L. Lindeboom; H. Raubenheimer

On regularities and Fredholm theory

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 3, 565–574

Persistent URL: `http://dml.cz/dmlcz/127744`

Czechoslovak Mathematical Journal, 52 (127) (2002), 565–574

ON REGULARITIES AND FREDHOLM THEORY

L. Lindeboom, Pretoria, and H. Raubenheimer, Johannesburg

(Received August 8, 1999)

Abstract. We investigate the relationship between the regularities and the Fredholm theory in a Banach algebra.

Keywords: regularities, Fredholm theory, inessential ideal

MSC 2000: 46H05, 46H10

1. Introduction

Regularities are introduced and studied in [12] and [15] to give an axiomatic theory for spectra in literature which do not fit into the axiomatic theory of Żelazko [22]. In this note we investigate the relationship between the regularities and the Fredholm theory in a Banach algebra.

All algebras in this paper are complex and unital. Denote by A^{-1} the group of invertible elements in a Banach algebra A and by $\sigma(a, A) = \{\lambda \in \mathbb{C} \mid a - \lambda \notin A^{-1}\}$ the ordinary spectrum of $a \in A$. When no confusion can arise we write simply $\sigma(a)$. If $K \subset \mathbb{C}$, we use the symbol $\operatorname{acc} K$ to indicate the set of accumulation points of K and the symbol iso K for the set of isolated points of K. The topological boundary is denoted by ∂K and the closure by $\overline{K}$. If K is a bounded subset of $\mathbb{C}$ then ηK designates the connected hull of $\overline{K}$. By an ideal in A we mean a two sided ideal in A. An ideal J in A is said to be *inessential* [1, p. 106] if

$$a \in J \Longrightarrow \operatorname{acc} \sigma(a) \subset \{0\},$$

so that the spectrum of an element of J is either finite or a sequence converging to zero. If J is a closed inessential ideal in A then by a result of Aupetit [1, Theorem 5.7.4 (iii)] and [17, Theorem 5.3] we have

$$a \in A \Longrightarrow \mathsf{acc}\,\sigma(a) \subset \eta\sigma(a+J, A/J). \tag{1.1}$$

We will say a closed ideal J in A is *s-inessential* whenever

$$a \in A \Longrightarrow \mathsf{acc}\,\sigma(a) \subset \sigma(a+J, A/J).$$

The radical of A will be denoted by $\operatorname{Rad} A$ and A is said to be *semisimple* if $\operatorname{Rad} A = \{0\}$. A Banach algebra A is called *semiprime* if $0 \neq u \in A$ implies there exists $x \in A$ such that $uxu \neq 0$. All semisimple Banach algebras are semiprime. An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$. The set of these elements will be denoted by $\mathrm{QN}(A)$. Recall that if J is a closed ideal in A then $b \in A$ is called *Riesz* relative to J if $b + J \in \mathrm{QN}(A/J)$, see [2, Section R.1]. The set $\mathsf{kh}\,J$ is defined by $\mathsf{kh}\,J = \{b \in A \mid b + J \in \operatorname{Rad} A/J\}$. Clearly, this set is contained in the set of Riesz elements relative to J. An element $a \neq 0$ in a semiprime Banach algebra A is called *rank one* if there exists a linear functional τ_a on A such that $axa = \tau_a(x)a$ for all $x \in A$. For properties of these elements we refer to [19]. The *finite elements* of A, denoted by $\mathcal{F}(A)$, is the set of all $a \in A$ of the form $a = \sum_{i=1}^{n} a_i$ with each a_i a rank one element. In the case of a semiprime Banach algebra the set of finite elements coincides with the socle of A, i.e. $\operatorname{Soc} A = \mathcal{F}(A)$. By [19, Lemma 2.7] $\mathcal{F}(A)$ is an ideal in A.

We call an element $a \in A$ *regular* if it has a generalized inverse in A, $b \in A$ for which $a = aba$, and write

$$\widehat{A} = \{a \in A \mid a \in aAa\}$$

for the set of regular elements. These include both the left and right invertible elements,

$$A^{-1}_{\text{left}} \cup A^{-1}_{\text{right}} \subset \widehat{A} \tag{1.2}$$

as well as the idempotents $A^{\bullet} = \{a \in A \mid a^2 = a\}$. The *decomposably regular* elements are those which admit invertible generalized inverses; they are those elements which can be written as the product of an invertible and an idempotent:

$$A^{-1}A^{\bullet} = A^{\bullet}A^{-1} = \{a \in A \mid a \in aA^{-1}a\} \subset \widehat{A}.$$

It is then familiar [8, Theorem 7.3.4] that

$$A^{-1}A^{\bullet} = \widehat{A} \cap \overline{A^{-1}}. \tag{1.3}$$

For properties of the regular and decomposably regular elements we refer to [7], [8], [10].

2. Regularities

In this section we gather basic information on regularities as developed in [12].

2.1. Definition [12, Definition 1.2]**.** A nonempty subset $\mathcal{R}$ of a Banach algebra A is called a regularity if

1. $a \in A$ and $n \in \mathbb{N}$ then $a \in \mathcal{R} \Leftrightarrow a^n \in \mathcal{R}$,
2. a, b, c, d are mutually commuting elements of A and $ac + bd = 1$ then $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$.

2.2. Proposition [12, Proposition 1.3]**.** *Let $\mathcal{R}$ be a regularity in a Banach algebra A.*

1) *If $a, b \in A$, $ab = ba$ and $a \in A^{-1}$ then $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$.*
2) *$A^{-1} \subset \mathcal{R}$.*

A regularity $\mathcal{R}$ in A defines a mapping $\widetilde{\sigma}_{\mathcal{R}}$ from A into subsets of $\mathbb{C}$ by $\widetilde{\sigma}_{\mathcal{R}}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \notin \mathcal{R}\}$ $(a \in A)$. This mapping will be called the *spectrum corresponding to* $\mathcal{R}$. When no confusion can arise we will write $\widetilde{\sigma}(a)$. For results on the spectrum arising from the regularities $\mathcal{R}_5$ and $\mathcal{R}_6$, [12, p. 111], we refer to [13].

Consider the following condition:

(P1) $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$ for all commuting elements $a, b \in A$.

Clearly a nonempty subset $\mathcal{R}$ of A satisfying (P1) is a regularity.

3. Subalgebras

In this section we investigate how the spectrum corresponding to a regularity depends on the algebra. For the regularity A^{-1} of invertible elements this dependence is familiar [21, Theorem VII.2.6] and [4].

3.1. Theorem. *Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose $\mathcal{R}_A$ is a regularity in A and $\mathcal{R}_B$ is a regularity in B such that $\mathcal{R}_{\mathcal{B}} \subset \mathcal{R}_{\mathcal{A}}$.*

1) *Then $\widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A) \subset \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ for every $b \in B$.*
2) *If $\partial \mathcal{R}_{\mathcal{B}} \cap \mathcal{R}_{\mathcal{A}} = \emptyset$ then $\partial \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B) \subset \widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A)$ for all $b \in B$ such that $\widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B) \neq \emptyset$.*

P r o o f. 1) Let $b \in B$. If $\lambda \notin \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ then $b - \lambda \in \mathcal{R}_{\mathcal{B}} \subset \mathcal{R}_{\mathcal{A}}$ and so $\lambda \notin \widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A)$.

2) Let $b \in B$ and $\lambda \in \partial \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$. Then there is a sequence (λ_n) in $\mathbb{C} \setminus \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ such that $\lambda_n \to \lambda$ and a sequence (μ_n) in $\widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ such that $\mu_n \to \lambda$. Then $(b - \lambda_n)$ is a sequence in $\mathcal{R}_{\mathcal{B}}$ such that $b - \lambda_n \to b - \lambda$ and $(b - \mu_n)$ is a sequence in $B \setminus \mathcal{R}_{\mathcal{B}}$ such that $b - \mu_n \to b - \lambda$. Consequently, $b - \lambda \in \partial \mathcal{R}_{\mathcal{B}}$ and since $\partial \mathcal{R}_{\mathcal{B}} \cap \mathcal{R}_{\mathcal{A}} = \emptyset$ it follows that $b - \lambda \notin \mathcal{R}_{\mathcal{A}}$ and so $\lambda \in \widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A)$. □

The above theorem applies to the regularity $\mathcal{R}_2 = A^{-1}$ [12, p. 111] of invertible elements: Let A and B be Banach algebras such that $1 \in B \subset A$. Then in general $B^{-1} \subset A^{-1}$ and if B is a closed subalgebra of A then it is well known that $\partial B^{-1} \cap A^{-1} = \emptyset$ [21, p. 398]. The proof of the next result follows from the definition of a regularity and will be omitted.

3.2. Proposition. *Let A and B be Banach algebras such that $1 \in B \subset A$. If $\mathcal{R}_\mathcal{A}$ is a regularity in A and $\mathcal{R}_\mathcal{B}$ is a regularity in B then $\mathcal{R}_\mathcal{A} \cap \mathcal{R}_\mathcal{B}$ is a regularity in B.*

3.3. Corollary. *Let A and B be Banach algebras such that $1 \in B \subset A$. If $\mathcal{R}_\mathcal{A}$ is a regularity in A then $\mathcal{R}_\mathcal{A} \cap \mathcal{B}$ is a regularity in B.*

For the regularity of invertible elements it is well known that if A is a C^* algebra and if B is a closed C^* subalgebra of A then $B^{-1} = A^{-1} \cap B$, see the proof of Theorem VII.6.5 in [21]. The proof of the next result follows from Corollary 3.3 and Theorem 3.1.1) and will be omitted.

3.4. Proposition. *Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose $\mathcal{R}_\mathcal{A}$ is a regularity in A. Then $\widetilde{\sigma}_{\mathcal{R}_\mathcal{A}}(b, A) = \widetilde{\sigma}_{\mathcal{R}_\mathcal{A} \cap \mathcal{B}}(b, B)$ for every $b \in B$.*

4. The radical

We provide a characterization of the radical in a Banach algebra involving a regularity in the algebra. The radical $\operatorname{Rad} A$ of A is the intersection of all maximal left (or right) ideals of A and it is familiar [1, Theorem 3.1.3] that

$$\operatorname{Rad} A = \{a \in A \mid 1 - Aa \subset A^{-1}\}.$$

It can also be shown that

$$\operatorname{Rad} A = \{a \in A \mid Aa \subset \mathrm{QN}(A)\}.$$

4.1. Proposition. *If $\mathcal{R}$ is a regularity in a Banach algebra A then* $\operatorname{Rad} A = \{a \in A \mid \mathcal{R}a \subset \mathrm{QN}(A)$.

P r o o f. Since $\mathcal{R} \subset A$ it follows that $\operatorname{Rad} A \subset \{a \in A \mid \mathcal{R}a \subset \mathrm{QN}(A)\}$. To prove the nontrivial inclusion suppose $a \in \{a \in A \mid \mathcal{R}a \subset \mathrm{QN}(A)\}$. Let $d \in A$. Since A is a complex Banach algebra, $A = A^{-1} + A^{-1}$ and so $d = d_1 + d_2$ with $d_i \in A^{-1}$ $(i = 1, 2)$. Since $A^{-1} \subset \mathcal{R}$ by Proposition 2.2.2), it follows from our assumption that $d_1a, (1 - d_1a)^{-1}d_2a \in \mathrm{QN}(A)$ and so $1 - da = (1 - d_1a)(1 - (1 - d_1a)^{-1}d_2a) \in A^{-1}$. We have shown that $a \in \{a \in A \mid 1 - Aa \subset A^{-1}\}$. □

Since A^{-1} is a regularity it follows at once from the above proposition that $\operatorname{Rad} A = \{a \in A \mid A^{-1}a \subset \mathrm{QN}(A)\}$. This result was proved in [18, Remark 4] by different methods.

Let X be a complex Banach space and let $\mathcal{T}$ be a subset of X satisfying $\alpha\mathcal{T} \subset \mathcal{T}$ for all $0 \neq \alpha \in \mathbb{C}$. Following [14] let $P(\mathcal{T}) = \{x \in X \mid x + \mathcal{T} \subset \mathcal{T}\}$. If A is a Banach algebra and $\mathcal{R}$ a regularity in A then by [14, Lemma 2.1] $P(\mathcal{R})$ is a linear subspace of A and if $\mathcal{R}$ is an open subset of A then $P(\mathcal{R})$ is closed in A. If in addition A is a commutative Banach algebra then by Proposition 2.2 $A^{-1}\mathcal{R} \subset \mathcal{R}$ and $\mathcal{R}A^{-1} \subset \mathcal{R}$. In view of [14, Lemma 2.3] $P(\mathcal{R})$ is an ideal in A.

4.2. Theorem. *Let $\mathcal{R}$ be a regularity in a Banach algebra A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$. Then*

1) $\partial\sigma(a, A) \subset \widetilde{\sigma}_{\mathcal{R}}(a, A) \subset \sigma(a, A)$ *for all* $a \in A$.
2) $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a, A) \subset \operatorname{acc} \sigma(a, A)$.
3) $\eta\sigma(a, A) = \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$.
4) $P(\mathcal{R}) \subset \operatorname{Rad} A$.

P r o o f. 1) Let $A = B$ in Theorem 3.1 and employ Proposition 2.2.2).

2) Follows from 1).

3) By 1) and the fact that the spectrum is closed it follows that $\overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)} \subset \sigma(a, A)$ and so $\eta\widetilde{\sigma}_{\mathcal{R}}(a, A) = \eta\overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)} \subset \eta\sigma(a, A)$, see the remarks preceding Lemma 1.1 in [11]. It also follows from 1) that $\partial\sigma(a, A) \subset \overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)}$ and so by [11, Theorem 1.2] $\sigma(a, A) \subset \eta\overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)} = \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$. Consequently, $\eta\sigma(a, A) \subset \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$. If we combine these remarks we obtain $\eta\sigma(a, A) = \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$.

4) Since $\mathcal{R}$ is a regularity it follows from Proposition 2.2 that $\alpha\mathcal{R} \subset \mathcal{R}$ for every $0 \neq \alpha \in \mathbb{C}$. Since $A^{-1} \subset \mathcal{R}$, by Proposition 2.2.2), and since A^{-1} is an open subset of A it follows from our assumption and Lemma 2.2 in [14] that $P(\mathcal{R}) \subset P(A^{-1}) = \operatorname{Rad} A$ [14, Theorem 2.5]. □

We mention illustrations of the above theorem: If A is a Banach algebra then for the regularities $\mathcal{R}_i$ ($i = 2, 3, 4, 5, 6$) [12, p. 111] it is familiar that $\partial A^{-1} \cap \mathcal{R}_i = \emptyset$, cf. [21, Theorem VII.2.5] and [3, Proposition].

5. Perturbation results

In this section we study the behaviour of elements belonging to a regularity under perturbations by rank one elements, inessential elements and Riesz elements.

5.1. Theorem. *Let A be a Banach algebra and suppose $\mathcal{R}$ is a regularity of A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$.*

1) *If J is a closed inessential ideal of $A, a \in A$ and $b \in J$ then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.
2) *If J is a closed inessential ideal of $A, a \in A$ and b is Riesz relative to J with $ab = ba$ then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

P r o o f. 1) Suppose J is a closed inessential ideal of A and $b \in J$. It follows from 1.1 that

$$\operatorname{acc} \sigma(a+b, A) \subset \eta\sigma(a+b+J, A/J) = \eta\sigma(a+J, A/J) \subset \eta\sigma(a, A).$$

If we combine this with Theorem 4.2.2) and 3) we obtain $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

2) The proof of this statement follows exactly in the same way as 1) if we observe that $b + J \in \mathrm{QN}(A/J)$ and $a + J$ and $b + J$ commute in A/J implies that $\sigma(a + b + J, A/J) = \sigma(a + J, A/J)$. □

5.2. Corollary. *Let A be a Banach algebra and suppose $\mathcal{R}$ is a regularity of A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$. If $a \in A$ and $b \in \operatorname{Rad} A$ then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

5.3. Corollary. *Let A be a semisimple Banach algebra and suppose $\mathcal{R}$ is a regularity of A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$. If $a \in A$ and if $b \in A$ is rank one then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

P r o o f. If $b \in A$ is rank one, then it belongs to the inessential ideal $\mathcal{F}(A)$ of finite elements [19, Sections 2 and 3]. By [1, Corollary 5.7.6] the closure $\overline{\mathcal{F}(A)}$ of $\mathcal{F}(A)$ is also an inessential ideal. □

One can also provide a direct proof of Corollary 5.3 if one combines [9, Theorem 5] and Theorem 4.2.2) and 3).

5.4. Theorem. *Let A and B be Banach algebras and $T\colon A \to B$ a bounded homomorphism with closed range. If $\mathcal{R}$ is a regularity of A and $\mathcal{M}$ is a regularity of B with $\partial B^{-1} \cap \mathcal{M} = \emptyset$ then for each $a \in A$*

$$\bigcap_{Tb=0} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{M}}(Ta, B).$$

P r o o f. This follows from [5, Theorem 3], Proposition 2.2.2) and Theorem 4.2 3). □

For the spectrum and singular spectrum the results in this section are familiar: e.g. [13, Section 3], [5, Theorem 5], [17, Theorem 5.3] and [1, Theorem 5.7.4 (iii)].

6. Regular elements

It is well known [7, Examples 4.5 and 4.6] and [10, Examples 1 and 2] that the elements of $\widehat{A}$ and $A^{-1}A^{\bullet}$ do not multiply well and so in general neither $\widehat{A}$ nor $A^{-1}A^{\bullet}$ is a regularity in A. However, we have the following

6.1. Proposition [12, Lemma 2.8]**.** *Let a, b, c, d be mutually commuting elements in a Banach algebra A with $ac + bd = 1$. Then $ab \in \widehat{A}$ if and only if $a, b \in \widehat{A}$.*

6.2. Lemma. *Let A be a semiprime Banach algebra. Then $\mathcal{F}(A) \subset A^{-1}A^{\bullet} \subset \widehat{A}$.*

Proof. We prove first that $\mathcal{F}(A) \subset \widehat{A}$. If $u \in \mathcal{F}(\mathcal{A})$ then by [19, Theorem 3.4] there is an idempotent $p \in \mathcal{F}(A) \cap uA$ such that $u = pu$. Since $p \in uA$, we have $p = uv$ for some $v \in A$. Consequently, $u = uvu$ which proves that u is regular. This together with $\mathcal{F}(\mathcal{A})$ being an inessential ideal in A gives $\mathcal{F}(A) \subset A^{-1}A^{\bullet}$ [10, Theorem 7 (7.2)]. □

6.3. Theorem. *Let A be a semiprime Banach algebra. Then $\widehat{A} + \mathcal{F}(A) \subset \widehat{A}$.*

Proof. By the last lemma $\mathcal{F}(A) \subset \widehat{A}$. The result now follows from [8, (7.3.2.6)]. □

This result was proved by Kordula and Müller [12, Lemma 2.9] in the algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X by different methods if one recalls that in the algebra $\mathcal{L}(X)$ the ideal of finite elements coincides with the ideal of finite rank operators, see [19].

Let J be an ideal in A. We say $a \in A$ is *J-Fredholm* if $a + J$ is invertible in the quotient algebra A/J. Recall [12, p. 111] that $\mathcal{R}_7 = \{a \in A \mid a \text{ is } J\text{-Fredholm}\}$ is a set satisfying (P1) and is therefore a regularity in A.

6.4. Proposition. *Suppose J is an ideal in A such that $J \subset \widehat{A}$. Then $\mathcal{R}_7 \subset \widehat{A}$.*

Proof. If $a \in \mathcal{R}_7$ then a is J-Fredholm and so by 1.2, we have $a + J \in \widehat{A/J}$. Since $J \subset \widehat{A}$, it follows from [8, Theorem 7.3.3] that $a \in \widehat{A}$. □

6.5. Theorem. *If J is a closed s-inessential ideal in A such that $J \subset \widehat{A}$ then $\mathcal{R}_7 \subset A^{-1}A^{\bullet}$.*

Proof. By Proposition 6.4 we have that $\mathcal{R}_7 \subset \widehat{A}$. Also, if $a \in \mathcal{R}_7$ then $0 \notin \sigma(a + J, A/J)$. In view of J being s-inessential it follows that $a \in \overline{A^{-1}}$. By 1.3 we conclude $a \in A^{-1}A^{\bullet}$. □

6.6. Theorem. *Let A be a semisimple Banach algebra and let J be an inessential ideal in A. Then $J \cap \widehat{A} \subset \mathcal{F}(A)$.*

P r o o f. Suppose $a = aa'a$ for some a' in A. If $a \in J$ then in view of [16, Theorem 1.4] the idempotent $a'a \in J \subset \mathsf{kh}\,\mathcal{F}(A)$. By [20, Theorem 4.6] we have $a'a \in \mathcal{F}(A)$. Since $\mathcal{F}(A)$ is an ideal in A it follows that $a \in \mathcal{F}(A)$. □

This result was proved by Harte [7, Theorem 4.2 (4.2.1)] in the algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X.

7. An example

In this section we provide an example of a regularity in a Banach algebra and investigate how this regularity is related to the set of decomposably regular elements.

An element $a \in A$ is said to be *almost invertible* if $0 \notin \mathsf{acc}\,\sigma(a)$ [6]. We have the following implications:

$$\text{invertible} \implies \text{almost invertible } J\text{-Fredholm} \implies J\text{-Fredholm}.$$

Let J be a closed ideal in a Banach algebra A. Denote

$$\mathcal{R}_0(J) = \{a \in A \mid a \text{ is almost invertible } J\text{-Fredholm}\}.$$

7.1. Proposition. *Suppose a closed ideal J in A is s-inessential. Then $\mathcal{R}_0(J)$ is a regularity in A.*

P r o o f. We prove that $\mathcal{R}_0(J)$ satisfies (P1). If $a, b \in \mathcal{R}_0(J)$ with $ab = ba$ then ab is J-Fredholm. Since $\sigma(ab) \subset \sigma(a) \cdot \sigma(b)$ it follows that $ab \in \mathcal{R}_0(J)$. Conversely, if $ab \in \mathcal{R}_0(J)$ then a and b are J-Fredholm because $ab = ba$. This together with J s-inessential gives $a, b \in \mathcal{R}_0(J)$. □

7.2. Corollary. $\widetilde{\sigma}_{\mathcal{R}_0(J)}(a) = \mathsf{acc}\,\sigma(a) \cup \sigma(a + J, A/J)$ *for every $a \in A$.*

P r o o f. This follows from the definition of $\mathcal{R}_0(J)$. □

We will prove later that $\mathcal{R}_0(J)$ is actually an open regularity, see Theorem 7.5. However, to prove a stronger result we need the following

7.3. Definition. Let J be a closed ideal in A and $a \in A$. We say that a is J-Browder if $a = x + y$ with $x \in A^{-1}$, $y \in J$ and $xy = yx$.

Then we have the following implications [6, 16]:

(7.4) invertible $\Longrightarrow$ almost invertible J-Fredholm $\Longrightarrow$ J-Browder $\Longrightarrow$ J-Fredholm.

If A and B are Banach algebras then the homomorphism $T\colon A \to B$ is said to have the *Riesz property* if its kernel $T^{-1}(0)$ is an inessential ideal. If J is a closed inessential ideal then the almost invertible J-Fredholm and J-Browder elements coincide [6, Theorem 1] or [17, Corollary 3.6].

7.5. Theorem. *Suppose J is a closed inessential ideal in A. Then $\mathcal{R}_0(J)$ is an open regularity in A.*

Proof. We prove that $\mathcal{R}_0(J)$ satisfies (P1). If $a, b \in \mathcal{R}_0(J)$ with $ab = ba$ then it follows as in the proof of Proposition 7.1 that $ab \in \mathcal{R}_0(J)$. Conversely, if $ab \in \mathcal{R}_0(J)$ then by 7.4 ab is J-Browder. In view of $ab = ba$ and J being inessential (meaning that the quotient map $A \to A/J$ has the Riesz property) it follows from [8, Theorem 7.7.6] that both a and b are J-Browder. By the remarks following 7.4 we have $a, b \in \mathcal{R}_0(J)$.

We prove finally that $\mathcal{R}_0(J)$ is open. Let $x \in \mathcal{R}_0(J)$ and let $\varepsilon > 0$ satisfy $\{\lambda \in \mathbb{C} \mid |\lambda| < 3\varepsilon\} \cap \sigma(x) \subset \{0\}$. Since $\sigma(\cdot)$ and $\sigma(\cdot, A/J)$ are both upper semicontinuous there exists $\delta > 0$ such that if $\|x - y\| < \delta$ then y is J-Fredholm,

$$\sigma(y) \subset \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\} \cup \{\lambda \in \mathbb{C} \mid |\lambda| > 2\varepsilon\}$$

and

$$\sigma(y + J, A/J) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \geqslant 2\varepsilon\}.$$

However, since J is inessential, $\sigma(y) \setminus \sigma(y + J, A/J)$ consists of isolated points and some of the holes of $\sigma(y + J, A/J)$ [4, Theorem 6.1]. Hence either $0 \notin \sigma(y)$ or $0 \in \operatorname{iso} \sigma(y)$ and so y is almost invertible. We have shown that $y \in \mathcal{R}_0(J)$. $\square$

The above theorem was proved in the operator algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X by Kordula and Müller [12, Theorem 2.1].

7.6. Theorem. *Suppose J is a closed inessential ideal in a semisimple Banach algebra A. Then $\mathcal{R}_0(J) \subset A^{-1}A^{\bullet}$.*

Proof. If $a \in \mathcal{R}_0(J)$ then a is almost invertible and so $a \in \overline{A^{-1}}$. Since a is J-Fredholm and since $J \subset \operatorname{kh} \mathcal{F}(A)$ [16, Theorem 4.6] it follows that a is $\operatorname{kh} \mathcal{F}(A)$-Fredholm. In view of $\mathcal{F}(A)$ and $\operatorname{kh} \mathcal{F}(A)$ having the same set of idempotents, see the remark following Lemma 5.7.1 in [1], we have by [1, Theorem 5.7.2] that a is $\mathcal{F}(A)$-Fredholm. By Lemma 6.2 and Proposition 6.4 we obtain $a \in \widehat{A}$. It follows from 1.3 that $a \in A^{-1}A^{\bullet}$. $\square$

Acknowledgement. The authors should like to thank the referee for several helpful suggestions.

References

[1] *B. Aupetit*: A Primer on Spectral Theory. Springer-Verlag, 1991.
[2] *B. A. Barnes, G. J. Murphy, M. R. F. Smyth and T. T. West*: Riesz and Fredholm Theory in Banach Algebras. Pitman, Boston-London-Melbourne, 1982.
[3] *P. G. Dixon*: Spectra of left approximate identities in Banach algebras. Bull. London Math. Soc. *19* (1987), 169–173.
[4] *J. J. Grobler and H. Raubenheimer*: Spectral properties of elements in different Banach algebras. Glasgow Math. J. *33* (1991), 11–20.
[5] *R. E. Harte*: The exponential spectrum in Banach algebras. Proc. Amer. Math. Soc. *58* (1976), 114–118.
[6] *R. E. Harte*: Fredholm theory relative to a Banach algebra homomorphism. Math. Z. *179* (1982), 431–436.
[7] *R. E. Harte*: Fredholm, Weyl and Browder theory. Proc. Royal Irish Academy Vol. 85A. 1985, pp. 151–176.
[8] *R. E. Harte*: Invertibility and Singularity for Bounded Linear Operators. Marcel Dekker, New York-Basel, 1988.
[9] *R. E. Harte*: On rank one elements. Studia Math. *117* (1995), 73–77.
[10] *R. E. Harte and H. Raubenheimer*: Fredholm, Weyl and Browder theory III. Proc. Royal Irish Academy Vol. 95A. 1995, pp. 11–16.
[11] *R. E. Harte and A. W. Wickstead*: Boundaries, hulls and spectral mapping theorems. Proceedings of the Royal Irish Academy Vol 81A. 1981, pp. 201–208.
[12] *V. Kordula and V. Müller*: Axiomatic theory of spectrum. Studia Math. *119* (1996), 109–128.
[13] *L. Lindeboom and H. Raubenheimer*: A note on the singular spectrum. Extracta Math. *13* (1998), 349–357.
[14] *A. Lebow and M. Schechter*: Semigroups of operators and measures of noncompactness. J. Funct. Anal. *7* (1971), 1–26.
[15] *M. Mbekhta and V. Müller*: On axiomatic theory of spectrum II. Studia Math. *119* (1996), 129–147.
[16] *H. du T. Mouton*: On inessential ideals in Banach algebras. Quaestiones Mathematicae *17* (1994), 59–66.
[17] *T. Mouton and H. Raubenheimer*: More Fredholm theory relative to a Banach algebra homomorphism. Proceedings of the Royal Irish Academy Vol. 93A. 1993, pp. 17–25.
[18] *T. Mouton and H. Raubenheimer*: On rank one and finite elements in Banach algebras. Studia Math. *104* (1993), 211–219.
[19] *J. Puhl*: The trace of finite and nuclear elements in Banach algebras. Czechoslovak Math. J. *28(103)* (1978), 656–676.
[20] *M. R. F. Smyth*: Riesz theory in Banach algebras. Math. Z. *145* (1975), 145–155.
[21] *A. E. Taylor and D. C. Lay*: Introduction to Functional Analysis. 2nd ed. John Wiley, New York, 1980.
[22] *W. Żelazko*: Axiomatic approach to joint spectra I. Studia Math. *64* (1979), 249–261.

Author's address: L. Lindeboom, Dept. of Mathematics, UNISA, PO Box 392, Pretoria 0001, South Africa, e-mail: lindel@alpha.unisa.ac.za; H. Raubenheimer, Dept. of Mathematics, Rand Afrikaans University, POB 524, Auckland Park, Johannesburg 2006, South Africa, e-mail: hein@hra.rau.ac.za.