frac{\lambda_1}{p}$. We have

$$\begin{aligned} M_1 \geqslant V_\lambda(x_n) &\geqslant \frac{1}{p}\|Dx_n\|_p^p - \lambda\beta_1|Z| - M_\varepsilon - \varepsilon\|x_n\|_p^p \\ &\geqslant \Big(\frac{1}{p} - \frac{\varepsilon}{\lambda_1}\Big)\|Dx_n\|_p^p - \lambda\beta_1|Z| - M_\varepsilon \text{ (Rayleigh quotient).} \end{aligned} \tag{25}$$

Since $1/p - \varepsilon/\lambda_1 > 0$ (recall the choice of $\varepsilon > 0$), the above inequality implies that $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and so $x_n \to x$ in $L^p(Z)$ as $n \to \infty$. We have

$$\langle A(x_n), x_n - x\rangle = \lambda\langle v_n^*, x_n - x\rangle.$$

From Theorem 2.2 of Chang [8] we have that $\{v_n^*\}_{n \geqslant 1} \subseteq L^q(Z)$ and is bounded. So we have

$$\lim\langle A(x_n), x_n - x\rangle = \lim \lambda(v_n^*, x_n - x)_{pq}.$$

Since $A$ is maximal monotone, we have that $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$ implies $\|Dx_n\|_p \to \|Dx\|_p$.

Since $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and $L^p(Z, \mathbb{R}^N)$ is uniformly convex, from the Kadec-Klee property (see Hu-Papageorgiou [16], Definition I.1.72 and Lemma I.1.74, p. 28) it follows that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, hence $x_n \to x$ in $W_0^{1,p}(Z)$. This proves the claim.

From (25) we have that $V_\lambda(\cdot)$ is coercive. This combined with claim 1, allows the use of Theorem 3.5 of Chang [8], which gives $y_1 \in W_0^{1,p}(Z)$ such that $0 \in \partial V_\lambda(y_1)$ and

$$c_\lambda = \inf_{W_0^{1,p}(Z)} V_\lambda = V_\lambda(y_1).$$

From hypothesis H(j) (iv) for $\widehat{x} = x_0$ we have $\widehat{\psi}(\widehat{x}) > 0$ where $\widehat{\psi}\colon L^r(Z) \to \mathbb{R}$ is defined by $\widehat{\psi}(y) = \int_Z j(z, y(z))\,\mathrm{d}z$. Evidently $\widehat{\psi}$ is locally Lipschitz and $\widehat{\psi}|_{W_0^{1,p}(Z)} = \psi$. Since $W_0^{1,p}(Z)$ is embedded continuously and densely in $L^r(Z)$, the continuity of $\widehat{\psi}$, implies that we can find $x \in W^{1,p}(Z)$ such that $\widehat{\psi}(x) = \psi(x) > 0$. Hence there exists $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ we have $V_\lambda(y_1) = \frac{1}{p}\|Dy\|_p^p - \lambda\psi(y_1) < 0 = V_\lambda(0)$. So $y_1 \neq 0$.

**Claim 2.** *There exists* $r > 0$ *such that* $\inf[V_\lambda(x) : \|x\| = r] > 0$.

By virtue of hypothesis H(j) (v), we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \leqslant \delta$ we have for some $\gamma < 0$

$$j(z, x) \leqslant \frac{\gamma |x|^p}{p}.$$

Also recall that $j(z, x) \leqslant \beta_1 + \beta_2 |x|^r$. Thus we can find $\beta_4 > 0$ large enough such that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$j(z, x) \leqslant \frac{\gamma |x|^p}{p} + \beta_4 |x|^s \ \text{ with } \ p < s \leqslant p^* = \frac{Np}{N - p}.$$

Therefore, we can write

$$V_\lambda(x) \geqslant \frac{1}{p}\Big(1 - \frac{\lambda\gamma}{\lambda_1}\Big)\|Dx\|_p^p - \lambda\beta_5\|Dx\|_p^s \ \text{ for some } \ \beta_5 > 0.$$

Note that $\big(1 - \lambda\gamma/\lambda_1\big) > 0$ (since $\gamma < 0$ and $0 < \lambda_0 \leqslant \lambda$, $\lambda_1 > 0$). Thus for every $\lambda \geqslant \lambda_0 > 0$ we can find $\|y\|_1 \varrho > 0$ (depending in general on $\lambda$) such that $\inf[V_\lambda(x)\colon \|x\| = \varrho] > 0$. Then $V_\lambda(y_1) < V_\lambda(0) < \inf[V_\lambda(x)\colon \|x\| = \varrho]$ and so we can apply Theorem 6 and obtain $y_2 \neq 0$, $y_2 \neq y_1$ such that $0 \in \partial V_\lambda(y_2)$.

Now let $y = y_1$ or $y = y_2$. From $0 \in \partial v_\lambda(y)$ we have

$$A(y) = \lambda v^*$$

for some $v^* \in \partial\psi(y)$.

From Clarke [9] we know that $v^* \in L^q(Z)$ and $v^*(z) \in \partial j(z, y(z))$ a.e. on $Z$. From the representation theorem for the elements in $W^{-1,q}(Z)$ (see Adams [1], Theorem 3.10, p. 50) we have that $\operatorname{div}(\|Dy\|^{p-1}Dy) \in W^{-1,q}(Z)$. So we have for all $u \in W_0^{1,p}(Z)$

$$\langle A(y), u\rangle = \langle -\operatorname{div}(\|Dy\|^{p-2}Dy), u\rangle = \lambda(v^*, u)_{pq},$$

consequently

$$-\operatorname{div}(\|Dy(z)\|^{p-2}Dy(z) = \lambda v^*(z) \in \lambda\partial j(z, y(z)) \text{ a.e.}$$

and hence $y_1$, $y_2$ are distinct, nontrivial solutions of (24). □

**Remark.** Our theorem extends Theorem 3.5 of Chang [8], who studies a semilinear problem and proves the existence of one solution for some $\lambda \in \mathbb{R}$. Moreover, in Chang $j(z, x) = \int_0^x h(z, s)\,\mathrm{d}s$. Our result also extends Theorem 5.35 of Ambrosetti-Rabinowitz [2] to nonlinear problems with multivalued terms.

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