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Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 1, 9–18

Persistent URL: `http://dml.cz/dmlcz/127777`

Czechoslovak Mathematical Journal, 53 (128) (2003), 9–18

INCIDENCE STRUCTURES OF TYPE (p, n)

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(Received October 4, 1999)

Abstract. Every incidence structure $\mathcal{J}$ (understood as a triple of sets (G, M, I), $\mathrm{I} \subseteq G \times M$) admits for every positive integer p an incidence structure $\mathcal{J}^p = (G^p, M^p, \mathrm{I}^p)$ where G^p (M^p) consists of all independent p-element subsets in G (M) and I^p is determined by some bijections. In the paper such incidence structures $\mathcal{J}$ are investigated the $\mathcal{J}^p$'s of which have their incidence graphs of the simple join form. Some concrete illustrations are included with small sets G and M.

Keywords: incidence structures, independent sets

MSC 2000: 06B05, 08A35

Definition 1. Let G and M be sets and $\mathrm{I} \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, \mathrm{I})$ is called an *incidence structure.*[1] If $A \subseteq G$, $B \subseteq M$, then we denote

$$A^{\uparrow} = \{m \in M \mid g \,\mathrm{I}\, m \ \ \forall g \in A\}, \ \ B^{\downarrow} = \{g \in G \mid g \,\mathrm{I}\, m \ \ \forall m \in B\}.$$

Moreover, we denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$ for $A \subseteq G$, $B \subseteq M$.

Definition 2. An incidence structure $\mathcal{J}_1 = (G_1, M_1, \mathrm{I}_1)$ is *embedded* into an incidence structure $\mathcal{J} = (G, M, \mathrm{I})$ if $G_1 \subseteq G$, $M_1 \subseteq M$ and $\mathrm{I}_1 \subseteq \mathrm{I} \cap (G_1 \times M_1)$. If $\mathrm{I}_1 = \mathrm{I} \cap (G_1 \times M_1)$, then $\mathcal{J}_1$ is a *substructure* of $\mathcal{J}$.

If we put $\mathcal{P}_G = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\}$, then the pair $\mathcal{G} = (G, \mathcal{P}_G)$ is a (lower) closure space in which $X^{\uparrow\downarrow}$ is a closure of any subset $X \subseteq G$ in $\mathcal{G}$. A set $A \subseteq G$ is *independent* in $\mathcal{G}$ if $a \notin (A - \{a\})^{\uparrow\downarrow}$ for all $a \in A$. In what follows we denote $A_a := A - \{a\}$.

Supported by the Council of the Government of the Czech Republic J14/98:153100011.

[1] The triple (G, M, I) is called an incidence structure with regard to consecutive applications. The name "context" is used more frequently in literature—see [1] where the notation is taken from.

If $A \subseteq G$, then we put $X^A(a) := A_a^{\uparrow} - \{a\}^{\uparrow}$ for $a \in A$. Then $X^A(a) = \emptyset$ iff $A_a^{\uparrow} \subseteq \{a\}^{\uparrow}$ iff $a \in A_a^{\uparrow\downarrow}$. Hence the set A is independent in $\mathcal{G}$ if and only if $X^A(a) \neq \emptyset$ for all $a \in A$. Moreover, $m \in X^A(a)$ iff $\{m\}^{\downarrow} \cap A = A_a$.

Let a non-empty subset $A \subseteq G$ be independent in $\mathcal{G}$. Then we put $\mathcal{X} = \{X^A(a) \mid a \in A\}$. For every choice $Q^A = \{m_a \in X^A(a) \mid X^A(a) \in \mathcal{X}\} \subseteq M$ from the set $\mathcal{X}$ (which exists according to the axiom of choice) we define a map $\alpha\colon A \to Q^A$ by the formula $\alpha(a) = m_a$. This map is called an A-*norming map*.

If we put $\mathcal{P}_M = \{B \subseteq M \mid B = B^{\downarrow\uparrow}\}$, then $\mathcal{M} = (M, \mathcal{P}_M)$ is a (upper) closure space. A set $B \subseteq M$ is independent in $\mathcal{M}$ if $m \notin (B - \{m\})^{\downarrow\uparrow} = B_m^{\downarrow\uparrow}$ for all $m \in M$. If $m \in B$, then we put $Y^B(m) = B_m^{\downarrow} - \{m\}^{\downarrow}$. B is independent in $\mathcal{M}$ if and only if $Y^B(m) \neq \emptyset$ for all $m \in B$. Moreover, $a \in Y^B(m)$ iff $\{a\}^{\uparrow} \cap B = B_m$.

Let a non-empty set $B \subseteq M$ be independent in $\mathcal{M}$. Then we put $\mathcal{Y} = \{Y^B(m) \mid m \in B\}$. For every choice $Q^B = \{a_m \in Y^B(m) \mid Y^B(m) \in \mathcal{Y}\} \subseteq G$ we consider a map $\beta\colon B \to Q^B$ given by the formula $\beta(m) = a_m$. It will be called a B-*norming map*.

Theorem 1. *Let $A \subseteq G$, $B \subseteq M$ be independent sets in $\mathcal{G}$, $\mathcal{M}$, respectively. Then each A-norming map $A \to Q^A$ (each B-norming map $B \to Q^B$) is injective and the sets Q^A, Q^B are independent in $\mathcal{M}$, $\mathcal{G}$, respectively. (See* [3]*.)*

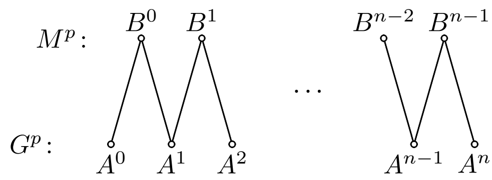
Remark 1. If $\alpha\colon A \to B$ is a map norming an independent set A of $\mathcal{G}$, then $\alpha^{-1}\colon B \to A$ is a map norming the independent set B of $\mathcal{M}$. Moreover, from $\alpha(a) = m_a$ for $a \in A$ we get $a \in Y^B(m_a)$.

Definition 3. Let us consider an incidence structure $\mathcal{J} = (G, M, \mathrm{I})$ and a positive integer $p \geqslant 2$. Let G^p and M^p be the sets of all independent sets of $\mathcal{G}$ and $\mathcal{M}$, respectively, of cardinality p. Then $\mathcal{J}^p = (G^p, M^p, \mathrm{I}^p)$ is an *incidence structure of independent sets* of $\mathcal{J}$, where $A\ \mathrm{I}^p\ B$ if and only if there exists an A-norming map $\alpha\colon A \to B$ for $A \in G^p$, $B \in M^p$.

Remark 2. If $A \in G^p$, then $X^A(a) \neq \emptyset$ for all $a \in A$ and there exists a set $B \in M^p$ and a norming map $\alpha\colon A \to B$. Similarly for a set $B \in M^p$. Hence $A^{\uparrow} \neq \emptyset$, $B^{\downarrow} \neq \emptyset$ in $\mathcal{J}^p$ for all $A \in G^p$, $B \in M^p$. If $G^p = \emptyset$, then $M^p = \emptyset$ and $\mathcal{J}^p = (\emptyset, \emptyset, \emptyset)$. For every incidence structure $\mathcal{J}$ and for every $p \geqslant 2$ there exists a unique incidence structure $\mathcal{J}^p$.

Definition 4. $\mathcal{J} = (G, M, \mathrm{I})$ is said to be an *incidence structure of type* (p, n), where $p > 1$, $n \geqslant 1$ are positive integers, if in $\mathcal{J}^p = (G^p, M^p, \mathrm{I}^p)$ we have $G^p = \{A^0, \ldots, A^n\}$, $M^p = \{B^0, \ldots, B^{n-1}\}$ and $A^i\ \mathrm{I}^p\ B^i$ iff $i = j$ or $i = j + 1$ for all $j \in \{0, \ldots, n - 1\}$.

Remark 3. If $\mathcal{J}$ is the structure of type (p, n), then the incidence graph of the structure $\mathcal{J}^p$ can be drawn in the form



and $\mathcal{J}^p$ is called a *simple join*.

Theorem 2. *If* $\mathcal{J} = (G, M, \mathrm{I})$ *is an incidence structure of type* (p, n)*, then*
(a) $|A^i \cap A^{i+1}| = p - 1$ *for all* $i \in \{0, \dots, n-1\}$,
(b) $|B^i \cap B^{i+1}| = p - 1$ *for all* $i \in \{0, \dots, n-2\}$.

P r o o f. (a) Since $A^i, A^{i+1}\ \mathrm{I}^p\ B^i$ for all $i \in \{0, \dots, n-1\}$, there exist norming mappings $\alpha_i \colon A^i \to B^i$, $\beta_i \colon B^i \to A^{i+1}$ and $\beta_i\alpha_i \colon A^i \to A^{i+1}$ is a bijective mapping of the sets A^i, A^{i+1}. We put $\alpha_i(a) = m_a$, $\beta_i(m_a) = a'$ for each $a \in A^i$. Since the inverse mapping $\alpha_i^{-1} \colon B^i \to A^i$, in which $\alpha_i^{-1}(m_a) = a$, is also norming, we get $a, a' \in Y^{B^i}(m_a)$ for each $a \in A^i$.

Let us suppose that there exist two distinct elements $b_1, b_2 \in A^{i+1} - A^i$. Then there exist distinct elements $a_1, a_2 \in A^i$ such that $\beta_i\alpha_i(a_1) = b_1$, $\beta_i\alpha_i(a_2) = b_2$. It is obvious that $a_1, b_1 \in Y^{B^i}(m_{a_1})$ and $a_2, b_2 \in Y^{B^i}(m_{a_2})$. If we put $A' = A^i_{a_i} \cup \{b_i\}$, then $|A'| = p$ and $A' \neq A^i, A^{i+1}$. We obtain $a \in Y^{B^i}(m_a)$ for all $a \in A^i_{a_1}$. The set A' is independent in $\mathcal{G}$ and $\alpha \colon a \mapsto m_a$ for all $a \in A^i_{a_1}$, $b_1 \mapsto m_{a_1}$, is a norming mapping of the set A' to B^i. Hence $A'\ \mathrm{I}^p\ B^i$. However, this contradicts $A' \neq A^i, A^{i+1}$.

(b) It can be proved similarly to (a).

Notation. Since $|A^i \cap A^{i+1}| = p-1$, we can put $R^i = A^i \cap A^{i+1}$, $A^i = \{a_i'\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i$ for all $i \in \{0, \dots, n-1\}$. In a similar way we put $Q^i = B^i \cap B^{i+1}$, $B^i = \{m_i'\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i$.

Remark 4. In Theorems 3–7 we suppose that an incidence structure $\mathcal{J} = (G, M, \mathrm{I})$ of type (p, n) is given, where $G^p = \{A^0, \dots, A^n\}$, $M^p = \{B^0, \dots, B^{n-1}\}$, and all former notation is respected.

Theorem 3.
1. $\{a_i'\}^{\uparrow} \cap B^i = \{a_{i+1}\}^{\uparrow} \cap B^i$ *for all* $i \in \{0, \dots, n-1\}$,
2. $\{m_i'\}^{\downarrow} \cap A^{i+1} = \{m_{i+1}\}^{\downarrow} \cap A^{i+1}$ *for all* $i \in \{0, \dots, n-2\}$.

P r o o f. 1. There exist norming maps $\alpha_i \colon A^i \to B^i$, $\beta_i \colon B^i \to A^{i+1}$ for each $i \in \{0, \dots, n-1\}$, where $\beta_i\alpha_i \colon A^i \to A^{i+1}$ is bijective. If $a \in A^i$, then we put

$\beta_i\alpha_i(a) = \beta_i(m_a) = \bar{a}$, where $a, \bar{a} \in Y^{B^i}(m_a)$. Assume that $a \in R^i$. If $a \neq \bar{a}$, then $a, \bar{a} \in A^{i+1}$ and $a, \bar{a} \in Y^{B^i}(m_a)$ implies a contradiction to the independence of A^{i+1}. Hence $a = \bar{a}$, $\beta_i\alpha_i(R^i) = R^i$ and $\beta_i\alpha_i(a'_i) = a_{i+1}$. This yields $a'_i, a_{i+1} \in Y^{B^i}(m_{a'_i})$ and thus $\{a'_i\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i = B^i_{m'_{a_i}}$.

2. Since A^{i+1} I^p B^i, B^{i+1} for each $i \in \{0, \ldots, n-2\}$ by Definition 4, there exist norming mappings $\beta_i\colon B^i \to A^{i+1}$, $\alpha_{i+1}\colon A^{i+1} \to B^{i+1}$, where $\alpha_{i+1}\beta_i\colon B^i \to B^{i+1}$ is bijective. If we put $\alpha_{i+1}\beta_i(m) = \alpha_{i+1}(a_m) = \overline{m}$ for $m \in B^i$, then $m, \overline{m} \in X^{A^{i+1}}(a_m)$. Similarly to 1 we can show that $m'_i, m_{i+1} \in X^{A^{i+1}}(a_{m'_i})$. Thus $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1} = A^{i+1}_{a_{m'_i}}$.

Theorem 4.

1. $a'_i \in \{m'_i\}^\downarrow \Longleftrightarrow a'_i \notin \{m_{i+1}\}^\downarrow$,
2. $m'_i \in \{a'_{i+1}\}^\uparrow \Longleftrightarrow m'_i \notin \{a_{i+2}\}^\uparrow$

for all $i \in \{0, \ldots, n-2\}$.

Proof. 1. There exists a norming mapping $\alpha_i\colon A^i \to B^i$ because A^i I^p B^i. Since $m'_i \in B^i$, there exists an element $a' \in A^i$ such that $\alpha_i(a') = m'_i$. Then $m'_i \in X^{A^i}(a')$ and $\{m'_i\}^\downarrow \cap A^i = A^i_{a'}$. If we put $\alpha_i(a) = m_a$ for $a \in A^i_{a'}$, then $m_a \in X^{A^i}(a)$ and $Q^i = B^i \cap B^{i+1} = \{m_a \mid a \in A^i_{a'}\}$.

Let us assume that $a'_i \in \{m'_i\}^\downarrow, \{m_{i+1}\}^\downarrow$ or $a'_i \notin \{m'_i\}^\downarrow, \{m_{i+1}\}^\downarrow$. From Theorem 3 we obtain $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1}$ and thus (by assumption) $\{m_{i+1}\}^\downarrow \cap A^i = \{m'_i\}^\downarrow \cap A^i = A^i_{a'}$. Hence $m_{i+1} \in X^{A^i}(a')$. From $B^{i+1} = \{m_{i+1}\} \cup Q^i$ it follows that $a' \mapsto m_{i+1}$, $a \mapsto m_a$ for $a \in A^i_{a'}$ is a norming mapping of A^i onto B^{i+1}. Thus A^i I^p B^{i+1}. It is a contradiction.

2. There exists a norming mapping $\beta_i\colon B^i \to A^{i+1}$ because A^{i+1} I^p B^i. Since $a'_{i+1} \in A^{i+1}$, there exists an element $m' \in B^i$ such that $\beta_i(m') = a'_{i+1}$. Then $a'_{i+1} \in Y^{B^i}(m')$ and $\{a'_{i+1}\}^\uparrow \cap B^i = B^i_{m'}$. If we put $\beta_i(m) = a_m$ for $m \in B^i_{m'}$, then $a_m \in Y^{B^i}(m)$ and $R^{i+1} = A^{i+1} \cap A^{i+2} = \{a_m \mid m \in B^i_{m'}\}$.

Let us assume that $m'_i \in \{a'_{i+1}\}^\uparrow, \{a_{i+2}\}^\uparrow$ or $m'_i \notin \{a'_{i+1}\}^\uparrow, \{a_{i+2}\}^\uparrow$. From Theorem 3 we obtain $\{a'_{i+1}\}^\uparrow \cap B^{i+1} = \{a_{i+2}\}^\uparrow \cap B^{i+1}$ and thus (by assumption) $\{a_{i+2}\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i = B^i_{m'}$. Hence $a_{i+2} \in Y^{B^i}(m')$. From $A^{i+2} = \{a_{i+2}\} \cup R^{i+1}$ it follows that $\beta\colon m'_i \mapsto a_{i+2}$, $m \mapsto a_m$ for $m \in B^i_{m'}$ is a norming mapping of B^i onto A^{i+2}. Thus A^{i+2} I^p B^i. It is a contradiction. □

Remark 5. Since $a'_i \in \{m'_i\}^\downarrow$ iff $a'_i \notin \{m_{i+1}\}^\downarrow$, we obtain $m'_i \in \{a'_i\}^\uparrow$ iff $m_{i+1} \notin \{a'_i\}^\uparrow$. Similarly $a'_{i+1} \in \{m'_i\}^\downarrow$ iff $a_{i+2} \notin \{m'_i\}^\downarrow$.

Theorem 5. *Let* $m'_{i+1} = m_{i+1}$. *If* $a'_{i+1} = a_{i+1}$, *then* $a'_i \in \{m'_i\}^\downarrow$ *iff* $a'_i \notin \{m_{i+2}\}^\downarrow$. *If* $a'_{i+2} = a_{i+2}$, *then* $m'_i \in \{a'_{i+1}\}^\uparrow$ *iff* $m'_i \notin \{a_{i+3}\}^\uparrow$.

P r o o f. Accepting the former notation we have $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i = \{m'_{i+1}\} \cup Q^{i+1}$, $B^{i+2} = \{m_{i+2}\} \cup Q^{i+1}$. Moreover, $Q^i = Q^{i+1}$ and $B^{i+2} = \{m_{i+2}\} \cup Q^i$ because of $m'_{i+1} = m_{i+1}$.

a) Let us assume that $a'_{i+1} = a_{i+1}$. Then $R^i = R^{i+1}$ and $A^{i+2} = \{a_{i+2}\} \cup R^i$. By Theorem 3 we obtain $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1}$, hence $\{m'_i\}^\downarrow \cap R^i = \{m_{i+1}\}^\downarrow \cap R^i$. Moreover, $\{m'_{i+1}\}^\downarrow \cap A^{i+2} = \{m_{i+2}\}^\downarrow \cap A^{i+2}$. Since $R^i \subseteq A^{i+2}$, we obtain $\{m'_{i+1}\}^\downarrow \cap R^i = \{m_{i+2}\}^\downarrow \cap R^i$ and the equality $m'_{i+1} = m_{i+1}$ implies that $\{m'_i\}^\downarrow \cap R^i = \{m_{i+2}\}^\downarrow \cap R^i$.

Let us assume that either $a'_i \in \{m'_i\}^\downarrow, \{m_{i+2}\}^\downarrow$ or $a'_i \notin \{m'_i\}^\downarrow, \{m_{i+2}\}^\downarrow$. Since $A^i = \{a'_i\} \cup R^i$, $A^{i+2} = \{a_{i+2}\} \cup R^i$, we get $\{m'_i\}^\downarrow \cap A^i = \{m_{i+2}\}^\downarrow \cap A^i$. Since $A^i \mathrm{I}^p B^i$, there exists a norming mapping $\alpha_i\colon A^i \to B^i$. Let $\alpha_i(a') = m'_i$, $\alpha_i(a) = m_a$ for $a \in A^i_{a'}$. Then $\alpha_i(A^i_{a'}) = Q^i$. From $\{m'_i\}^\downarrow \cap A^i = A^i_{a'}$ it follows that $m'_i, m_{i+2} \in X^{A^i}(a')$. Hence $a' \mapsto m_{i+2}$, $a \mapsto m_a$ for $a \in A^i_{a'}$ is a norming mapping of the set A^i onto $B^{i+2} = \{m_{i+2}\} \cup Q^i$, i.e. $A^i \ \mathrm{I}^p \ B^{i+2}$. It is a contradiction.

b) Let us assume that $a'_{i+2} = a_{i+2}$. Then $R^{i+1} = R^{i+2}$ and $A^{i+3} = \{a_{i+3}\} \cup R^{i+1}$. By Theorem 3 we obtain $\{a'_{i+1}\}^\uparrow \cap B^{i+1} = \{a_{i+2}\}^\uparrow \cap B^{i+1}$, hence $\{a'_{i+1}\}^\uparrow \cap Q^i = \{a_{i+2}\}^\uparrow \cap Q^i$. Moreover, $\{a'_{i+2}\}^\uparrow \cap B^{i+2} = \{a_{i+3}\}^\uparrow \cap B^{i+2}$. Since $Q^i \subseteq B^{i+2}$, we obtain $\{a'_{i+2}\}^\uparrow \cap Q^i = \{a_{i+3}\}^\uparrow \cap Q^i$ and the equality $a'_{i+2} = a_{i+2}$ implies that $\{a'_{i+1}\}^\uparrow \cap Q^i = \{a_{i+3}\}^\uparrow \cap Q^i$.

Let us assume that either $m'_i \in \{a'_{i+1}\}^\uparrow, \{a_{i+3}\}^\uparrow$ or $m'_i \notin \{a'_{i+1}\}^\uparrow, \{a_{i+3}\}^\uparrow$. Then $\{a'_{i+1}\}^\uparrow \cap B^i = \{a_{i+3}\}^\uparrow \cap B^i$. By assumption $A^{i+1} \ \mathrm{I}^p \ B^i$, where $A^{i+1} = \{a'_{i+1}\} \cup R^{i+1}$. Hence there exists a norming mapping $\beta_i\colon B^i \to A^{i+1}$, where $\beta_i(m') = a'_{i+1}$ for a certain $m' \in B^i$ and $\beta_i(m) = a_m$ for $m \in B^i_{m'}$. Then $\beta_i(B^i_{m'}) = R^{i+1}$. From $\{a'_{i+1}\}^\uparrow \cap B^i = B^i_{m'} = \{a_{i+3}\}^\uparrow \cap B^i$ we get $a'_{i+1}, a_{i+3} \in Y^{B^i}(m')$. From $A^{i+3} = \{a_{i+3}\} \cup R^{i+1}$ we obtain that $m' \mapsto a_{i+3}$, $m \mapsto a_m$ for $m \in B^i_{m'}$ is a norming mapping of the set B^i onto A^{i+3}, i.e. $A^{i+3} \ \mathrm{I}^p \ B^i$. It is a contradiction. □

Theorem 6. *If* $0 \leqslant i \leqslant n-2$, *then* $a'_i \neq a_{i+1}, a'_{i+1}, a_{i+2}, a'_{i+2}$.

P r o o f. Let us recall that $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i = \{a'_{i+1}\} \cup R^{i+1}$, $A^{i+2} = \{a_{i+2}\} \cup R^{i+1} = \{a'_{i+2}\} \cup R^{i+2}$.

1. If $a'_i = a_{i+1}$, then $A^{i+1} = A^i$. This is a contradiction.

2. Let $a'_i = a'_{i+1}$. If $a_{i+1} = a'_{i+1}$, then $a'_i = a_{i+1}$, a contradiction. If $a_{i+1} \neq a'_{i+1}$, then $a'_{i+1} \in R^i$ and $a'_i \in R^i$. This is a contradiction again.

3. We prove that $a'_i \neq a_{i+2}$. Since $A^i \ \mathrm{I}^p \ B^i$, there exists a norming mapping $\alpha_i\colon A^i \to B^i$, where $\alpha_i(a) = m_a$ for $a \in A^i$.

a) Let $a_{i+1} \neq a'_{i+1}$. From $a'_i \neq a'_{i+1}$ we obtain $m_{a'_i} \neq m_{a'_{i+1}}$. Hence $m_{a'_i} \neq m'_i$ or $m_{a'_{i+1}} \neq m'_i$. First assume that $m_{a'_{i+1}} \neq m'_i$. This yields $m_{a'_{i+1}} \in Q^i$ and $m_{a'_{i+1}} \in B^{i+1} = \{m_{i+1}\} \cup Q^i$. From $m_{a'_i} \in X^{A^i}(a'_i)$, $m_{a'_{i+1}} \in X^{A^i}(a'_{i+1})$ we obtain

$a'_i \mathrm{I}\, m_{a'_{i+1}}$ and $a'_{i+1} \not\mathrm{I}\, m_{a'_{i+1}}$. By Theorem 3 $\{a'_{i+1}\}^{\uparrow} \cap B^{i+1} = \{a_{i+2}\}^{\uparrow} \cap B^{i+1}$, thus $a_{i+2} \not\mathrm{I}\, m_{a'_{i+1}}$. Since $a'_i \mathrm{I}\, m_{a'_{i+1}}$, we get $a_{i+2} \neq a'_i$. If $m_{a'_i} \neq m'_i$, then we can proceed similarly.

b) Let $a_{i+1} = a'_{i+1}$. First we assume that $m'_i \in \{a'_i\}^{\uparrow}$. According to Theorem 3, $\{a'_i\}^{\uparrow} \cap B^i = \{a_{i+1}\}^{\uparrow} \cap B^i$, which implies $m'_i \in \{a_{i+1}\}^{\uparrow}$ and $m'_i \in \{a'_{i+1}\}^{\uparrow}$. From Theorem 4 we get $m'_i \notin \{a_{i+2}\}^{\uparrow}$. Hence $a'_i \mathrm{I}\, m'_i$, $a_{i+2} \not\mathrm{I}\, m'_i$ and thus $a'_i \neq a_{i+2}$. If $m'_i \in \{a'_i\}^{\uparrow}$, then we can proceed similarly.

4. We show that $a'_i \neq a'_{i+2}$. If $a'_{i+2} = a_{i+2}$, then $a'_i \neq a'_{i+2}$ according to 3. Let $a'_{i+2} \neq a_{i+2}$. Then $a'_{i+2} \in R^{i+1}$. If $a_{i+1} = a'_{i+2}$, then $a'_i = a'_{i+2}$ implies $a'_i = a_{i+1}$. This is a contradiction to 1. Hence $a_{i+1} \neq a'_{i+2}$. From $a'_{i+2} \in R^{i+1}$ we obtain $a'_{i+2} \in R^i$ and thus $a'_i \neq a'_{i+2}$. □

Remark 6. In an incidence structure of type (p, n) the case $a'_i = a_{i+3}$ is possible, as is shown in Fig. 5.

Theorem 7. *If* $0 \leqslant i \leqslant n-3$, *then* $m'_i \neq m_{i+1}, m'_{i+1}, m_{i+2}, m'_{i+2}$.

Proof. Analogous to Theorem 6. □

Theorem 8. *Let* $\mathcal{J} = (G, M, \mathrm{I})$ *be an incidence structure and* $p > 1$ *a positive integer. Let* $A^i \subseteq G$, $|A^i| = p$ *for* $i \in \{0, \ldots, n\}$ *and* $B^i \subseteq M$, $|B^i| = p$ *for* $i \in \{0, \ldots, n-1\}$, *where* $n \geqslant 1$. *Let the following conditions be valid:*

1. *The sets* A^0, B^0 *are independent in* $\mathcal{G}$, $\mathcal{M}$, *respectively, and there exists a norming mapping* $\alpha_0\colon A^0 \to B^0$.
2. $|A^i \cap A^{i+1}| = p-1$, $|B^i \cap B^{i+1}| = p-1$ *for all possible* i.
3. (a) $\{a'_i\}^{\uparrow} \cap B^i = \{a_{i+1}\}^{\uparrow} \cap B^i$, $i \in \{0, \ldots, n-1\}$.
   (b) $\{m'_i\}^{\downarrow} \cap A^{i+1} = \{m_{i+1}\}^{\downarrow} \cap A^{i+1}$, $i \in \{0, \ldots, n-2\}$
   with respect to the former notation.

Then all sets A^i, B^i *are independent in* $\mathcal{G}$, $\mathcal{M}$, *respectively, and* $A^i \,\mathrm{I}^p\, B^j$ *for* $i = j$, $i = j+1$, $j \in \{0, \ldots, n-1\}$.

Proof. Let all the assumptions hold. If $A^i \in G^p$, $B^i \in M^p$ for a certain $i \in \{0, \ldots, n-2\}$ and a norming mapping $\alpha_i\colon A^i \to B^i$ exists, then $A^{i+1} \in G^p$, $B^{i+1} \in M^p$ and there exist norming mappings $\beta_i\colon B^i \to A^{i+1}$, $\alpha_{i+1}\colon A^{i+1} \to B^{i+1}$. We have $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i$, $R^i = A^i \cap A^{i+1}$ with respect to our notation. If we put $\alpha_i(a) = m_a$ for $a \in A^i$, then $a \in Y^{B^i}(m_a)$ and $\{a\}^{\uparrow} \cap B^i = B^i_{m_a}$. According to 3(a), $\{a'_i\}^{\uparrow} \cap B^i = \{a_{i+1}\}^{\uparrow} \cap B^i = B^i_{m_{a'_i}}$ and thus $a'_i, a_{i+1} \in Y^{B^i}(m_{a'})$. Since $a \in Y^{B^i}(m_a)$ for $a \in R^i$, the set A^{i+1} is independent in $\mathcal{G}$ and $\beta_i\colon m_a \mapsto a$ for $a \in R^i$, $m_{a'_i} \mapsto a_{i+1}$ is a norming mapping of the set B^i onto A^{i+1}. Hence $A^{i+1} \,\mathrm{I}^p\, B^i$. Moreover, $m_a \in X^{A^{i+1}}(a)$ for $a \in R^i$ and $m_{a'_i} \in X^{A^{i+1}}(a_{i+1})$.

If we put $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i$, where $Q^i = B^i \cap B^{i+1}$, then $\alpha_i(a') = m'_i$ for a certain $a' \in A^i$. According to 3(b) we have $\{m'_i\}^{\downarrow} \cap A^{i+1} = \{m_{i+1}\}^{\downarrow} \cap A^{i+1} = A^{i+1}_{a'}$, which implies $m'_i, m_{i+1} \in X^{A^{i+1}}(a')$. Let $a' \in R^i$. Then $\alpha_{i+1}\colon a \mapsto m_a$ for $a \in R^i_{a'}$, $a' \mapsto m_{i+1}$, $a_{i+1} \mapsto m_{a'_i}$ is a norming mapping of the set A^{i+1} onto B^{i+1} and B^{i+1} is independent in $\mathcal{M}$. If $a' = a_{i+1}$, then $\alpha_{i+1}\colon a \mapsto m_a$ for $a \in R^i$, $a_{i+1} \mapsto m_{i+1}$ is a norming mapping of A^{i+1} onto B^{i+1} again. Thus $A^{i+1}\ \mathrm{I}^p\ B^{i+1}$.

By assumption 1 we get $A^0 \in G^p$, $B^0 \in M^p$ and $A^0 \mathrm{I}^p B^0$. Hence $A^1 \in G^p$, $B^1 \in M^p$ and $A^1 \mathrm{I}^p B^0, B^1$. This yields $A^2 \in G^p$, $B^2 \in M^p$, $A^2\ \mathrm{I}^p\ B^1, B^2$ and so on. □

Remark 7. Let the assumptions from Theorem 8 be valid. If we put $G^p_1 = \{A^0, \dots, A^n\}$, $M^p_1 = \{B^0, \dots, B^{n-1}\}$ and $A^i\ \mathrm{I}^p_1\ B^j$ iff $i = j$, $i = j+1$, then the incidence structure $\mathcal{J}^p_1 = (G^p_1, M^p_1, \mathrm{I}^p_1)$ is embedded into $\mathcal{J}^p$.

Theorems 2–7 can be used to construct incidence structures of type (p, n), as is shown in the following example.

Example. Let us construct the incidence tables of some incidence structures of type (3,3). Let $\mathcal{J} = (G, M, \mathrm{I})$ be an incidence structure of type (3.3). Then $G^3 = \{A^0, A^1, A^2, A^3\}$, $M^3 = \{B^0, B^1, B^2\}$, where $A_i \subset G$ for $i \in \{0, 1, 2, 3\}$ and $B^i \subset M$ for $i \in \{0, 1, 2\}$. In what follows we suppose that $G = \bigcup_{i=0}^{3} A^i$, $M = \bigcup_{i=0}^{2} B^i$. From Theorem 2 we obtain $A^0 = \{a'_0\} \cup R^0$, $A^1 = \{a_1\} \cup R^0 = \{a'_1\} \cup R^1$, $A^2 = \{a_2\} \cup R^1 = \{a'_2\} \cup R^2$, $A^3 = \{a_3\} \cup R^2$ and $B^0 = \{m'_0\} \cup Q^0$, $B^1 = \{m_1\} \cup Q^0 = \{m'_1\} \cup Q^1$, $B^2 = \{m_2\} \cup Q^1$.

Moreover, we will assume that the following conditions are satisfied:

(P1) $R^0 \neq R^1 \neq R^2 \neq R^0$,

(P2) $Q^0 \neq Q^1$,

(P3) $a_3 \neq a'_0$, $a_1 \neq a'_2$.

According to (P1), (P3) and Theorem 6, a'_i, a_j are distinct elements for all possible i, j. From $R^0 \neq R^1$ and $R^1 \neq R^2$ we obtain $a'_1 \in R^0$ and $a'_2 \in R^1$. The condition $a_1 \neq a'_2$ implies $a'_2 \in R^0$. Hence $R^0 = \{a'_1,\ a'_2\}$, $R^1 = \{a_1, a'_2\}$, $R^2 = \{a_1, a_2\}$. Similarly $m'_1 \in Q^0$. If we put $Q^0 = \{m'_1, m'_2\}$, then m'_i, m_j are distinct elements and $Q^1 = \{m_1, m'_2\}$. There exist a norming set $\alpha\colon A^0 \to B^0$ by assumptions.

1. Assume that $\alpha(a'_0) = m'_0$. We select such a notation that $\alpha(a'_1) = m'_1$, $\alpha(a'_2) = m'_2$ (see Tab. 1). By Theorem 3 we get $\{a'_0\}^{\uparrow} \cap B^0 = \{a_1\}^{\uparrow} \cap B^0$ and $\{m'_0\}^{\downarrow} \cap A^1 = \{m_1\}^{\downarrow} \cap A^1$. From Theorem 4, $a'_0 \notin \{m'_0\}^{\downarrow}$ implies $a'_0 \in \{m_1\}^{\downarrow}$ and thus $a'_0\ \mathrm{I}\ m_1$. Moreover, $\{a'_1\} \cap B^1 = \{a_2\}^{\uparrow} \cap B^1$ by Theorem 3 and $m'_0 \in \{a'_1\}^{\uparrow}$ implies $m'_0 \notin \{a_2\}^{\uparrow}$ by Theorem 4. Thus $a_2 \not\mathrm{I}\, m'_0$.

We know that $\{m_1'\}^\downarrow \cap A^2 = \{m_2\}^\downarrow \cap A^2$ and $a_1' \notin \{m_1'\}^\downarrow$ implies $a_1' \in \{m_2\}^\downarrow$. Thus $a_1' \mathrm{I}\, m_2$. Finally, we obtain $\{a_2'\}^\uparrow \cap B^2 = \{a_3\}^\uparrow \cap B^2$ and $m_1' \notin \{a_3\}^\uparrow$ because of $m_1' \in \{a_2'\}^\uparrow$. Thus $a_3 \not\mathrm{I}\, m_1'$.

It remains to decide about the incidence of elements a_0', m_2 and a_3, m_0'. If $a_0' \not\mathrm{I} m_2$, then for instance $A^0\ \mathrm{I}^p\ B^1$. This is a contradiction and hence $a_0' \mathrm{I}\, m_2$.

I	m_0'	m_1'	m_2'	m_1	m_2
a_0'		–	–	–	–
a_1'	–		–	–	–
a_2'	–	–		–	–
a_1		–	–		–
a_2			–	–	
a_3	?			–	–

Table 1.

My colleague Dr. V. Tichý has devised a computer program assigning to every incidence structure $\mathcal{J} = (G, M, \mathrm{I})$ for $|G|, |M| < 12$ all incidence structures $\mathcal{J}^p$ of independent sets of $\mathcal{J}$. In the figures enclosed part a) shows the incidence table of the structure $\mathcal{J}$, parts b), c) show all independent sets in $\mathcal{G}$, $\mathcal{M}$, respectively, and part d) ahows the incidence graph of the structure $\mathcal{J}^p$. Fig. 1 illustrates the described incidence structure $\mathcal{J}$ for $a_3 \not\mathrm{I}\, m_0'$ and Fig. 2 for $a_3 \mathrm{I}\, m_0'$. Both structures are of type (3,3).

2. Assume that $\alpha(a_0') \neq m_0'$. Let for instance $\alpha(a_0') = m_2'$, $\alpha(a_1') = m_1'$, $\alpha(a_2') = m_0'$. Fig. 3 shows such an incidence structure $\mathcal{J}$ of type (3,3) which is assigned similarly to 1.

Incidence structures in Figs. 1, 2, 3 are not isomorphic.

Figs. 4, 5 illustrate incidence structure of type (3,3), in which conditions (P_1), (P_2) are satisfied but $a_3 \neq a_0'$, $a_1 = a_2'$, and $a_3 = a_0'$, $a_1 \neq a_2'$, respectively.

An incidence structure of type (5,4), where $R^0 = R^1$ and $Q^1 = Q^2$, is in Fig. 6.

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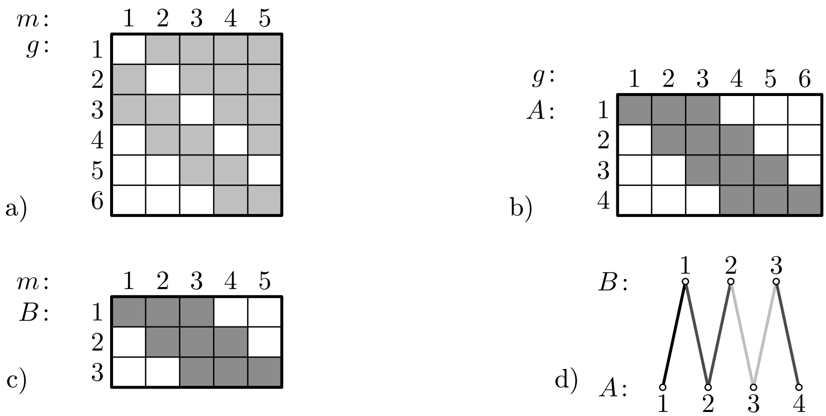

Fig. 1

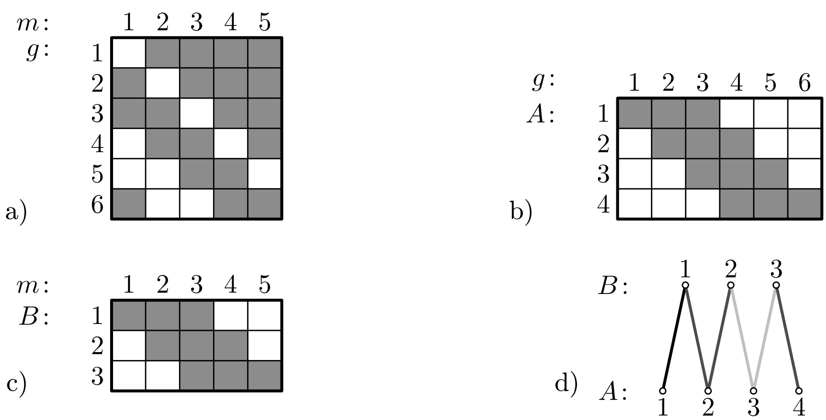

Fig. 2

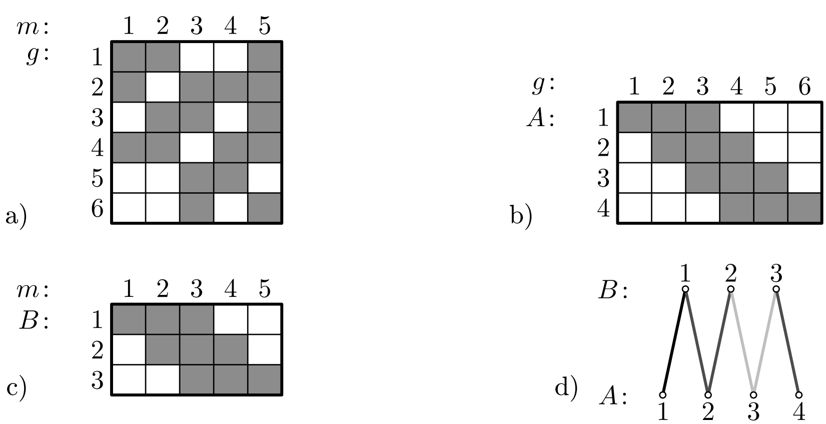

Fig. 3

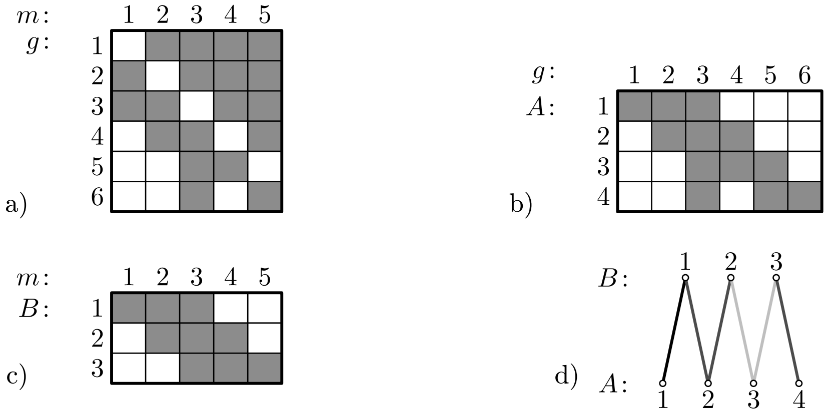

Fig. 4

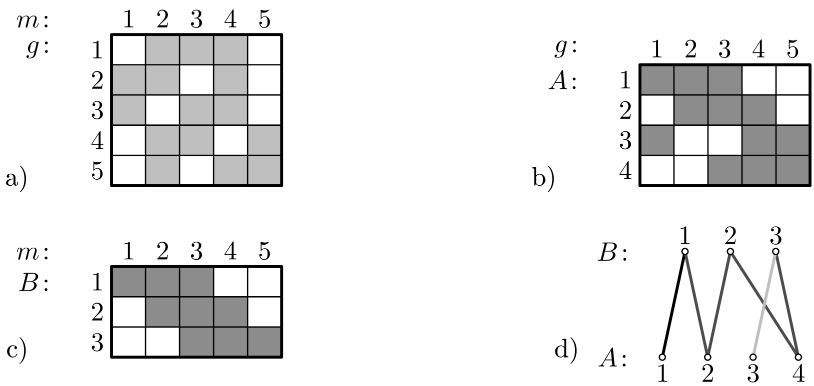

Fig. 5

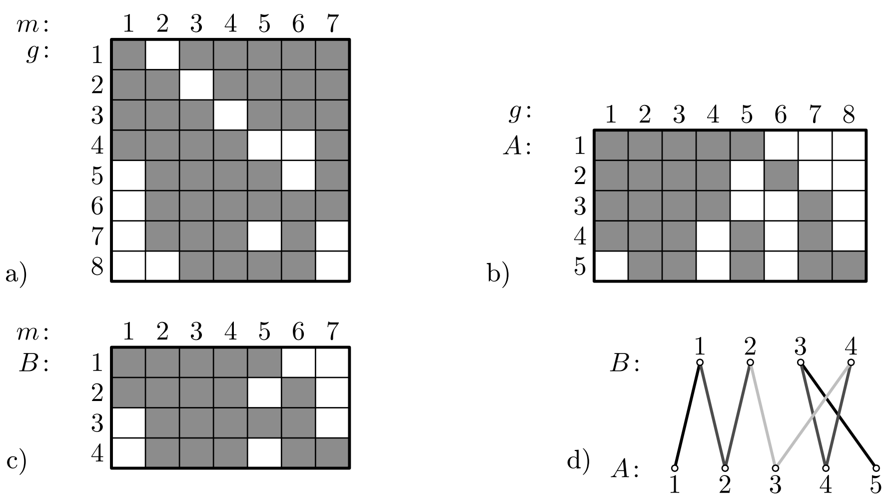

Fig. 6