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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 3, 643–650

Persistent URL: `http://dml.cz/dmlcz/127750`

Czechoslovak Mathematical Journal, 52 (127) (2002), 643–650

A FUNCTION RELATED TO A LAGRANGE-BÜRMANN SERIES

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(Received August 30, 1999)

Abstract. An infinite series which arises in certain applications of the Lagrange-Bürmann formula to exponential functions is investigated. Several very exact estimates for the Laplace transform and higher moments of this function are developed.

Keywords: Lagrange-Bürmann, Laplace transform

MSC 2000: 32A05, 40A25

The Lagrange-Bürmann formula [1], [2] has found many applications, such as evaluating roots of certain transcendental equations and obtaining expansions of a function in powers of a related but different function. Here, we would like to discuss some properties of a function which appears naturally in some of these problems, for example, expanding the function $e^{\alpha z}$ in powers of $w = ze^{-z}$ where α is an arbitrary constant. The function $F(x)$ is defined by the series

$$F(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}x^{n-1}}{n!}\, e^{-nx}. \tag{1}$$

From this expression, one can obtain the usual Laplace transform and higher moment integrals of the Laplace transform of this function. This function is also found in the work of S. Ramanujan [3] in a form which is different from that which appears in (1), and also in the Questions and Solutions of his collected papers [4].

Lemma 1. *For* $0 \leqslant x \leqslant 1$,

$$\sum_{n=1}^{\infty} e^{-nx} \frac{(nx)^{n-1}}{n!} = 1. \tag{2}$$

P r o o f. Clearly, this holds at $x = 0$. Suppose that $x > 0$, then the equation $w = x/\varphi(x)$ where $\varphi(x)$ is regular in a neighbourhood of $x = 0$, $\varphi(0) \neq 0$ implies that a Lagrange series for x in powers of w can be written as follows

$$x = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{\mathrm{d}^{n-1}[\varphi(y)]^n}{\mathrm{d}y^{n-1}}\right]_{y=0}.$$

In the case in which one puts $\varphi(x) = \mathrm{e}^x$ so that $w = x\mathrm{e}^{-x}$, one obtains

$$x = \sum_{n=1}^{\infty} \frac{x^n \mathrm{e}^{-nx}}{n!} n^{n-1}.$$

Removing a factor of x on both sides of this expression gives (2).

Letting the variable in the region $x > 1$ be called y, any $y > 1$ can be associated to a particular $x < 1$ by means of the equation $x\mathrm{e}^{-x} = y\mathrm{e}^{-y}$. Thus

$$x = \sum_{n=1}^{\infty} \frac{n^{n-1} y^n}{n!} \mathrm{e}^{-ny}.$$

If $s(y)$ is defined by $x = ys(y)$ for $y > 1$, then

$$F(y) = s(y) = \sum_{n=1}^{\infty} \frac{n^{n-1} y^{n-1}}{n!} \mathrm{e}^{-ny}.$$

These results are summarized in Theorem 1. □

Theorem 1. *The function $F(x)$ defined by equation* (1) *is given by*

(3) $$F(x) = 1,\ \ x \leqslant 1, \qquad F(x) = s(x),\ \ x > 1.$$

Define the related functions $g_m(x)$ and $f_m(x)$ on $(0,1)$ and $I = [0,1]$ respectively as follows

(4) $$g_m(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-xn} \frac{(xn)^{n-m}}{n!}, \quad f_m(x) = x^m g_m(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-xn} \frac{x^n n^{n-m}}{n!}.$$

The terms in the series for $f_m(x)$ are defined and continuous for all $x \geqslant 0$. The function $x^n \mathrm{e}^{-xn}$ has a maximum at $x = 1$ for all $n \geqslant 1$ and a minimum of zero at $x = 0$. Setting

$$a_n(x) = \mathrm{e}^{-nx} \frac{n^{n-m}}{n!} x^n,$$

then the bound

$$\lim_{n\to\infty} \frac{a_{n+1}(x)}{a_n(x)} = x\mathrm{e}^{1-x} < 1,$$

holds for $x \neq 1$ since $x\mathrm{e}^{1-x}$ has a maximum value of 1 at $x = 1$. Therefore $f_m(x)$ in (4) converges on $(0,1)$ and $(1,\infty)$, respectively. Thus the series for $f_0(x)$ converges uniformly for all $x \in [0,1)$. Using Stirling's formula,

$$\frac{1}{n!} < \frac{1}{\sqrt{2\pi n}} n^{-n}\mathrm{e}^{n} \exp\Bigl(-\frac{1}{12n+\frac{1}{4}}\Bigr),$$

it follows that

$$a_n(x) < \frac{1}{\sqrt{2\pi n}} n^{-m} \exp\Bigl(-\frac{1}{12n+\frac{1}{4}}\Bigr) < \frac{1}{n^{m+1/2}}.$$

The series on the right hand side converges for all integers $m \geqslant 1$, since it is a p-series. The Weierstrass test then implies that the series in (4) for $f_m(x)$ converges uniformly on $\mathbb{R}$, for all integers $m \geqslant 1$, hence on I. Thus $f_m(x)$ can be differentiated term by term with respect to x in this interval. Also, $g_m(x)$ is obtained by multiplying $f_m(x)$ by x^{-m} when $x \neq 0$.

Theorem 2. *For $x \in I$ and $m \geqslant 1$,*

$$f_m(x) = \int_0^x \frac{(1-u)}{u} f_{m-1}(u)\,\mathrm{d}u, \tag{5}$$

and for $x \neq 0$,

$$g_m(x) = x^{-m} \int_0^x (1-u)u^{m-2} g_{m-1}(u)\,\mathrm{d}u. \tag{6}$$

Proof. Suppose $m \geqslant 1$ and let $x \in I$. Uniform convergence allows one to differentiate $f_m(x)$ with respect to x term by term to obtain

$$f_m'(x) = -\sum_{n=1}^{\infty} n\mathrm{e}^{-nx} \frac{x^n n^{n-m}}{n!} + \sum_{n=1}^{\infty} n\mathrm{e}^{-nx} \frac{x^{n-1} n^{n-m}}{n!} = \frac{(1-x)}{x} f_{m-1}(x).$$

Using (4), the function on the right hand side of this equation is continuous on I, so this differential equation can be integrated to give

$$f_m(x) = \int_0^x \frac{(1-u)}{u} f_{m-1}(u)\,\mathrm{d}u.$$

This is just (5). Using the definition $f_m(x) = x^m g_m(x)$, one obtains (6), and we are done. □

From Lemma 1, it is clear that $g_1(x) = 1$, hence $f_1(x) = x$ and the differential equation for $f_m(x)$ gives

$$f_0(x) = g_0(x) = \frac{x}{1-x}.$$

One can now use (6) to work out $g_2(x)$ for $m = 2$ as follows

$$g_2(x) = \frac{1}{x} - \frac{1}{2}.$$

In fact, a formula which gives $g_m(x)$ can be obtained and is given in the following Theorem.

Theorem 3. *For $x \in (0,1]$ and $m \geqslant 2$, the function $g_m(x)$ can be written as a finite series in the form*

$$g_m(x) = \frac{1}{x^{m-1}} + \sum_{i=1}^{m-1} \frac{a_{m,i}}{x^{m-1-i}}. \tag{7}$$

The coefficients $a_{m,i}$ can be determined recursively by using the relations

$$\begin{aligned} a_{m,1} &= \frac{1}{2}(a_{m-1,1} - 1), \\ a_{m,i} &= \frac{1}{i+1}\,(a_{m-1,i} - a_{m-1,i-1}), \quad i = 2, \dots, m-2 \\ a_{m,m-1} &= -\,\frac{1}{m}\,a_{m-1,m-2}. \end{aligned} \tag{8}$$

The recursion is initialized by using the value $a_{2,1} = -1/2$.

Proof. The proof is by induction. The form of $g_m(x)$ given by (7) is clear from the calculated form of $g_2(x)$, which agrees with (7) using the given initialization. Suppose that

$$g_{m-1}(x) = \frac{1}{x^{m-2}} + \sum_{i=1}^{m-2} \frac{a_{m-1,i}}{x^{m-i-2}}.$$

Substituting this into (6), one obtains

$$\begin{aligned} g_m(x) &= x^{-m} \int_0^x (1-t)\left(1 + \sum_{i=1}^{m-2} a_{m-1,i} t^i\right) \mathrm{d}t \\ &= x^{-m+1} + \frac{1}{2}(a_{m-1,1} - 1)x^{-m+2} \\ &\quad + \sum_{i=2}^{m-2} \frac{1}{i+1}\,(a_{m-1,i} - a_{m-1,i-1})x^{i-m+1} - \frac{1}{m}\,a_{m-1,m-2}. \end{aligned} \tag{9}$$

This is of the form (7), and the result follows by induction under the identification (8). □

Define the function $P_r(x)$ for $x > 0$ by means of the integral

$$P_r(x) = \int_0^\infty t^r e^{-xt} F(t)\,dt \tag{10}$$

in terms of the function F, which is defined by equation (1). The case $r = 0$ is just the Laplace transform of $F(x)$. Substituting the function F from (1) into the integral (10), one can obtain a series representation for the function $P_r(x)$ as follows

$$\begin{aligned} P_r(x) &= \int_0^\infty e^{-xt} \sum_{n=1}^\infty \frac{n^{n-1} t^{n+r-1}}{n!} e^{-nt}\,dt \\ &= \sum_{n=1}^\infty \int_0^\infty \frac{n^{n-1} t^{n+r-1}}{n!} e^{-(n+x)t} \\ &= \sum_{n=1}^\infty \frac{n^{n-1}}{n!(n+x)^{n+r-1}} \int_0^\infty z^{n+r-1} e^{-z}\,dz \\ &= \sum_{n=1}^\infty \frac{n^{n-1}}{n!(n+x)^{n+r-1}} \Gamma(n+r). \end{aligned}$$

For example, let us write the cases corresponding to $r = 0, 1$ explicitly in order to see their structure. They are

$$P_0(x) = \sum_{n=1}^\infty \frac{n^{n-2}}{(n+x)^{n-1}}, \qquad P_1(x) = \sum_{n=1}^\infty \frac{n^{n-1}}{(n+x)^n}.$$

The behaviour of $P_r(x)$ for larger values of x can be studied and we will do this here for the cases $r = 0$ and 1 proceeding directly from the integral (10).

Putting $r = 0$ in the integral (10), one can write

$$P_0(x) = \int_0^\infty e^{-xt} F(t)\,dt = \int_0^1 e^{-xt}\,dt + \int_1^\infty e^{-xt} s(t)\,dt = \frac{1}{x} + \frac{1}{x}\int_1^\infty e^{-xt}\,ds.$$

Taking $x = ys(y)$ in the equation $xe^{-x} = ye^{-y}$, one can solve the equation $(s-1)y = \log s$ for y. Therefore, $P_0(x)$ can be written as follows,

$$P_0(x) = \frac{1}{x} - \frac{e^{-x}}{x}\int_0^1 \left(e(1-v)^{1/v}\right)^x dv = \frac{1}{x} + R_0(x) \tag{11}$$

where $R_0(x)$ is defined in the obvious way.

Lemma 2. *The following inequalities hold for* $0^+ \leqslant v \leqslant 1$,

$$(1-v) \leqslant e(1-v)^{1/v} \leqslant (1-v)^{1/2}, \tag{12}$$

and

$$\text{(13)}\qquad \mathrm{e}(1-v)^{1/v} \geqslant \begin{cases} (1-4v/3)^{3/8}, & 0 \leqslant v \leqslant 3/4, \\ 0, & 3/4 \leqslant v \leqslant 1. \end{cases}$$

P r o o f. Only the proof of the first inequality in (12) will be given here. To do this define the function

$$f(v) = \log(1-v) - v\log(1-v) + v.$$

Then $f(0) = 0$ and $f'(v) = -\log(1-v) \geqslant 0$, so this function is strictly increasing on $[0,1)$, and $f(v) \geqslant 0$. Dividing by v and exponentiating, we are done. The other inequalities can be done in the same way. □

Theorem 4. *The following bounds for* $R_0(x)$ *defined in* (11) *hold,*

$$\text{(14)}\qquad -\frac{2\mathrm{e}^{-x}}{x(x+2)} \leqslant R_0(x) \leqslant -\frac{\mathrm{e}^{-x}}{x(x+1)}.$$

In fact, one can write

$$\text{(15)}\qquad P_0(x) = \frac{1}{x} - \frac{2\mathrm{e}^{-x}}{x(x+2+\theta)}, \quad 0 < \theta < \frac{2}{3}.$$

P r o o f. To prove these bounds, it is easy to see that the inequalities in (12) can be integrated with respect to v after raising the inequalities to the power x. This gives us

$$\frac{1}{x+1} \leqslant \int_0^1 \left(\mathrm{e}(1-v)^{1/v}\right)^x \,\mathrm{d}v \leqslant \frac{2}{x+2}.$$

On the other hand, using (13), one can obtain a slightly different lower bound as follows,

$$\frac{2}{x+8/3} \leqslant \int_0^1 \left(\mathrm{e}(1-v)^{1/v}\right)^x \,\mathrm{d}v \leqslant \frac{2}{x+2}.$$

Consequently, one can write $P_0(x)$ in the form (15). The limits $8/3$ and 2 are actually obtained at $x = \infty$ and $x = 0$. □

Consider the case $r = 1$ in the integral given by (10). Using the properties of $F(t)$ given in Theorem 1, one can write

$$\int_0^\infty t\mathrm{e}^{-xt} F(t)\,\mathrm{d}t = \int_0^1 t\mathrm{e}^{-xt}\,\mathrm{d}t + \int_1^\infty t\mathrm{e}^{-xt} s(t)\,\mathrm{d}t.$$

The first integral can be done by parts to give

$$\int_0^1 t\mathrm{e}^{-xt}\,\mathrm{d}t = -\frac{e^{-x}}{x} - \frac{1}{x^2}\,(\mathrm{e}^{-x}-1).$$

Similarly one obtains for the second integral,

$$\int_1^\infty t\mathrm{e}^{-xt}s(t)\,\mathrm{d}t = \frac{\mathrm{e}^{-x}}{x} + \frac{\mathrm{e}^{-x}}{x^2} + \int_1^\infty \Big(\frac{t}{x}+\frac{1}{x^2}\Big)\mathrm{e}^{-xt}\,\mathrm{d}s.$$

Adding these two equations, one obtains the following expression for $P_1(x)$

$$\int_0^\infty t\mathrm{e}^{-xt}F(t)\,\mathrm{d}t = \frac{1}{x^2} + \frac{1}{x}\int_1^\infty \Big(t+\frac{1}{x}\Big)\mathrm{e}^{-xt}\,\mathrm{d}s = \frac{1}{x^2} + R_1(x),$$

where $R_1(x)$ is defined in the obvious way. Again putting $x = ys(y)$ in the equation $x\mathrm{e}^{-x} = y\mathrm{e}^{-y}$, one can solve the equation $(s-1)y = \log s$ for y, and introducing the variable $s = 1-v$, one obtains the following expression for $R_1(x)$,

$$R_1(x) = -\frac{\mathrm{e}^{-x}}{x}\int_0^1 \Big(-\frac{\log(1-v)}{v}+\frac{1}{x}\Big)\big(\mathrm{e}(1-v)^{1/v}\big)^x\,\mathrm{d}v. \tag{16}$$

Theorem 5. *The following bounds for $R_1(x)$ hold*

$$-\frac{2\mathrm{e}^{-x}}{x^2}\,p_1(x) \leqslant R_1(x) \leqslant -\frac{2\mathrm{e}^{-x}}{x^2}\,p_2(x), \tag{17}$$

where

$$p_1(x) = \frac{x^4+21x^3+158x^2+504x+192}{(x+2)(x+4)(x+6)(x+8)}, \tag{18}$$

$$p_2(x) = \frac{x^3+\frac{46}{3}x^2+\frac{200}{3}x+\frac{128}{3}}{\big(x+\frac{8}{3}\big)\big(x+\frac{16}{3}\big)(x+8)}.$$

P r o o f. The following inequality is easy to show

$$g(v) = -\frac{\log(1-v)}{v} \leqslant (1+v)(1+v^2),$$

for v in the interval $[0,1]$. Using this fact and the inequality on the right of (12), it follows by integrating that

$$\begin{aligned}
&\frac{1}{x}\int_0^1 \Big(-\frac{\log(1-v)}{v}+\frac{1}{x}\Big)\big(\mathrm{e}(1-v)^{1/v}\big)^x\,\mathrm{d}v \\
&\quad\leqslant \frac{1}{x}\int_0^1 \Big((1+v)(1+v^2)+\frac{1}{x}\Big)(1-v)^{x/2}\,\mathrm{d}v \\
&\quad= \frac{2}{x^2}\,\frac{x^4+21x^3+158x^2+504x+192}{(x+2)(x+4)(x+6)(x+8)}
\end{aligned}$$

which is of the form $p_1(x)$ in (18).

Similarly, using the fact that $-\log(1-v)/v \geqslant 1 + v/2 + v^2/3$ on $[0,1)$, one can obtain a bound in the other direction using inequality (13),

$$\begin{aligned}
&\frac{1}{x}\int_0^1 \left(-\frac{\log(1-v)}{v} + \frac{1}{x}\right)\left(\mathrm{e}(1-v)^{1/v}\right)^x \,\mathrm{d}v \\
&\quad \geqslant \frac{1}{x}\int_0^1 \left(1 + \frac{v}{2} + \frac{v^2}{3} + \frac{1}{x}\right)\left(1 - \frac{4v}{3}\right)^{3x/8} \,\mathrm{d}v \\
&\quad = \frac{1}{9x^2}\,\frac{162x^3 + 2484x^2 + 10800x + 6912}{9x^3 + 144x^2 + 704x + 1024}
\end{aligned}$$

which is of the form $p_2(x)$ in (18) after rearranging constants. This gives the stated upper bound for $R_1(x)$ given in (17). □

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