Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 4, 771–780

Persistent URL: `http://dml.cz/dmlcz/127763`

Czechoslovak Mathematical Journal, 52 (127) (2002), 771–780

EXTREME GEODESIC GRAPHS

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(Received November 1, 1999)

Abstract. For two vertices u and v of a graph G, the *closed interval* $I[u,v]$ consists of u, v, and all vertices lying in some u–v geodesic of G, while for $S \subseteq V(G)$, the set $I[S]$ is the union of all sets $I[u,v]$ for $u, v \in S$. A set S of vertices of G for which $I[S] = V(G)$ is a geodetic set for G, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A vertex v in G is an extreme vertex if the subgraph induced by its neighborhood is complete. The number of extreme vertices in G is its extreme order $\mathrm{ex}(G)$. A graph G is an extreme geodesic graph if $g(G) = \mathrm{ex}(G)$, that is, if every vertex lies on a u–v geodesic for some pair u, v of extreme vertices. It is shown that every pair a, b of integers with $0 \leqslant a \leqslant b$ is realizable as the extreme order and geodetic number, respectively, of some graph. For positive integers r, d, and $k \geqslant 2$, it is shown that there exists an extreme geodesic graph G of radius r, diameter d, and geodetic number k. Also, for integers n, d, and k with $2 \leqslant d < n$, $2 \leqslant k < n$, and $n - d - k + 1 \geqslant 0$, there exists a connected extreme geodesic graph G of order n, diameter d, and geodetic number k. We show that every graph of order n with geodetic number $n - 1$ is an extreme geodesic graph. On the other hand, for every pair k, n of integers with $2 \leqslant k \leqslant n - 2$, there exists a connected graph of order n with geodetic number k that is not an extreme geodesic graph.

Keywords: geodetic set, geodetic number, extreme order, extreme geodesic graph

MSC 2000: 05C12

1. Introduction

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. A u–v path of length $d(u,v)$ is also referred to as a u–v *geodesic*. A vertex w is said to *lie in* a u–v geodesic P if w is an internal vertex of P, that is, w is a vertex of P distinct from u and v. The *closed interval* $I[u,v]$ consists of u, v, and all vertices lying in some u–v geodesic of G, while for $S \subseteq V(G)$, the set $I[S]$ is the union of all sets $I[u,v]$ for $u, v \in S$.

[1] Research supported in part by the Western Michigan University Research Development Award Program.

A set S of vertices of G is defined in [5] to be a *geodetic set* in G if $I[S] = V(G)$, while S is called *convex* if $I[S] = S$. A geodetic set of minimum cardinality is a *minimum geodetic set* and the cardinality of such a set is the *geodetic number* $g(G)$. For every nontrivial connected graph G order n, it follows that $2 \leqslant g(G) \leqslant n$. For example, the graph G_1 of Figure 1 has geodetic number 2 as $S_1 = \{w_1, y_1\}$ is the unique minimum geodetic set of G_1. On the other hand, each 2-element subset S of the vertex set of G_2 has the property that $I[S]$ is properly contained in $V(G_2)$. Thus $g(G_2) \geqslant 3$. Since $S_2 = \{u_2, v_2, x_2\}$ is a geodetic set, $g(G_2) = 3$.

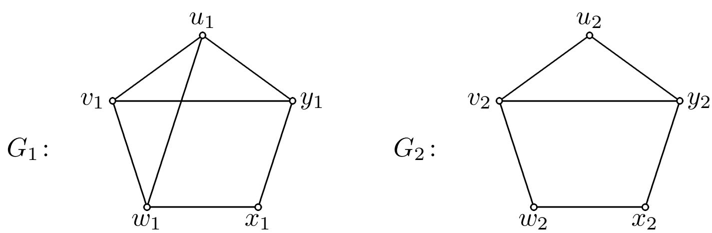


Figure 1. The geodetic number of a graph

The closed intervals $I[u, v]$ in a connected graph G were studied and characterized by Nebeský [17, 18] and were also investigated extensively in the book by Mulder [16], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The geodetic number of a graph was introduced by Harary, Loukakis, and Tsouros in [14], who showed that determining the geodetic number of a graph is a NP-hard problem. The geodetic number of graphs and oriented graphs have been studied further in [5, 10]. Two classes of graphical games concerning convex sets, called *achievement and avoidance*, were presented by Harary in [13]. These games were examined for the geodetic number by Buckley and Harary in [4] and by Nečásková in [19]. Some related geodetic games as well as related sequential and closed sequential geodetic numbers for graphs were studied in [3, 13].

Geodetic concepts in graphs are closely related to convexity concepts. A fundamental concept occurring in geometry, topology, and functional analysis is that of convex sets. We quote from the initial paragraph of the foreword to the book by Bonnesen and Fenchel [1]: "Convex figures have always played an important role in geometry… (Minkowski) created the formal tools appropriate for problems about convex regions and bodies… (and) above all opened the way to various applications…". Convexity in graphs is discussed in the book by Buckley and Harary [2] and also by Harary and Nieminen in [15]. These concepts were studied further in [8, 9]. We refer to the book [2] for concepts and results on distance in graphs and to the books [7, 12] for terminology and notation in graph theory.

A vertex v in a graph G is an *extreme vertex* if the subgraph induced by its neighborhood is complete. The *extreme order* $\mathrm{ex}(G)$ of G is the number of extreme vertices in G. Since every extreme vertex of a graph is an end-vertex of every geodesic containing it, we have the following result (see [6]).

Theorem A. *Every geodetic set in a graph contains its extreme vertices. In particular, every geodetic set in a graph contains its end-vertices.*

By Theorem A, $0 \leqslant \mathrm{ex}(G) \leqslant g(G)$ for every graph G. A graph G is an *extreme geodesic graph* if $g(G) = \mathrm{ex}(G)$, that is, if G has a unique minimum geodetic set, consisting of the extreme vertices of G. An *extreme geodesic* is u–v geodesic for some extreme vertices u and v. Extreme geodesic graphs are then characterized as those graphs every vertex of which lies on an extreme geodesic. The graph G_1 in Figure 2 has two extreme vertices, namely u_1 and w_1. Since $\{u_1, w_1\}$ is also a geodetic set, $\mathrm{ex}(G_1) = 2 = g(G_1)$ and G_1 is an extreme geodesic graph. On the other hand, the graph G_2 in Figure 2 has two extreme vertices v_2 and y_2 and so $\mathrm{ex}(G_2) = 2$. Since $\{v_2, w_2, y_2\}$ is a minimum geodetic set of G_2, it follows that $g(G_2) = 3$. The graph G_3 in Figure 2 contains no extreme vertices, but $\{u_3, w_3\}$ is its unique minimum geodetic set. So $\mathrm{ex}(G_3) = 0$ and $g(G_3) = 2$. Consequently, the graphs G_2 and G_3 are not extreme geodesic graphs.

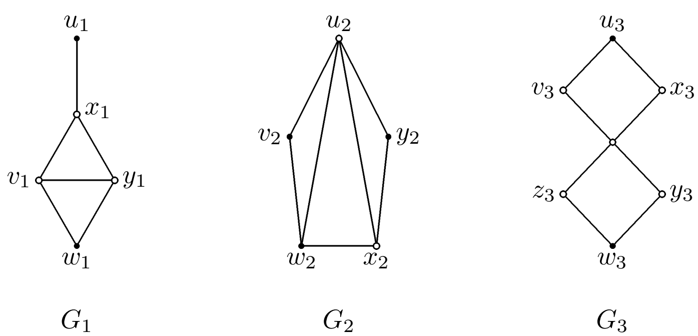


Figure 2. Graphs G_1, G_2, and G_3 and minimum geodetic sets

For $n \geqslant 2$, the complete graph K_n is the only connected graph of order n having the largest possible geodetic number, namely n. Since every vertex of K_n is an extreme vertex, $\mathrm{ex}(K_n) = g(K_n) = n$. So K_n is an extreme geodesic graph. A path P_n of order $n \geqslant 2$ has two extreme vertices, namely, its two end-vertices. So $\mathrm{ex}(P_n) = 2$. Since $g(P_n) = 2$, it follows that P_n is also an extreme geodesic graph. Obviously, a cycle C_n $(n \geqslant 4)$ contains no extreme vertices and so C_n is not an extreme geodesic graph. Similarly, no complete bipartite graph $K_{r,s}$ with $2 \leqslant r \leqslant s$ is an extreme geodesic graph.

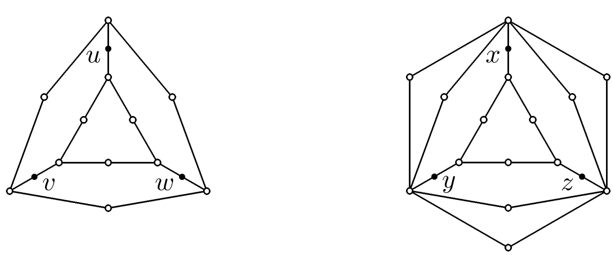
If G is a nontrivial connected graph with $\mathrm{ex}(G) = a$ and $g(G) = b$, then $0 \leqslant a \leqslant b$. In fact, every pair a, b of integers with $0 \leqslant a \leqslant b$ is realizable as the extreme order and geodetic number, respectively, of some graph, as we now show.

Theorem 1.1. *For every pair a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 2$, there exists a connected graph G with $\operatorname{ex}(G) = a$ and $g(G) = b$.*

P r o o f. If $a = b$, then $a \geqslant 2$ and $G = K_a$ has the desired properties. Thus we assume that $a < b$. We construct a graph $G_{a,b}$ with the required extreme order a and geodetic number b. Let F_i $(1 \leqslant i \leqslant a)$ be a copy of K_2 with $V(F_i) = \{s_i, t_i\}$ and let $H_j\colon\ u_j, v_j, x_j, y_j, u_j$ $(1 \leqslant j \leqslant b-a)$ be a copy of the cycle C_4. Then the graph $G_{a,b}$ is obtained from the graphs F_i and H_j $(1 \leqslant i \leqslant a,\ 1 \leqslant j \leqslant b-a)$ by identifying the a vertices t_i and the $b-a$ vertices y_j and denoting this vertex by v. Since the graph $G_{a,b}$ has a extreme vertices, namely, $s_1, s_2, \ldots, s_a$, it follows that $\operatorname{ex}(G_{a,b}) = a$. Moreover, it can be verified that the set $S = \{s_1, s_2, \ldots, s_a, v_1, v_2, \ldots, v_{b-a}\}$ is the unique minimum geodetic set. Therefore, $g(G_{a,b}) = b$, as desired. □

For a vertex v of in a connected graph G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* $\operatorname{rad} G$ and the maximum eccentricity is its *diameter* $\operatorname{diam} G$. A vertex v is called a *peripheral vertex* of G if $e(G) = \operatorname{diam} G$. In every example we have seen thus far, each vertex in every minimum geodetic set is a peripheral vertex. This may not seem surprising; in fact, one may suspect that this is true in general. However, this is not the case. Indeed, there are graphs possessing a minimum geodetic set in which no vertex is a peripheral vertex.

Figure 3 shows the subdivision graph $S(K_3 \times K_2)$ of the Cartesian product $K_3 \times K_2$. The diameter of $S(K_3 \times K_2)$ is 5 and $\{u, v, w\}$ is a minimum geodetic set, each of whose vertices has eccentricity 4. In fact, this minimum geodetic set consists exactly of those vertices that are *not* the peripheral vertices of $S(K_3 \times K_2)$. While $S(K_3 \times K_2)$ does contain minimum geodetic sets consisting entirely of peripheral vertices, the graph H of Figure 3, which is a modification of $S(K_3 \times K_2)$, has a unique minimum geodetic set, namely $\{x, y, z\}$, and so no peripheral vertex of H belongs to any minimum geodetic set.



$S(K_3 \times K_2)$ $\qquad$ H

Figure 3. Minimum geodetic sets containing no peripheral vertices

It was shown in [5] that if G is a nontrivial connected graph of order n and diameter d, then $g(G) \leqslant n - d + 1$. Since $\mathrm{ex}(G) \leqslant g(G)$ for every nontrivial connected graph G, we have the following result.

Theorem 1.2. *If G is a nontrivial connected graph of order n and diameter d, then*

$$\mathrm{ex}(G) \leqslant n - d + 1.$$

The upper bound in Theorem 1.2 is sharp. Observe that by Theorem A, the geodetic number of a tree T is the number of its end-vertices. In fact, the set of all end-vertices of T is the unique minimum geodetic set of T. Therefore, $\mathrm{ex}(T) = g(T)$, implying that all trees are extreme geodesic graphs. Let the graph G be obtained from the path $P\colon\ v_0, v_1, \ldots, v_d$ by joining $n-(d+1)$ new vertices to P at the vertex v_1. Then the graph G is a tree of order n, diameter d, with $n-d+1$ extreme vertices, namely, the end-vertices of G. So $\mathrm{ex}(G) = g(G) = n - d + 1$.

If G is a tree of order n with a end-vertices, then $\mathrm{ex}(G) = g(G) = a$. This together with the fact that $\mathrm{ex}(K_n) = g(K_n) = n$ implies that every pair a, n of integers with $2 \leqslant a \leqslant n$ is realizable as the order and geodetic number, respectively, of a extreme geodesic graph of order n.

Theorem 1.3. *For every pair a, n of integers with $2 \leqslant a \leqslant n$, there exists a connected extreme geodesic graph of order n with geodetic number a.*

Of course, $\mathrm{rad}\, G \leqslant \mathrm{diam}\, G \leqslant 2\,\mathrm{rad}\, G$ for every connected graph G. Ostrand [20] showed that every two positive integers a and b with $a \leqslant b \leqslant 2b$ are realizable as the radius and diameter, respectively, of some connected graph. In [5] Ostrand's theorem was extended so that the geodetic number can be prescribed as well. We now present the corresponding result for extreme geodesic graphs.

Theorem 1.4. *For positive integers r, d, and $k \geqslant 2$ with $r \leqslant d \leqslant 2r$, there exists a connected extreme geodesic graph G with*

$$\mathrm{rad}\, G = r, \quad \mathrm{diam}\, G = d, \quad \textit{and} \quad g(G) = k.$$

Proof. When $r = 1$, we let $G = K_k$ or $G = K_{1,k}$ according to whether $d = 1$ or $d = 2$, respectively. For $r \geqslant 2$, we construct an extreme geodesic graph G with the desired property. Let $C\colon\ v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P\colon\ u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of length $d-r$. Let H be the graph obtained from C and P by identifying v_1 in C and u_0 in P and adding the edge $v_r v_{r+2}$. The graph G is then obtained by adding $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ to H and joining each vertex w_i $(1 \leqslant i \leqslant k-2)$ to the vertex u_{d-r-1}. The graph G is shown in Figure 4. Certainly, $\operatorname{ex}(G) = k$ as $u_{d-r}, v_{r+1}, w_1, w_2, \ldots, w_{k-2}$ are the extreme vertices of G, $\operatorname{rad} G = r$, and $\operatorname{diam} G = d$. Let $S = \{u_{d-r}, v_{r+1}, w_1, w_2, \ldots, w_{k-2}\}$ denote the set consisting of all k extreme vertices of G. Since $I[S] = V(G)$, it follows that $g(G) = k$. $\square$

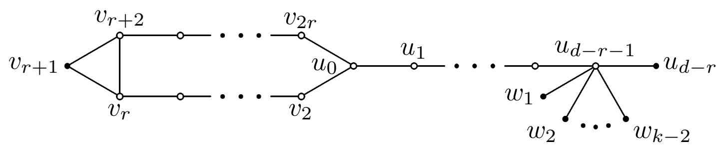


Figure 4. A extreme geodesic graph G with $\operatorname{rad} G = r$, $\operatorname{diam} G = d$, and $g(G) = k$

The graph G of Figure 4 is the smallest extreme geodesic graph (in terms of order) with the properties described in Theorem 1.4. Under similar conditions, we may simultaneously prescribe the order, diameter, and the geodetic number of an extreme geodesic graph G.

Theorem 1.5. *If n, d, and k are integers such that $2 \leqslant d < n$, $2 \leqslant k < n$, and $n-d-k+1 \geqslant 0$, then there exists an extreme geodesic graph G of order n, diameter d, and geodetic number k.*

Proof. Let $F = \overline{K}_k + K_{n-d-k+2}$, where $V(\overline{K}_k) = \{v_1, v_2, \ldots, v_k\}$ and $V(K_{n-d-k+2}) = \{w_1, w_2, \ldots, w_{n-d-k+2}\}$, and let $P\colon\ u_0, u_1, u_2, \ldots, u_{d-2}$ be a path of length $d-2$. Then the graph G is obtained by identifying v_1 in F and u_0 in P. Then G has order n and diameter d. Moreover, the set $\{u_d, v_2, \ldots, v_k\}$ of all extreme vertices of G is also a geodetic set of G. Therefore, $\operatorname{ex}(G) = g(G) = k$, as desired. $\square$

2. Graphs with prescribed order, extreme order, and geodetic number

A nontrivial complete graph is an extreme geodesic graph and so every graph of order n with geodetic number n is an extreme geodesic graph. In fact, this statement is true for graphs of order n with geodetic number $n-1$ as well. It was shown in [6] that a connected graph G of order $n \geqslant 3$ has geodetic number $n-1$ if and only if G is the join of K_1 and pairwise disjoint complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, that is,

$$G = (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_r}) + K_1, \tag{1}$$

where r $(\geqslant 2)$, $n_1, n_2, \ldots, n_r$ are positive integers with $n_1 + n_2 + \ldots + n_r = n-1$. Since the number of extreme vertices of G in (1) is $n-1$, we have the following result.

Theorem 2.1. *Every connected graph of order $n \geqslant 2$ with geodetic number $n-1$ is an extreme geodesic graph.*

In general, for each integer $\ell \geqslant 2$,

$$G = (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_r}) + K_\ell \tag{2}$$

is an extreme geodesic graph since the unique minimum geodetic set S of G is the set of its extreme vertices, that is, $S = V(K_{n_1}) \cup V(K_{n_2}) \cup \ldots \cup V(K_{n_r})$. So G is an extreme geodesic graph with $\mathrm{ex}(G) = g(G) = n - \ell$. In particular, if $\ell = 2$, then the graph G in (2) has geodetic number $n-2$. Since every graph of order n with geodetic number n or $n-1$ is an extreme geodesic graph, it is natural to ask if this is true for graphs with other geodetic numbers as well. We show next that such is not the case.

Theorem 2.2. *For every pair k, n of integers with $2 \leqslant k \leqslant n-2$, there exists a connected graph of order n with geodetic number k that is not an extreme geodesic graph.*

P r o o f. We consider two cases.

Case 1. $k = n-2$. Then $n \geqslant 4$. For $n = 4, 5$, let $G = C_n$, which is not an extreme geodesic graph. Since $g(C_4) = 2$ and $g(C_5) = 3$, it follows that $g(G) = n-2$. For $n \geqslant 6$, let G be obtained from the join $P + K_1$ of a path of order $n-1$ and the trivial graph, where $P\colon v_1, v_2, \ldots, v_{n-1}$ and $V(K_1) = \{v\}$, by adding the edges $v_i v_{i+2}$ for $3 \leqslant i \leqslant n-3$. For $n = 7$ and $k = 5$, the graph G is shown in Figure 5. Then G has exactly $n-3$ extreme vertices, namely all vertices of G except v, v_2, and v_3; while $g(G) = n-2$. Therefore, G is not an extreme geodesic graph.

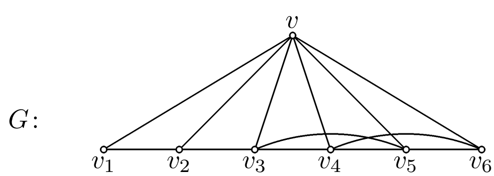


Figure 5. A graph of order 7 and geodetic number 5 that is not an extreme geodesic graph

Case 2. $2 \leqslant k \leqslant n-3$. Then $n \geqslant 5$. Let $F = K_{2,n-k}$ with partite sets $\{u, v\}$ and $\{u_1, u_2, \ldots, u_{n-k}\}$. Then the graph G is obtained from F by adding the $k-1$ vertices $v_1, v_2, \ldots, v_{k-1}$ and $k-1$ edges vv_i, where $1 \leqslant i \leqslant k-1$. Then G has $k-1$ extreme vertices, namely $v_1, v_2, \ldots, v_{k-1}$, and so $\mathrm{ex}(G) = k-1$. Since $\{u, v_1, v_2, \ldots, v_{k-1}\}$ is the unique minimum geodetic set of G, it follows that $g(G) = k$. Therefore, G is not an extreme geodesic graph and has the desired property. $\square$

Theorem 1.3 and its proof suggest another problem, namely, whether it is possible to extend Theorem 2.2 by prescribing an extreme order as well. We have no solution to this problem, which we state as a conjecture.

For every triple a, b, n of integers with $a \leqslant b \leqslant n-2$, $b \geqslant 2$, and n sufficiently large, there exists a connected graph G of order n with $\mathrm{ex}(G) = a$ and $g(G) = b$.

Of course, Conjecture 2.3 is true when $a = b$ by Theorem 1.3. By Theorem 2.2, this conjecture is also true when $a = b - 1 = n - 3$. For $a = b - 2 = n - 4$, let G be obtained from the graph K_{n-2}, where $V(K_{n-2}) = \{u_1, u_2, \ldots, u_{n-2}\}$, by adding two new vertices x, y and the three edges u_1x, xy, yu_2. Since u_1, u_2, x, y are the only nonextreme vertices of G, it follows that $\mathrm{ex}(G) = n - 4$. Next we show that $g(G) = n - 2$. Since $V(G) - \{u_1, u_2\}$ is a geodetic set, $g(G) \leqslant n - 2$. On the other hand, let $S = V(G) - \{u_1, u_2, x, y\}$ be the set of extreme vertices of G. By Theorem A, every geodetic set of G contains S. However, $S \cup \{w\}$ for $w \in \{u_1, u_2, x, y\}$ is not a geodetic set of G, implying that $g(G) \geqslant n - 2$. Therefore, $g(G) = n - 2$.

3. Geodetic ratios and extreme order ratios

It is common for a parameter f defined in terms of the vertices of a connected graph G of order n to satisfy $0 < f(G) \leqslant n$ or $0 < f(G) < n$. There are numerous instances in the literature of two such parameters f_1 and f_2 being studied, where $0 < f_1(G) \leqslant f_2(G) \leqslant n$ for every graph G. A common problem concerns whether every two integers a and b with $0 < a \leqslant b$ are realizable as the values of f_1 and f_2, respectively, for some graph. Normally, a considerably more challenging problem involves, for a given integer $n \geqslant 2$, determining those pairs a, b of integers with $0 < a \leqslant b \leqslant n$ (or $0 < a \leqslant b < n$) for which there exists a graph G of order n such that $f_1(G) = a$ and $f_2(G) = b$. Often, only partial results of this nature exist. Of course, if for some pair a, b of integers with $0 < a \leqslant b < n$, say, there exists a graph G of order n such that $f_1(G) = a$ and $f_2(G) = b$, then $0 \leqslant a/n \leqslant b/n < 1$. In this case, we say that the rational numbers a/n and b/n are realizable as the f_1-ratio and f_2-ratio, respectively, of some graph. This suggests a new, less restrictive problem when considering such pairs f_1, f_2 of parameters. These ideas were introduced in [11].

For a connected graph G of order $n \geqslant 2$, the *geodetic ratio* of G is defined in [11]as

$$r_g(G) = \frac{g(G)}{n}.$$

Certainly, $2 \leqslant g(G) \leqslant n$ for every nontrivial connected graph G. Therefore, $0 < r_g(G) \leqslant 1$. It was shown in [5] that if k and n are integers with $2 \leqslant k \leqslant n$, then there exists a connected graph G of order n with geodetic number k. Therefore, every rational number $r \in (0,1]$ is realizable as the geodetic ratio for some connected graph.

Similarly, we define the *extreme order ratio* of a graph G of order $n \geqslant 2$ as

$$r_{\rm ex}(G) = \frac{{\rm ex}(G)}{n}.$$

Since ${\rm ex}(G) \leqslant g(G)$ for every nontrivial connected graph G, it follows that $r_{\rm ex}(G) \leqslant r_g(G)$. By Theorem 1.3, for every rational number $s \in (0,1]$, there exists a graph G with $r_{\rm ex}(G) = r_g(G) = s$. If Conjecture 2.3 is true, then we have a solution to the following weaker problem.

Problem 3.1. *Determine all rational numbers s and t with $0 \leqslant s \leqslant t < 1$, for which there exists a graph G of order n such that $r_{\rm ex}(G) = s$ and $r_g(G) = t$.*

Although we have only been able to verify Conjecture 2.3 in some special cases, we can solve Problem 3.1 for a much more general range of rational numbers. For graphs with distinct prescribed geodetic and extreme order ratios, we present the following result.

Theorem 3.2. *For every pair s, t of rational numbers with $0 \leqslant s < t < (1+s)/2 < 1$, there exists a connected graph G with $r_{\rm ex}(G) = s$ and $r_g(G) = t$.*

Proof. First, we assume that $s > 0$. Let $s = s_1/s_2$ and $t = t_1/t_2$, where s_1, s_2, t_1, t_2 are positive integers. Since $0 < s < t < (1+s)/2$, it follows that $s_2t_1 - s_1t_2 > 0$ and $s_2t_2 - 2s_2t_1 + s_1t_2 > 0$. For an integer $k \geqslant 2$, let

$$\begin{aligned} a &= ks_1t_2 \\ 2b &= k(s_2t_1 - s_1t_2) \\ c &= k(s_2t_2 - 2s_2t_1 + s_1t_2). \end{aligned}$$

Let F_i $(1 \leqslant i \leqslant a-1)$ be a copy of K_2 with $V(F_i) = \{x_i, y_i\}$, $P\colon\ v_1, v_2, \ldots, v_c, v_{c+1}$ a path, and H_j $(1 \leqslant j \leqslant b)$ a copy of $K_{2,3}$ with partite sets $\{u_{j1}, u_{j2}\}$ and $\{w_{j1}, w_{j2}, w_{j3}\}$. Then the graph G is obtained from the graphs F_i, P and H_j by identifying the $a-1$ vertices y_i, the vertex v_{c+1}, and the b vertices w_{j1} and denoting this vertex by v. The order of G is $a + 4b + c = ks_2t_2$. Since G contains $a = ks_1t_2$ extreme vertices, $r_{\rm ex}(G) = s$. Moreover, it can be verified that the set

$$\{u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{b1}, u_{b2}, v, x_1, x_2, \ldots, x_{a-1}\}$$

is a minimum geodetic set, $g(G) = a + 2b = ks_2t_1$. Therefore, $r_g(G) = t$, as desired.

For $s = 0$, the graph G is obtained from the b graphs $H_1, H_2, \ldots, H_b$ only and by identifying the b vertices w_{j1} $(1 \leqslant j \leqslant b)$. The proof is otherwise identical. $\square$

Curiously enough, we are not aware of a single graph with $r_{\text{ex}}(G) = s$ and $r_g(G) = t$ with $s < t$ for which $0 \leqslant s < t < (1+s)/2 < 1$ does not hold. Consequently, it may be that Theorem 3.2 cannot be improved. We leave this as an open problem.

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