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NOTE ON A VARIATION OF THE SCHRÖDER-BERNSTEIN PROBLEM FOR FIELDS

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Abstract. In this note we study fields F with the property that the simple transcendental extension $F(u)$ of F is isomorphic to some subfield of F but not isomorphic to F. Such a field provides one type of solution of the Schröder-Bernstein problem for fields.

Keywords: field, subfield, isomorphism, transcendental extension, algebraic extension

MSC 2000: 12E99, 12F05, 12F20

In [2] there is an abelian group G that contains subgroups G_1 and G_2, $G \supset G_1 \supset G_2$, such that G is isomorphic to G_2 but not to G_1. This solution to the Schröder-Bernstein problem for abelian groups has the additional feature that G_1 is a direct summand of G and G_2 is a direct summand of G_1.

In functional analysis, Gowers [1] provided an analogous solution for Banach spaces. He constructed Banach spaces B, B_1, B_2 such that $B \supset B_1 \supset B_2$, B is isomorphic to B_2 but not to B_1, B_1 is a direct summand of B and B_2 is a direct summand of B_1.

In this note, we discuss one type of solution to the Schröder-Bernstein problem for fields. We cannot provide the direct summands because the direct sum of two fields is generally a ring but not a field.

By an SB-field we mean a field F such that the simple transcendental extension $F(u)$ of F is isomorphic to a subfield of F but not isomorphic to F. Thus F and $F(u)$ are a solution to the Schröder-Bernstein problem for fields. Recall that the simple transcendental extension of F is just the field of rational functions over F ([4], Section 32). Routine arguments ([4], Section 64) show that an SB-field must be of infinite degree of transcendence (over its prime subfield). We say that a field F is cube root complete (square root complete) if for each $y \in F$ there is an $x \in F$ such that $x^3 = y$ ($x^2 = y$).

In Theorem I we find that a cube root complete or square root complete field F of infinite degree of transcendence must contain an SB-subfield. It has been known among some algebraists that if F is algebraically closed, then F must be an SB-field. (For an easy proof, consult the secondary argument in the proof of Theorem I.) Hence, the field of real numbers $\mathbb{R}$ contains an SB-subfield that is not algebraically closed (the polynomial $x^2 + 1$ has no zero in $\mathbb{R}$), so an SB-field need not be algebraically closed.

Any uncountable field must be of infinite degree of transcendence, and it follows that the field of complex numbers C is an SB-field (Theorem I). We also show that $\mathbb{R}$ is not an SB-field. We seek cube root complete fields of infinite degree of transcendence that are not SB-fields. Of course $\mathbb{R}$ is one such field, but we also will construct such a countable field (Proposition 1).

Theorem I. *Let F be a field of infinite degree of transcendence that is either cube root complete or square root complete. Then there is a subfield K of F that is an SB-field. Moreover, if F is algebraically closed, then F is an SB-field.*

Proof. We will give the proof for cube root complete F. The proof for square root complete F is analogous, so we leave it. Let P be the result of adjoining to the prime subfield of F all the cube roots of unity in F (there are one or three). Let $y, x_1, x_2, x_3, \ldots, x_n, \ldots$ be countably infinitely many algebraically independent elements of F. Let F_0 denote $P(y, x_1, x_2, x_3, \ldots, x_n, \ldots)$.

Let W denote the family of all cube root complete subfields of F containing F_0. Then $F \in W$. By the Hausdorff Maximum Principle ([3], p. 32) there is a maximal chain of members of W; call it $\{F_a\}_a$. Because no element can have more than 3 cube roots, we deduce that $\bigcap\limits_a F_a$ is the smallest member of this maximal chain. Any field G such that $\bigcap\limits_a F_a \supset G \supset F_0$ and $G \neq \bigcap\limits_a F_a$ cannot be cube root complete. Put $F_b = \bigcap\limits_a F_a$.

Let φ_0 be the isomorphism of F_0 onto $P(x_1, x_2, x_3, \ldots, x_n, \ldots)$ which leaves each element of P fixed and maps y to x_1 and x_j to x_{j+1} for all j. Let $\{\varphi\}$ denote the family of all isomorphisms extending φ_0 whose domain is a subfield of F_b and whose range is a subfield of F_b algebraic over $P(x_1, x_2, x_3, \ldots, x_n, \ldots)$. Then $\varphi_0 \in \{\varphi\}$. We partially order $\{\varphi\}$ as follows: $\varphi_1 \leqslant \varphi_2$ means that φ_2 extends φ_1. Again by the Hausdorff Maximum Principle, there is a maximal chain $\{\varphi_a\}_a$ in $\{\varphi\}$. It follows that the greatest common extension φ_b of all the φ_a is the greatest member of $\{\varphi_a\}_a$.

We claim that the domain of φ_b is F_b. Assume, to the contrary, that it is not. Then the domain of φ_b is a proper subfield of F_b and hence is not cube root complete. There is a $v \in$ domain of φ_b such that the polynomials $x^3 - v$ and $x^3 - \varphi(v)$ are irreducible over (domain φ_b) and (range φ_b) respectively. We extend φ_b to an isomorphism φ' by mapping a zero of $x^3 - v$ in F_b to a zero of $x^3 - \varphi_b(v)$ in F_b, and this conflicts with the maximality of φ_b. It follows that φ_b is an isomorphism of F_b onto a subfield of F_b that is algebraic over $P(x_1, x_2, x_3, \ldots, x_n, \ldots)$. Put $K = \varphi_b(F_b)$.

Now y is transcendental and K is algebraic over $P(x_1, x_2, x_3, \ldots, x_n, \ldots)$ so y is transcendental over K. Moreover $K(y) \subset F_b$ so $\varphi_b(K(y)) \subset \varphi_b(F_b) = K$. It remains to prove that $K(y)$ is not isomorphic to K. Note that K is isomorphic to the cube root complete field F_b, so K is cube root complete. Now suppose $K(y)$ is isomorphic to K. Then $K(y)$ is cube root complete. There must exist polynomials $p(y)$ and $q(y)$ in y with coefficients in K such that $(p(y)/q(y))^3 = y$ and

$$(p(y))^3 = y(q(y))^3$$

where the degree of the left side is a multiple of 3 and the degree of the right side is not a multiple of 3. This contradiction proves that $K(y)$ is not isomorphic to K. Hence K is an SB-subfield of F.

Now let F be algebraically closed. Let A be a (necessarily infinite) algebraic basis of F ([4], Section 64). Let B be the result of deleting from A one particular element w. Let $P(B)^*$ denote an algebraic closure of $P(B)$ inside the algebraically closed field F. Then w is transcendental and $P(B)^*$ is algebraic over $P(B)$, so w is transcendental over $P(B)^*$. But $P(B)$ is isomorphic to $P(A)$ because A and B have the same cardinality. Thus $P(B)^*$ is isomorphic to the algebraic closure of $P(A)$ which in turn is isomorphic to F. It follows that $P(B)^*(w)$ is a subfield of F that is isomorphic to the simple transcendental extension of F. That this extension is not isomorphic to F is proved by the same argument used in the preceding paragraph, so we leave it. □

A cardinality argument can be used to prove that any uncountable field has infinite degree of transcendence. From Theorem I we deduce that the real and complex fields have SB-subfields. Moreover C is an SB-field. We have:

Corollary 1. *The algebraic closure of any uncountable field is an SB-field.*

We seek fields of infinite degree of transcendence that are cube root complete and yet are not SB-fields. We find both countable and uncountable fields with these properties.

Proposition 1. *The real field $\mathbb{R}$ is not an SB-field. Moreover, there is a countable subfield H of $\mathbb{R}$ that is cube root complete and of infinite degree of transcendence but is not an SB-field.*

P r o o f. Let H_0 denote a countable subfield of $\mathbb{R}$ of infinite degree of transcendence. Let H_1 be the subfield of $\mathbb{R}$ generated by the set $\{x \in \mathbb{R}\colon\ x^3 \in H_0\}$. Let H_2 be the subfield of $\mathbb{R}$ generated by the set $\{x \in \mathbb{R}\colon\ x^2 \in H_1\}$. Let H_3 be the subfield of $\mathbb{R}$ generated by the set $\{x \in \mathbb{R}\colon\ x^3 \in H_2\}$. Let H_4 be the subfield of $\mathbb{R}$ generated by the set $\{x \in \mathbb{R}\colon\ x^2 \in H_3\}$. In general H_{n+1} is the subfield of $\mathbb{R}$ generated by the set $\{x \in \mathbb{R}\colon\ x^2 \in H_n\}$ if n is odd and generated by the set $\{x \in \mathbb{R}\colon\ x^3 \in H_n\}$ if n is even. By induction we obtain an expanding sequence of countable subfields of $\mathbb{R}$. Let H be the greatest common extension of all the H_n. It is clear from the construction that H is cube root complete, and countable. Moreover, if $y \in H$ and y is positive, then H contains the square root of y. Of course H is of infinite degree of transcendence because H_0 is.

Let φ be an isomorphism of H into H. If $r \in H$, $s \in H$ and $r < s$, then $s - r$ is positive, $(s-r)^{\frac{1}{2}} \in H$, $\varphi((s-r)^{\frac{1}{2}})^2 = \varphi(s-r) = \varphi(s) - \varphi(r) > 0$ and $\varphi(s) > \varphi(r)$. Thus φ preserves order on H. But φ maps each rational number to itself. For any $h \in H$, h and $\varphi(h)$ exceed the same rational numbers and are exceeded by the same rational numbers, so $h = \varphi(h)$. It follows that there cannot be any proper extension of H isomorphic to a subfield of H. So H is not an SB-field. By essentially the same argument, $\mathbb{R}$ is not an SB-field. $\square$

We sum up:
The field of complex numbers is an SB-field, but the field of real numbers is not. Any algebraically closed field of infinite degree of transcendence is an SB-field, but an SB-field need not be algebraically closed. A cube root complete field of infinite degree of transcendence need not be an SB-field, but it must contain an SB-subfield. We leave open the question whether there exists a square root complete field of infinite degree of transcendence that is not an SB-field. I conjecture yes, but the matter could be the topic of further study. Another problem is to find a necessary and sufficient condition for a field to be an SB-field.

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