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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A $2n^{\text{th}}$ ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper we prove two results. The first is an extension of the result of G. D. Jones [4]:

(A) Every nontrivial solution for

$$\begin{cases} (-1)^n u^{(2n)} + f(t,u) = 0, \ \text{ in } \ (\alpha,\infty), \\ u^{(i)}(\xi) = 0, \quad i = 0,1,\dots,n-1, \quad \text{and} \quad \xi \in (\alpha,\infty), \end{cases}$$

must be unbounded, provided $f(t,z)z \geqslant 0$, in $E \times \mathbb{R}$ and for every bounded subset I, $f(t,z)$ is bounded in $E \times I$.

(B) Every bounded solution for $(-1)^n u^{(2n)} + f(t,u) = 0$, in $\mathbb{R}$, must be constant, provided $f(t,z)z \geqslant 0$ in $\mathbb{R} \times \mathbb{R}$ and for every bounded subset I, $f(t,z)$ is bounded in $\mathbb{R} \times I$.

Keywords: asymptotic behavior, higher order differential equation

MSC 2000: 34D05

1. Introduction

Asymptotic behavior of solution to differential equations has been widely studied. For example Hastings and Lazer [2] proved that assuming

$$p(t) \in C^1[\alpha,\infty), \ \ p'(t) \geqslant 0 \ \text{ and } \ \lim_{t\to\infty} p(t) = \infty, \tag{1.1}$$

all oscillatory solutions of

$$y^{(4)} - p(t)y = 0 \tag{1.2}$$

tend to zero. G. D. Jones [3] showed that, assuming

$$p(t) \in C^1[\alpha, \infty), \ \ p'(t) \leqslant 0 \ \text{ and } \ \lim_{t\to\infty} p(t) = 0, \tag{1.3}$$

all oscillatory solutions of (1.2) are unbounded. Biernacki [1] proved that, assuming (1.1),

$$y^{(4)} + p(t)y = 0 \tag{1.4}$$

has at least one oscillatory solution tending to zero. This result was generalized by Švec [6]. Švec proved that (1.4) has two linearly independent oscillatory solutions that tend to zero assuming only $0 < m \leqslant p(t)$. Švec [6] also proved that if $0 < m \leqslant p(t) \leqslant M$ and (1.4) is oscillatory, then (1.4) has a pair of unbounded solutions.

Recently G. D. Jones [4] extended this result to the following: if $0 \leqslant p(t) \leqslant M$ and (1.4) is oscillatory, then it has a pair of solutions such that every linear combination of them is unbounded. In this paper we extend the result of G. D. Jones [4] and show that Liouville's theorem holds for (1.5.2). We now list our conclusions:

(A) Every nontrivial solution of (1.5.1) with assumptions (1.6.1) and (1.7) is unbounded, which is stated in Theorem 3.1.

$$Lu = (-1)^n u^{(2n)} + f(t,u) = 0 \ \text{ in } \ E = (\alpha, \infty). \tag{1.5.1}$$

$$(-1)^n u^{(2n)} + f(t,u) = 0 \quad \text{in } \mathbb{R}. \tag{1.5.2}$$

$$f(t,z)z \geqslant 0 \ \text{ in } \ E \times \mathbb{R} \ \text{ and } \ f(t,z) \ \text{ is bounded in } \ E \times I \ \text{ for every bounded subset } \ I \ \text{ of } \mathbb{R}. \tag{1.6.1}$$

$$f(t,z)z \geqslant 0 \quad \text{in } \mathbb{R} \times \mathbb{R} \text{ and } f(t,z) \text{ is bounded in } \mathbb{R} \times I \text{ for every bounded subset } I \text{ of } \mathbb{R}. \tag{1.6.2}$$

$$\text{There exists a } \xi \text{ in the domain of } u \text{ that } u^{(i)}(\xi) = 0 \text{ for } i = 0, 1, \ldots, n-1. \tag{1.7}$$

In particular, let $f(t,u) = p(t)u$. Then we have the following generalization of the result of G. D. Jones [4]: Assume $p(t)$ is nonnegative and bounded in E. Then there are n linearly independent solutions of (1.8) such that every linear combination of them is unbounded, except the trivial solution, which is stated in Theorem 3.2,

$$(-1)^n u^{(2n)} + p(t)u = 0 \ \text{ in } \ E = (\alpha, \infty). \tag{1.8}$$

(B) Every bounded solution u of (1.5.2) with assumption (1.6.2) is constant.

2. Preliminary

We begin by defining some functionals and showing their relations.

Definition 2.1. Let $u \in C^{2m}(\Omega)$, $\Omega = [\beta,\gamma]$. We define

$$P_{2m}(u,\Omega) = \int_\beta^\gamma (-1)^m u u^{(2m)}\,\mathrm{d}t \quad \text{for } m = 0,1,\ldots,$$

$$G_{2m}(u) = \begin{cases} 0, & \text{if } m = 0\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{u^2}{2}\Big), & \text{if } m = 1,\\ (-1)^{m-1}\dfrac{\mathrm{d}}{\mathrm{d}t}\big(uu^{(2m-2)}\big) + 2G_{2m-2}(u') - G_{2m-4}(u''), & \text{if } m \geqslant 2,\end{cases}$$

and

$$H_{2m}(u) = \begin{cases} 0, & \text{if } m = 0\\ \dfrac{u^2}{2}, & \text{if } m = 1,\\ (-1)^{m-1}\big(uu^{(2m-2)}\big) + 2H_{2m-2}(u') - H_{2m-4}(u''), & \text{if } m \geqslant 2.\end{cases}$$

In the following lemmas we now show their relations and properties.

Lemma 2.2. *If $u \in C^{2m}(\Omega)$ and $\Omega = [\beta,\gamma]$, then*

$$P_{2m}(u,\Omega) = -G_{2m}(u)\big|_\beta^\gamma + \int_\beta^\gamma \big(u^{(m)}\big)^2\,\mathrm{d}t, \quad \textit{where } m = 0,1,\ldots.$$

Proof. The proof is done by induction on m. For $m = 0$ it is evident. For $m = 1$, by integration by parts, we have

$$\begin{aligned} P_2(u,\Omega) &= \int_\beta^\gamma -uu''\,\mathrm{d}t\\ &= -\,uu'\big|_\beta^\gamma + \int_\beta^\gamma (u')^2\,\mathrm{d}t\\ &= -\,G_2(u)\big|_\beta^\gamma + \int_\beta^\gamma (u')^2\,\mathrm{d}t. \end{aligned}$$

Suppose that the assertion holds for $m = 0, 1, \ldots, k$. We shall show that it is true for $m = k+1$. By repeating integration by parts we obtain

$$\begin{aligned}
P_{2k+2}(u,\Omega) = {} & \int_\beta^\gamma (-1)^{k+1} u u^{(2k+2)}\,\mathrm{d}t \\
& = (-1)^{k+1} u u^{(2k+1)}\big|_\beta^\gamma - \int_\beta^\gamma (-1)^{k+1} u' u^{(2k+1)}\,\mathrm{d}t \\
& = (-1)^{k+1} u u^{(2k+1)}\big|_\beta^\gamma - (-1)^{k+1} u' u^{(2k)}\big|_\beta^\gamma + \int_\beta^\gamma (-1)^{k+1} u'' u^{(2k)}\,\mathrm{d}t \\
& = P_{2k-2}(u'',\Omega) + (-1)^{k+1} \frac{\mathrm{d}}{\mathrm{d}t}\big(u u^{(2k)}\big)\Big|_\beta^\gamma - 2(-1)^{k+1} u' u^{(2k)}\big|_\beta^\gamma \\
& = P_{2k-2}(u'',\Omega) + (-1)^{k+1} \frac{\mathrm{d}}{\mathrm{d}t}\big(u u^{(2k)}\big)\Big|_\beta^\gamma \\
& \quad - 2\Big[\int_\beta^\gamma (-1)^{k+1} u'' u^{(2k)}\,\mathrm{d}t + \int_\beta^\gamma (-1)^{k+1} u' u^{(2k+1)}\,\mathrm{d}t\Big] \\
& = P_{2k-2}(u'',\Omega) + (-1)^{k+1} \frac{\mathrm{d}}{\mathrm{d}t}\big(u u^{(2k)}\big)\big|_\beta^\gamma - 2[P_{2k-2}(u'',\Omega) - P_{2k}(u',\Omega)] \\
& = (-1)^{k+1} \frac{\mathrm{d}}{\mathrm{d}t}(u u^{(2k)})\Big|_\beta^\gamma - P_{2k-2}(u'',\Omega) + 2P_{2k}(u',\Omega) \\
& = (-1)^{k+1} \frac{\mathrm{d}}{\mathrm{d}t}(u u^{(2k)})\Big|_\beta^\gamma - 2G_{2k}(u')\big|_\beta^\gamma + G_{2k-2}(u'')\big|_\beta^\gamma + \int_\beta^\gamma (u^{(k+1)})^2\,\mathrm{d}t \\
& = \; - G_{2k+2}(u)\big|_\beta^\gamma + \int_\beta^\gamma (u^{(k+1)})^2\,\mathrm{d}t,
\end{aligned}$$

where the last identity holds by virtue of the definition of $G_{2n}(u)$. Hence the proof of the lemma is complete. □

The following lemma is often used in the proofs of the main theorems.

Lemma 2.3. *Let $i = 1, 2$. If u is a solution of* (1.5.i) *satisfying assumption* (1.6.i), *then*

(1) $\mathrm{d}/\mathrm{d}t\, H_{2n}(u) = G_{2n}(u)$.
(2) *$G_{2n}(u)$ is increasing.*
(3) *$H_{2n}(u)(\xi) = 0$ and $G_{2n}(u)(\xi) = 0$ provided u satisfies condition* (1.7).
(4) *There exists $c \in [\xi, \infty)$ such that $G_{2n}(u)(t) > 0$ if $t > c$ provided u satisfies condition* (1.7) *and does not vanish in $[\xi, \infty)$.*

Proof. (1) By the definitions of $H_{2n}(u)$ and $G_{2n}(u)$, and using the induction on n, it is easy to check that part (1) is true.

(2) Multiplying both sides of $Lu = 0$ by u, integrating the resulting expression over any closed subset $\Omega = [\beta, \gamma]$ of the domain of u and using Lemma 2.2, we have

$$
\begin{aligned}
(2.1) \qquad 0 = \int_\beta^\gamma uLu\,\mathrm{d}t &= \int_\beta^\gamma (-1)^n uu^{(2n)}\,\mathrm{d}t + \int_\beta^\gamma f(t,u)u\,\mathrm{d}t \\
&= P_{2n}(u,\Omega) + \int_\beta^\gamma f(t,u)u\,\mathrm{d}t \\
&= -\,G_{2n}(u)\big|_\beta^\gamma + \int_\beta^\gamma (u^{(n)})^2\,\mathrm{d}t + \int_\beta^\gamma f(t,u)u\,\mathrm{d}t,
\end{aligned}
$$

and this implies that $G_{2n}(u)\big|_\beta^\gamma \geqslant 0$ for every $\gamma > \beta$.

Hence $G_{2n}(u)$ is increasing and we have completed the proof of part (2).

(3) We assume that the identities hold for $n = 0, 1, \ldots, k-1$. We shall show that $G_{2k}(u)(\xi) = 0$ provided $u^{(i)}(\xi) = 0$, $i = 0, 1, \ldots, k-1$. By Definition 2.1, it is easy to verify that

$$G_{2k}(u)(\xi) = \left[(-1)^{(k-1)}\frac{\mathrm{d}}{\mathrm{d}t}(uu^{(2k-2)}) + 2G_{2k-2}(u') - G_{2k-4}(u'')\right](\xi) = 0,$$

since $G_{2k-2}(u')(\xi) = 0$ and $G_{2k-4}(u'')(\xi) = 0$ provided $u^{(j)}(\xi) = 0$, $j = 1, 2, \ldots, k-1$. Similarly we have $H_{2n}(u)(\xi) = 0$. Hence the proof of part (3) is complete.

(4) We denote the domain of u by D. By parts (2) and (3) we have

$$(2.2) \qquad G_{2n}(u)(t) \geqslant 0 \ \text{ in } \ [\xi, \infty).$$

Suppose the result is not true. Then we have $G_{2n}(u)(t) = 0$ in $[\xi, \infty)$ by virtue of (2.2) and Lemma 2.3, part (2). Multiplying both sides of $Lu = 0$ by u and integrating over any subset $[\beta, \gamma)$ of D, we get

$$
\begin{aligned}
0 = \int_\beta^\gamma uLu\,\mathrm{d}t &= \int_\beta^\gamma (-1)^n uu^{(2n)}\,\mathrm{d}t + \int_\beta^\gamma f(t,u)u\,\mathrm{d}t \\
&= P_{2n}(u,\Omega) + \int_\beta^\gamma f(t,u)u\,\mathrm{d}t \\
&= -\,G_{2n}(u)\big|_\beta^\gamma + \int_\beta^\gamma [(u^{(n)})^2 + f(t,u)u]\,\mathrm{d}t \\
&= \int_\beta^\gamma [(u^{(n)})^2 + f(t,u)u]\,\mathrm{d}t.
\end{aligned}
$$

This show that $u^{(n)}$ vanishes in $[\xi, \infty)$. So u must be a polynomial function of degree less than n with n roots at ξ, since $u^{(i)}(\xi) = 0$, $i = 0, 1, \ldots, n-1$, and this impies that u vanishes in $[\xi, \infty)$, which contradicts our hypothesis. Hence part (4) is done. $\square$

In order to show that $H_{2n}(u)$ and $G_{2n}(u)$, which are used in the main theorems, are both bounded provided u is bounded, we quote the result of [5].

Lemma 2.4 ([5]). *Let $1 \leqslant k \leqslant \infty$, let i, j be integers with $1 \leqslant j \leqslant i$, and let J be any interval of the real line bounded or unbounded. Given any $\varepsilon > 0$ there exists a positive $k(\varepsilon)$ such that if $y \in L^k(J)$, $y^{(i-1)}$ is locally absolutely continuous and $y^{(i)} \in L^k(J)$, then $y^{(j)} \in L^k(J)$ and*

$$\|y^{(j)}\|_k \leqslant \varepsilon \|y^{(i)}\|_k + k(\varepsilon)\,\|y\|_k,$$

where $k(\varepsilon)$ depends only on ε and the length of J and $\|y\|_k$ denotes the L^k norm of y.

Remark 2.5. If u is a bounded solution of (1.5. i) satisfying the assumption (1.6. i), then $u^{(2n)}$ is bounded. According to Lemma 2.4, we have that $u^{(i)}$ is bounded, $i = 0, 1, \ldots, 2n$. Hence $H_{2n}(u)$ and $G_{2n}(u)$ also are bounded by virtue of the definitions of $H_{2n}(u)$ and $G_{2n}(u)$.

3. Main result

We are now ready to show our main theorems.

Theorem 3.1. *Every nontrivial solution of* (1.5.1) *satisfying assumptions* (1.6.1) *and* (1.7) *is unbounded.*

Proof. Suppose that a solution u is bounded in E. Then we have

$$H_{2n}(u) \text{ is bounded in } E, \tag{3.1}$$

according to Remark 2.5. By Lemma 2.3, parts (2) and (4), there exists a number c in E such that

$$G_{2n}(u)(t) \geqslant G_{2n}(u)(c) > 0 \quad \text{for} \quad t > c, \tag{3.2}$$

and using (3.1), Lemma 2.3. part (1) and the mean value theorem, we have

$$|H_{2n}(u)(t) - H_{2n}(u)(c)| = |G_{2n}(u)(d)(t - c)| \geqslant G_{2n}(c)(t - c),$$

where $d \in (c, t)$, since the last inequality follows by (3.2). So $H_{2n}(u)(t) \to \infty$ as $t \to \infty$. Thus $H_{2n}(u)(t)$ is unbounded, which contradicts (3.1).

Hence we have completed the proof of the theorem. $\square$

The following result which is an extension of the result of G. D. Jones [4], is a special case of Theorem 3.1.

Theorem 3.2. *Suppose $p(t)$ is nonnegative and bounded in E. There are n linearly independent solutions of* (1.8) *such that every linear combination of them is unbounded, except the trivial solution.*

P r o o f. Let u_i, $i = 0, 1, \ldots, 2n-1$, be $2n$ linearly independent solutions of (1.8) that satisfy $u_i^{(k)}(\xi) = \delta_{ik}$, $i, k = 0, 1, \ldots, 2n-1$, where δ_{ik} is the Kronecker symbol. And let $u = \sum\limits_{i=n}^{2n-1} b_i u_i$, where b_i, $i = n, n+1, \ldots, 2n-1$, be constants such that at least one b_i is not zero. It is easy to verify that u satisfies assumption (1.7) in Theorem 3.1 and by virtue of Theorem 3.1, u is unbounded in E. Hence the theorem is proved. □

Now we show the last theorem.

Theorem 3.3. *Every bounded solution u of* (1.5.2) *satisfying assumption* (1.6.2) *is a constant.*

P r o o f. According to Remark 2.5, $H_{2n}(u)$ is bounded. We claim that $G_{2n}(u)(t)$ vanishes in $\mathbb{R}$.

Suppose there is a $c \in \mathbb{R}$ such that $G_{2n}(u)(c) > 0$. According to Lemma 2.3, parts (1), (2) and the fundamental theorem of calculus, we have

$$|H_{2n}(u)(t) - H_{2n}(u)(c)| = \left|\int_c^t G_{2n}(u)(s)\,\mathrm{d}s\right| \geqslant |t-c|\,|G_{2n}(u)(c)|, \qquad \text{for} \quad t \geqslant c,$$

and this implies that $|H_{2n}(u)(t)| \to \infty$ as $t \to \infty$. This contradicts the fact that $H_{2n}(u)$ is bounded. If there is a $c \in \mathbb{R}$ such that $G_{2n}(u)(c) < 0$, then by the same argument we have

$$|H_{2n}(u)(c) - H_{2n}(u)(t)| = \left|\int_t^c G_{2n}(u)(s)\,\mathrm{d}s\right| \geqslant |t-c|\,|G_{2n}(u)(c)| \qquad \text{for} \quad c \geqslant t,$$

and this implies that $|H_{2n}(u)(t) \to \infty$ as $t \to \infty$. This is also a contradiction.

Hence $G_{2n}(u)(t)$ vanishes in $\mathbb{R}$. According to (2.1), we conclude that $u^{(n)} = 0$ in $\mathbb{R}$. This means u is a polynomial function of degree less than n. It is well known that a bounded polynomial function must be constant. Hence we have completed the proof of the theorem. □

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