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CHEBYSHEV CENTERS IN HYPERPLANES OF c_0

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Abstract. We give a full characterization of the closed one-codimensional subspaces of c_0, in which every bounded set has a Chebyshev center. It turns out that one can consider equivalently only finite sets (even only three-point sets) in our case, but not in general. Such hyperplanes are exactly those which are either proximinal or norm-one complemented.

Keywords: Chebyshev centers, proximinal hyperplanes, space c_0

MSC 2000: 41A65, 46B25

In various concrete problems, there is a need to approximate simultaneously a bounded set A of data in a given metric space (X, d) by a single point of X. One of the possibilities is to look for a point for which the supremum of its distances from the points of A is minimal ("least deviation principle"), or geometrically, the center of a closed ball B of minimal radius such that $B \supset A$. The study of such points ("Chebyshev centers" or "best simultaneous approximants") is one of important subjects in Approximation Theory.

Precisely speaking, a point $x_0 \in X$ is called a *Chebyshev center* of a set A whenever $\varphi(x_0) = \inf \varphi(X)$, where

$$\varphi(x) = \sup_{a \in A} d(x, a).$$

Chebyshev centers were studied mostly in the case where X is a normed (or Banach) space with the usual distance $d(x, y) = \|x - y\|$. We refer the reader to [5], [6], [3], [2], [1] for the basic properties and references. For example, Garkavi [5] proved that each bounded set in X has at most one Chebyshev center if and only if X is uniformly rotund in every direction. He also gave the first example of nonexistence of centers: a closed hyperplane X in $C[0, 1]$ and a three-point set $A \subset X$ that has no Chebyshev center in X. While the question about uniqueness is completely solved, the same cannot be said about the problem of existence. It is known that, in reflexive or dual spaces (more generally: in spaces which are norm-one complemented in their biduals), as well as in the classical spaces ℓ_p, $L_p[0,1]$ $(1 \leqslant p \leqslant \infty)$, $C(K)$, c_0, every bounded set has a Chebyshev center. But no satisfactory characterization is known, and many other problems concerning the existence of Chebyshev centers remain open.

One of the questions studied by the author was the following: given a Banach space X in which every finite (or every compact) set has a Chebyshev center, is it true that every bounded set in X has a Chebyshev center? The answer is negative in general: a counterexample of the form $X = c_0(Y)$ was given in [8]. The present paper is a by-product of the author's looking for such a counterexample. Our results imply (cf. Theorem 2) that, if X is a 1-codimensional closed subspace of the classical sequence space c_0, such that every three-point set in X has a Chebyshev center, then already every bounded subset of X has a Chebyshev center. (Observe that a two-point set $\{x, y\}$ always has at least one Chebyshev center, namely the point $(x+y)/2$.)

The main idea for the proof of our Theorem 1 has its roots in the above mentioned example by A. L. Garkavi. His example can be easily modified to obtain the following example of nonexistence of Chebyshev centers in a hyperplane of c_0 (which we state without proof).

Example. Let $f = (f_i) \in \ell_1 = (c_0)^*$ have infinite support, $f_1 = f_2 = f_3 = 1$ and $\sum\limits_{i=4}^{\infty} |f_i| = 1$. Then the three-point set, whose elements are

$$(-1,1,1,0,0,\ldots), \quad (1,-1,1,0,0,\ldots), \quad (1,1,-1,0,0,\ldots),$$

belongs to $f^{-1}(1)$ and has no Chebyshev center in $f^{-1}(1)$.

Let us start with two technical lemmas and an easy fact.

Lemma 1. *Let $f = (f_i) \in \ell_1$ be a sequence with infinite support* $\operatorname{supp} f$. *Suppose $2\|f_\infty\| < \|f\|_1$. Then there exist three disjoint finite nonempty subsets A_1, A_2, A_3 of* $\operatorname{supp} f$ *such that*

$$\min\{-\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 - \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 - \alpha_3\} \geqslant \beta > 0,$$

where

$$\alpha_k = \sum_{i \in A_k} |f_i| \quad (k = 1,2,3) \quad \textit{and} \quad \beta = \sum_{i \in B} |f_i|$$

with $B = \mathbb{N} \setminus (A_1 \cup A_2 \cup A_3)$.

P r o o f. Without any loss of generality we can (and do) suppose that $\|f\|_\infty = |f_1|$; hence $0 < |f_1| < \sum\limits_{i=2}^{\infty} |f_i|$. Put

$$m = \max\Big\{k \in \mathbb{N}\colon \sum_{i=1}^{k} |f_i| < \sum_{i=k+1}^{\infty} |f_i|\Big\}$$

and observe that maximality of m implies $|f_{m+1}| > 0$. So we have

$$\sum_{i=1}^{m} |f_i| < |f_{m+1}| + \sum_{i=m+2}^{\infty} |f_i|, \tag{1}$$

$$|f_{m+1}| \leqslant |f_1| < \sum_{i=2}^{\infty} |f_i| \leqslant \sum_{i=1}^{m} |f_i| + \sum_{i=m+2}^{\infty} |f_i| \tag{2}$$

and, again by maximality of m,

$$\sum_{i=m+2}^{\infty} |f_i| \leqslant \sum_{i=1}^{m} |f_i| + |f_{m+1}|. \tag{3}$$

The number

$$\varepsilon := \frac{1}{2} \min\Big\{|f_{m+1}| + \sum_{i=m+2}^{\infty} |f_i| - \sum_{i=1}^{m} |f_i|,\ \sum_{i=1}^{m} |f_i| + \sum_{i=m+2}^{\infty} |f_i| - |f_{m+1}|\Big\}$$

is greater than 0 by (1), (2). Let $n \in \mathbb{N}$ be such that

$$\sum_{i=n+1}^{\infty} |f_i| < \varepsilon \quad \text{and} \quad \sum_{i=m+2}^{n} |f_i| > 0.$$

Let us show that the sets

$$\begin{aligned} A_1 &= \{1, 2, \ldots, m\} \cap \operatorname{supp} f, \\ A_2 &= \{m+1\}, \\ A_3 &= \{m+2, m+3, \ldots, n\} \cap \operatorname{supp} f \end{aligned}$$

satisfy the required conditions.

First, note that the sets A_k are nonempty disjoint subsets of $\operatorname{supp} f$. Moreover,

$$\alpha_1 = \sum_{i=1}^{m} |f_i|, \quad \alpha_2 = |f_{m+1}|, \quad \alpha_3 = \sum_{i=m+2}^{n} |f_i|, \quad \beta = \sum_{i=n+1}^{\infty} |f_i|,$$

and $0 < \beta < \varepsilon$ ($\operatorname{supp} f$ is infinite!). Using the definition of ε we get $2\varepsilon \leqslant \alpha_2 + \alpha_3 + \beta - \alpha_1$, which implies

$$\beta \leqslant \alpha_2 + \alpha_3 - \alpha_1 + 2\beta - 2\varepsilon < \alpha_2 + \alpha_3 - \alpha_1.$$

Similarly, $2\varepsilon \leqslant \alpha_1 + \alpha_3 + \beta - \alpha_2$ implies

$$\beta \leqslant \alpha_1 + \alpha_3 - \alpha_2 + 2\beta - 2\varepsilon < \alpha_1 + \alpha_3 - \alpha_2.$$

Moreover, (3) is equivalent to $\alpha_3 + \beta \leqslant \alpha_1 + \alpha_2$, hence

$$\beta \leqslant \alpha_1 + \alpha_2 - \alpha_3.$$

□

The following easy fact is well known and belongs, more or less, to the mathematical folklore. We present a sketch of its proof for the sake of completeness. (Note that the equivalence (i) $\Leftrightarrow$ (ii) holds in every normed space.) Recall that a subset H of a metric space is called *proximinal* if every $x \in X$ has a nearest point in H (i.e., a point $y_0 \in H$ such that $d(x, y_0) \leqslant d(x, y)$ for every $y \in H$).

Fact 1. *Let $f = (f_i) \in \ell_1 = (c_0)^*$. Then the following assertions are equivalent:*
(i) *f attains its norm;*
(ii) *$f^{-1}(0)$ is proximinal;*
(iii) *f has a finite support* $\operatorname{supp} f = \{i\colon\ f_i \neq 0\}$.

S k e t c h o f p r o o f. It is sufficient to give a proof for $\|f\| = 1$.

a) Let us prove (i) $\Leftrightarrow$ (ii) for a general normed space X (instead of for c_0). Suppose there exists $v \in X$ with $\|v\| = 1$ and $f(v) = 1$. Given $x \in X$, put $y_0 = x - f(x)v$ and observe that $y_0 \in f^{-1}(0)$ and, for each $y \in f^{-1}(0)$, $\|x - y\| \geqslant |f(x - y)| = |f(x)| = \|x - y_0\|$. Thus $f^{-1}(0)$ is proximinal.

Now, suppose that $f^{-1}(0)$ is proximinal. Take $x \in f^{-1}(1)$ and $y_0 \in f^{-1}(0)$ such that $\|x - y_0\| = \operatorname{dist}(x, f^{-1}(0))$. Putting $v_0 = x - y_0$, we have $f(v_0) = 1$ and $\|v_0\| = \operatorname{dist}(0, f^{-1}(1))$. Denote by U the open unit ball of X and observe that the definition of $\|f\|$ implies that $U \cap f^{-1}(1) = \emptyset$ and $tU \cap f^{-1}(1) \neq \emptyset$ whenever $t > 1$. Consequently, $\|v_0\| = \operatorname{dist}(0, f^{-1}(1)) = 1 = f(v_0)$.

b) Let us show (i) $\Leftrightarrow$ (iii). If $\operatorname{supp} f$ is finite then f attains its norm at the point $v \in c_0$ given by $v(i) = \operatorname{sgn} f_i$. On the other hand, if there is $v \in c_0$ such that $\|v\| = 1 = f(v)$, we have $\sum\limits_{i=1}^{\infty} |f_i| \cdot |v(i)| = \sum\limits_{i=1}^{\infty} |f_i|$ since

$$1 = f(v) = \sum_{i=1}^{\infty} f_i v(i) \leqslant \sum_{i=1}^{\infty} |f_i| \cdot |v(i)| \leqslant \sum_{i=1}^{\infty} |f_i| = 1.$$

But this, since $|f_i| \cdot |v(i)| \leqslant |f_i|$ for all i, implies that $|f_i| \cdot |v(i)| = |f_i|$ for all i. Since $v \in c_0$, we must have $f_i = 0$ for all sufficiently large i. □

Lemma 2. *Let $f = (f_i) \in \ell_1 = (c_0)^*$ have an infinite support and let (λ_i) be a sequence of nonnegative numbers such that $\lambda_i \to 0$. Let $\varphi\colon\ c_0 \to \mathbb{R}$ be given by*

$$\varphi(x) = \max_{i\in\mathbb{N}}[\lambda_i + |x(i)|].$$

If $y \in c_0$ is such that $\varphi(y) > \varphi(0)$, then there exists $z \in c_0$ such that

$$f(z) = f(y) \quad \text{and} \quad \varphi(z) < \varphi(y).$$

Proof. Denote $I = \{i \in \mathbb{N}\colon\ \lambda_i + |y(i)| = \varphi(y)\}$ and observe that I is finite and $|y(i)| > 0$ whenever $i \in I$ (since $\varphi(y) > \varphi(0) \geqslant 0$). Put $a = \max\{\lambda_i + |y(i)|\colon\ i \in \mathbb{N} \setminus I\}$. Since $a < \varphi(y)$, there exists $\varepsilon > 0$ so small that

$$a + \varepsilon < \varphi(y) - \varepsilon \quad \text{and} \quad \varepsilon < \min_{i\in I} |y(i)|.$$

Since f has infinite support, there exists $i_0 \in (\operatorname{supp} f) \setminus I$. Put

$$\delta = \varepsilon \cdot \min\{1, |f_{i_0}|/\|f\|_1\}$$

and define

$$z(i) = \begin{cases} (\operatorname{sgn} y(i))(|y_i| - \delta) & \text{if } i \in I; \\ y(i_0) + \dfrac{\delta}{f_{i_0}} \displaystyle\sum_{j\in I} f_j \operatorname{sgn} y(j) & \text{if } i = i_0; \\ y(i) & \text{otherwise.} \end{cases}$$

We have

$$f(z) - f(y) = -\delta \sum_{i\in I} f_i \operatorname{sgn} y(i) + f_{i_0} \frac{\delta}{f_{i_0}} \sum_{j\in I} f_j \operatorname{sgn} y(j) = 0.$$

Moreover, we have

$$\lambda_i + |z(i)| = \lambda(i) + |y(i)| - \delta = \varphi(y) - \delta \geqslant \varphi(y) - \varepsilon$$

for $i \in I$, and

$$\lambda_{i_0} + |z(i_0)| \leqslant \lambda_{i_0} + |y_{i_0}| + \frac{\delta}{|f_{i_0}|} \sum_{j\in I} |f_j| \leqslant a + \delta \frac{\|f\|_1}{|f_{i_0}|} \leqslant a + \varepsilon.$$

The choice of ε implies that $\varphi(z) = \varphi(y) - \delta < \varphi(y)$. $\square$

As was already mentioned in the introduction of the present paper, the proof of the following theorem was inspired by the first example of nonexistence of Chebyshev centers given by Garkavi [5].

Theorem 1. *Let* $f = (f_i) \in \ell_1 = (c_0)^*$ *be such that* $\operatorname{supp} f$ *is infinite and* $2\|f\|_\infty < \|f\|_1$. *Then there exists* $\sigma \in \mathbb{R}$ *and a three-point set* $S = \{u, v, w\} \subset f^{-1}(\sigma)$ *such that* S *has no Chebyshev center in* $f^{-1}(\sigma)$.

P r o o f. Let A_k, α_k ($k = 1, 2, 3$), B and β be as in Lemma 1. Put $A = A_1 \cup A_2 \cup A_3$,

$$\sigma = \min\{-\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 - \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 - \alpha_3\},$$

$$\xi = \frac{\sigma + \alpha_2 - \alpha_3}{\alpha_1}, \qquad \eta = \frac{\sigma - \alpha_1 + \alpha_3}{\alpha_2}, \qquad \zeta = \frac{\sigma + \alpha_1 - \alpha_2}{\alpha_3}.$$

Define $u, v, w \in c_0$ by

$$u(i) = \begin{cases} \xi \operatorname{sgn} f_i & \text{if } i \in A_1, \\ -\operatorname{sgn} f_i & \text{if } i \in A_2, \\ \operatorname{sgn} f_i & \text{if } i \in A_3, \\ 0 & \text{if } i \in B; \end{cases} \qquad v(i) = \begin{cases} \operatorname{sgn} f_i & \text{if } i \in A_1, \\ \eta \operatorname{sgn} f_i & \text{if } i \in A_2, \\ -\operatorname{sgn} f_i & \text{if } i \in A_3, \\ 0 & \text{if } i \in B; \end{cases}$$

$$w(i) = \begin{cases} -\operatorname{sgn} f_i & \text{if } i \in A_1, \\ \operatorname{sgn} f_i & \text{if } i \in A_2, \\ \zeta \operatorname{sgn} f_i & \text{if } i \in A_3, \\ 0 & \text{if } i \in B. \end{cases}$$

Observe that $f(u) = \xi\alpha_1 - \alpha_2 + \alpha_3 = \sigma$, $f(v) = \alpha_1 + \eta\alpha_2 - \alpha_3 = \sigma$, $f(w) = -\alpha_1 + \alpha_2 + \zeta\alpha_3 = \sigma$, and $\sigma > 0$.

Claim. $\xi, \eta, \zeta \in (-1, 1]$.

P r o o f. Indeed, since each of α_k's is positive and smaller than the sum of the other two, we have

$$\begin{aligned}
-1 < \frac{\sigma}{\alpha_1} - 1 &= \frac{\sigma + \alpha_2 - (\alpha_1 + \alpha_2)}{\alpha_1} < \xi \leqslant \frac{(\alpha_1 - \alpha_2 + \alpha_3) + \alpha_2 - \alpha_3}{\alpha_1} = 1, \\
-1 < \frac{\sigma}{\alpha_2} - 1 &= \frac{\sigma - (\alpha_2 + \alpha_3) + \alpha_3}{\alpha_2} < \eta \leqslant \frac{(\alpha_1 + \alpha_2 - \alpha_3) - \alpha_1 + \alpha_3}{\alpha_2} = 1, \\
-1 < \frac{\sigma}{\alpha_3} - 1 &= \frac{\sigma + \alpha_1 - (\alpha_1 + \alpha_3)}{\alpha_3} < \zeta \leqslant \frac{(-\alpha_1 + \alpha_2 + \alpha_3) + \alpha_1 - \alpha_3}{\alpha_3} = 1.
\end{aligned}$$

This proves our claim. □

The claim and the definition of u, v, w imply that

$$\lambda_i := \max\{u(i), v(i), w(i)\} = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \in B; \end{cases}$$

and

$$\min\{u(i), v(i), w(i)\} = -\lambda_i \quad (i \in \mathbb{N}).$$

Suppose that $y \in f^{-1}(\sigma)$ is a Chebyshev center for $\{u, v, w\}$ in $f^{-1}(\sigma)$. This means that y minimizes over $f^{-1}(\sigma)$ the function

$$\begin{aligned} \varphi(x) &= \max\{\|x-u\|, \|x-v\|, \|x-w\|\} \\ &= \max_{i\in\mathbb{N}} \max\bigl\{x(i)-u(i), u(i)-x(i), x(i)-v(i), v(i)-x(i), x(i)-w(i), \\ &\qquad\qquad w(i)-x(i)\bigr\} \\ &= \max_{i\in\mathbb{N}} \max\bigl\{x(i)-\min\{u(i), v(i), w(i)\}, -x(i)+\max\{u(i), v(i), w(i)\}\bigr\} \\ &= \max_{i\in\mathbb{N}}[\lambda_i + |x(i)|] = \max\bigl\{\max_{i\in A}[1+|x(i)|], \max_{i\in B}|x(i)|\bigr\}. \end{aligned}$$

By Lemma 2, $\varphi(y) = \varphi(0) = 1$.

If $|y(i)| > 0$ for some $i \in A$, then $\varphi(y) \geqslant 1 + |y(i)| > 1$. Hence we must have $y(i) = 0$ for all $i \in A$; consequently, $\varphi(y) = \|y\|_\infty = 1$. Define $\tilde{f} = (\tilde{f}_i) \in \ell_1$ by

$$\tilde{f}_i = \begin{cases} 0 & \text{if } i \in A \\ f_i & \text{if } i \in B. \end{cases}$$

Then $\tilde{f}$ does not attain its norm (by Fact 1) and

$$\sigma = f(y) = \tilde{f}(y) < \|\tilde{f}\|_1 \cdot \|y\|_\infty = \sum_{i\in B} |f_i| = \beta \leqslant \sigma.$$

This contradiction completes the proof. □

Some preliminaries are needed before stating the main result of the present paper.

Fact 2 (Cf. [4]). *Let $f = (f_i) \in \ell_1 = (c_0)^*$. Then $f^{-1}(0)$ is norm-one complemented in c_0 if and only if $2\|f\|_\infty \geqslant \|f\|_1$.*

We will say that a Banach space X *admits generalized centers for finite sets* if, for every finite set $\{a_1, \ldots, a_n\} \subset X$ and every continuous monotone coercive function $f\colon [0, +\infty)^n \to \mathbb{R}$, the function

$$\psi(x) = f(\|x-a_1\|, \ldots, \|x-a_n\|)$$

attains its infimum over X. (Here "monotone" means monotone w.r.t. the coordinate-wise partial ordering on $[0,+\infty)^n$, and "coercive" means that $f(\xi) \to +\infty$ as $\|\xi\|_\infty \to \infty$, $\xi \in [0,+\infty)^n$.) The class of such Banach spaces was called (GC) and studied in [7].

Fact 3. *Let $\mathcal{C}$ be either the class of all Banach spaces that admit generalized centers for finite sets, or the class of all Banach spaces in which each bounded set has a Chebyshev center. Then*
(a) *$\mathcal{C}$ contains c_0 and all finite-dimensional spaces;*
(b) *$\mathcal{C}$ is stable under making arbitrary direct ℓ_∞-sums;*
(c) *$\mathcal{C}$ is stable under applying linear projections of norm one.*

Proof. (a) See [5] and [7].

(b) This was proved in [7] for generalized centers; for Chebyshev centers of bounded sets a simple coordinate-wise argument works.

(c) See e.g. [6] for Chebyshev centers; for generalized centers the same simple argument works (cf. also the proof of Proposition 2.2 in [7]). □

Theorem 2. *Let $f = (f_i) \in \ell_1 = (c_0)^*$, $H = f^{-1}(0)$. Then the following assertions are equivalent.*
(i) $\operatorname{supp} f$ *is finite or* $2\|f\|_\infty \geqslant \|f\|_1$.
(ii) *H is proximinal or norm-one complemented.*
(iii) *H admits generalized centers for finite sets.*
(iv) *Every bounded subset of H has a Chebyshev center in H.*
(v) *Every finite subset of H has a Chebyshev center in H.*
(vi) *Every three-point subset of H has a Chebyshev center in H.*

Proof. (i) $\Leftrightarrow$ (ii) holds by Fact 1 and Fact 2.

Let us prove that (i) implies (iii) and (iv). First, suppose that $\operatorname{supp} f$ is finite, i.e., for some $n \in \mathbb{N}$, $f_i = 0$ whenever $i > n$. Then it is easy to see that H is isometric with the ℓ_∞-sum

$$Y \oplus_\infty c_0 \quad \text{where} \quad Y = \Big\{ \xi \in \mathbb{R}^n : \sum_{i=1}^{n} f_i \xi_i = 0 \Big\}.$$

Then (iii) and (iv) follow from Fact 3(a,b). Second, if H is norm-one complemented in c_0, the properties (iii), (iv) follow from Fact 3(a,c).

The implications [(iii) or (iv)] $\Rightarrow$ (v) $\Rightarrow$ (vi) are obvious ((iii) implies (v) since the function $\xi \mapsto \max\{\xi_1, \dots, \xi_n\}$ is continuous, monotone and coercive on $[0,+\infty)^n$).

The remaining implication (vi) $\Rightarrow$ (i) follows from Theorem 1: if (i) does not hold, there exists a three-point set in $f^{-1}(\sigma)$ without Chebyshev centers in $f^{-1}(\sigma)$; it suffices to apply a translation that maps $f^{-1}(\sigma)$ onto H to show that (vi) does not hold. □

Concluding remarks.

(a) We have learned recently that N. V. Zamyatin proved in [9] the following characterization of hyperplanes $H \subset C(K)$ in which every bounded subset has a Chebyshev center (recall that the elements of $C(K)^*$ can be identified with signed regular Borel measures on K of bounded variation):

Let K be a compact topological space, $\mu \in C(K)^*$, $H = \mu^{-1}(0)$. Then the following assertions are equivalent:

(α) *Either H is proximinal and* $\operatorname{supp} \mu$ *is extremally disconnected in K, or $2|\mu(\{k\})| \geqslant \|\mu\|_{C(K)^*}$ for some $k \in K$.*

(β) *Every bounded subset of H has a Chebyshev center in H.*

While the methods of [9] are completely different from ours, the condition (α) presents analogues to conditions (i), (ii) of Theorem 2.

(b) Let us remark that in [8] we have studied (among others) the Banach spaces X for which every bounded subset of the space $c_0(X)$ of all null sequences in X (with the supremum norm) has a Chebyshev center. An example therein shows that the conditions (iv) and (v) from Theorem 2 are not equivalent for a general Banach space H.

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