P. J. de Paepe

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EXAMPLES OF INFINITELY GENERATED FUNCTION ALGEBRAS

P. J. de Paepe, Amsterdam

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Abstract. Examples of non-finitely generated function algebras on planar sets with small maximal ideal spaces are given.

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1. Introduction

In this paper we are interested in function algebras on a compact Hausdorff space X, i.e. subalgebras of the Banach algebra $C(X)$ (provided with the supremum norm on X) of continuous complex-valued functions on X which separate points of X, contain the constant functions and are closed in $C(X)$. If $f_1, \ldots, f_n$ are continuous on X, then $[f_1, \ldots, f_n; X]$ denotes the smallest uniformly closed algebra of functions on X generated by $f_1, \ldots, f_n$ and the constant functions. Note that $[f_1, \ldots, f_n; X]$ is a function algebra on X if $f_1, \ldots, f_n$ separate the points of X. The finitely generated function algebras are (isomorphic to) the algebras of uniform limits of analytic polynomials on compact subsets of $\mathbb{C}^n$. As far as we know, not much is known about non-finitely generated algebras. For instance one might think that the maximal ideal space of such an algebra should be big. This is not necessarily the case. We give examples of infinitely generated algebras which have a disk as its maximal ideal space.

By D we denote the closed unit disk in the complex plane and by Γ its boundary. The algebra $A(D)$ is the disk algebra and consists of all continuous functions on D which are analytic on the interior of D.

For more facts on function algebras, consult [1] and [5].

The main idea for our examples is the simple linear algebraic proof of the following two propositions.

2. Propositions

Proposition 1. *Let $B = \{f \in A(D)\colon\ f'(0) = \ldots = f^{(n)}(0) = 0\}$, which is a function algebra on D. Then the number of generators of B is at least $n+1$.*

Proof. Let $f_1, \ldots, f_n$ be elements of B, let $C = [f_1, \ldots, f_n; D]$ and define a linear map $\Lambda\colon\ B \longrightarrow \mathbb{C}^{n+1}$ by

$$\Lambda(f) = (f^{(n+1)}(0), \ldots, f^{(2n+1)}(0)).$$

Note that $\Lambda(C)$ equals the linear span of $\Lambda(f_1), \ldots, \Lambda(f_n)$. Also $\Lambda(B)$ equals $\mathbb{C}^{n+1}$. This can be seen using elements of B of the form $\sum\limits_{k=n+1}^{2n+1} \alpha_k z^k$. Hence $C \neq B$. $\square$

Proposition 2. *Let $a_1, \ldots, a_{n+1}$ be different points of the interior of D. Let $B = \{f \in A(D)\colon\ f(a_1) = \ldots = f(a_{n+1})\}$ (which can be considered as a function algebra on Γ). Then the number of generators of B is at least $n+1$.*

Proof. Let $f_1, \ldots, f_n$ be elements of B, let $C = [f_1, \ldots, f_n; D]$ and define a linear map $\Lambda\colon\ B \longrightarrow \mathbb{C}^{n+1}$ by

$$\Lambda(f) = (f'(a_1), \ldots, f'(a_{n+1})).$$

Again $\Lambda(C)$ equals the linear span of $\Lambda(f_1), \ldots, \Lambda(f_n)$.

Using elements of B of the form $p(z)\Pi_{k=1}^{n+1}(z - a_k)$ where p is a polynomial it follows that $\Lambda(B)$ equals $\mathbb{C}^{n+1}$, hence $C \neq B$. $\square$

3. An example

The maximal ideal space ΔB of the algebra B in Proposition 1 coincides with D. This follows from the fact that the kernel of a continuous point derivation of a function algebra is a function algebra itself with the same maximal ideal space. The maximal ideal space ΔB of the algebra B in Proposition 2 is gotten from the disk D by pinching together the points $a_1, \ldots, a_{n+1}$ (see [3] for a discussion on these facts).

Theorem. *There exists a function algebra on the unit disk D in the complex plane with maximal ideal space D which is not finitely generated.*

P r o o f. Choose a sequence of closed disks $D_1, D_2, \dots$ (with centres $a_1, a_2, \dots$) in the interior of D, mutually disjoint, which converge to a point. Let B_n be the function algebra on D_n consisting of all functions in the disk algebra on the disk D_n with vanishing derivatives of order up to n at a_n. Let B consist of all continuous functions f on D such that its restriction $f_{|D_n}$ belongs to B_n for all n.

The maximal sets of antisymmetry of B are the disks D_n and the singletons outside the sets D_n. Also, if $\varphi \in \Delta B$, then $|\varphi(f)| \leqslant \|f\|_L$ for all $f \in B$ and for some maximal set L of antisymmetry. Since $B_{|D_n} = B_n$ and $\Delta(B_n) = D_n$ it follows that $\Delta B = D$.

Suppose that B has N generators $f_1, \dots, f_N$, i.e. $B = [f_1, \dots, f_N; D]$, then

$$B_N = B_{|D_N} = [f_1, \dots, f_N; D]_{|D_N} \subset [f_{1|D_N}, \dots, f_{N|D_N}; D_N] \subset B_N$$

in contradiction with Proposition 1. $\square$

4. More examples

Using Propositions 1 and 2 we can give two more examples of non-finitely generated function algebras.

Example 1. Suppose $a_1, a_2, \dots$ form a sequence of points in the interior of D converging sufficiently fast to 1 so that the corresponding Blaschke product g converges. Note that by [2, page 75] the function $(z-1)g(z)$ belongs to $A(D)$ and it follows that

$$B = \{f \in A(D)\colon\ f \text{ is constant on the sequence } a_1, a_2, \dots\}$$

is a function algebra on Γ. Now let $f_1, \dots, f_n$ belong to B and $C = [f_1, \dots, f_n; D]$. The map $\Lambda\colon\ B \longrightarrow \mathbb{C}^{n+1}$ defined as in the proof of Proposition 2 is surjective (use elements of B of the form $p(z)(z-1)g(z)$ where p is a polynomial) but on the other hand $\Lambda(C) \neq \mathbb{C}^{n+1}$. Hence $C \neq B$, so the algebra B is not finitely generated.

Note that the maximal ideal space of B is the disk D in which the points $1, a_1, a_2, \dots$ are identified.

Example 2. Next, consider the Blaschke product h with zeroes of order $n+1$ at the points a_n (of course assuming the sequence $a_1, a_2, \dots$ converges sufficiently fast to 1). Again $H(z) = (z-1)h(z)$ belongs to $A(D)$. Let

$$B = \{f \in A(D)\colon\ f^{(k)}(a_n) = 0,\ 1 \leqslant k \leqslant n,\ n = 1, 2, \dots\}.$$

If $f_1, \ldots, f_n$ are functions in B, we define $C = [f_1, \ldots, f_n; D]$. As in Proposition 1 we define a map $\Lambda\colon\ B \longrightarrow \mathbb{C}^{n+1}$ by

$$\Lambda(f) = (f^{(n+1)}(a_n), \ldots, f^{(2n+1)}(a_n)).$$

One can prescribe the values $q^{(k)}(a_n)$, $0 \leqslant k \leqslant 2n+1$, of the functions

$$q(z) = (z-1)h(z)(z-a_n)^{-n-1} \sum_{k=0}^{2n+1} \alpha_k z^k$$

at will. This implies that B is a function algebra on D and that Λ is surjective. But $\Lambda(C) \neq \mathbb{C}^{n+1}$, so the algebra B is not finitely generated.

We will show that the maximal ideal space of B is the disk D. Let $\varphi \in \Delta B$. If $\varphi(H) \neq 0$ for the function H above, we define Φ on $A(D)$ by $\Phi(f) = \frac{\varphi(fH)}{\varphi(H)}$. It is easily seen that $\Phi \in \Delta A(D)$, so its action on $A(D)$ is point evaluation at some point x of D. In particular, $\varphi(f) = f(x)$ for all $f \in B$.

Let $U = \Delta B \setminus D$ and suppose U is non-empty. Let $\varphi \in U$, then $\varphi(H) = 0$. By Rossi's local maximum modulus principle it follows that $|\varphi(f)| \leqslant \|f\|_F$ for all $f \in B$ where F is the set $\{1, a_1, a_2, \ldots\}$. So φ determines a homomorphism of $[B_{|F}; F]$. From [5, page 119] it follows that $[B_{|F}; F] = C(F)$. This also follows from the Stone-Weierstrass theorem since $[B_{|F}; F]$ is conjugate-closed (just apply Mergelyan's theorem to $g(F)$ if $g \in [B_{|F}; F]$). So $\Delta[B_{|F}; F] = F$. As a consequence U is empty, hence $\Delta B = D$.

Remark. See [4] for another use of Blaschke products in a function algebra problem.

Postscript by the author. In Applied Mathematics and Computation *130* (2002), 1–4 has appeared an article Non-finitely generated function algebras by Laila E. M. Rashid, a publication which is practically identical to the preprint version of my paper.

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In Rashid's version the changes due to the revision do not appear. Also there are (in comparison with the preprint) only some minor changes (the title for instance). For reviews of Rashid's "papers" see Mathematical Reviews or Zentralblatt.

References

[1] *Th. Gamelin*: Uniform Algebras. Prentice-Hall, Englewood Cliffs, N.J., 1969.
[2] *J. B. Garnett*: Bounded Analytic Functions. Academic Press, N.Y., 1981.
[3] *P. J. de Paepe*: Maximality in sequences of function algebras. Indag. Math. *41* (1979), 163–170.
[4] *P. J. de Paepe and J. J. O. O. Wiegerinck*: A note on pervasive function algebras. Czechoslovak Math. J. *41(116)* (1991), 61–63.
[5] *E. L. Stout*: The Theory of Uniform Algebras. Bogden & Quigley, Tarrytown-on-Hudson, N.Y., 1971.

Author's address: Korteweg-de Vries Instituut, Universiteit van Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands, e-mail: `depaepe@science.uva.nl`.