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A FAMILY OF NOETHERIAN RINGS WITH THEIR FINITE LENGTH MODULES UNDER CONTROL

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Dedicated to Helmut Lenzing on the occasion of his 60th birthday.

Abstract. We investigate the category modΛ of finite length modules over the ring $\Lambda = A \otimes_k \Sigma$, where Σ is a V-ring, i.e. a ring for which every simple module is injective, k a subfield of its centre and A an elementary k-algebra. Each simple module E_j gives rise to a quasiprogenerator $P_j = A \otimes E_j$. By a result of K. Fuller, P_j induces a category equivalence from which we deduce that mod$\Lambda \simeq \coprod_j \operatorname{mod}\operatorname{End} P_j$. As a consequence we can

(1) construct for each elementary k-algebra A over a finite field k a nonartinian noetherian ring Λ such that mod$A \simeq$ modΛ,
(2) find twisted versions Λ of algebras of wild representation type such that Λ itself is of finite or tame representation type (in mod),
(3) describe for certain rings Λ the minimal almost split morphisms in modΛ and observe that almost all of these maps are not almost split in ModΛ.

Keywords: V-ring, progenerator, almost split morphisms

MSC 2000: 16D50, 16G60, 16G70

When passing from commutative to noncommutative rings one might expect that the representation theory is getting more complicated. Here we present a family of noetherian rings Λ with a very small centre for which the category $\operatorname{mod}\Lambda$ of *finite length* Λ-modules is well under control.

The main ingredient in our construction is a right V-ring Σ, that is a ring for which every simple right module is injective. For charactizations and examples see [4, 7.32ff] and [8]. We denote by $\{E_j : j \in J\}$ a set of representatives of the isomorphism classes of the simple injective right Σ-modules and assume that k is a subfield of the centre of Σ. The rings under consideration are of type

$$\Lambda = A \otimes_k \Sigma$$

where $A = kQ/I$ is an elementary k-algebra, that is the factor algebra of a path algebra kQ of a finite quiver Q modulo an admissible ideal I (cf. [2, III.1]). In particular, A has a finite k-dimension; hence if Σ is noetherian, then so is Λ.

In the following result we can see that the modules $P_j = A \otimes_k E_j$ come close to projective generators and induce an equivalence of categories of finite length modules. Using this equivalence we can describe the "complicated" category $\operatorname{mod}\Lambda$ in terms of categories $\operatorname{mod}\Gamma_j$ where $\Gamma_j = \operatorname{End} P_j$ is an artinian ring, often a finite dimensional algebra (for a suitable choice of Σ and j) or a ring of finite representation type (for a suitable choice of A and Σ).

Theorem A. *With the above notation the following assertions hold.*

1. *The* $(\Gamma_j - \Lambda)$*-bimodule* P_j *is a free left* Γ_j*-module (but not necessarily finitely generated),* P_j *is a projective object in* $\operatorname{mod}\Lambda$ *(but not necessarily a projective* Λ*-module) and* $P = \bigoplus_{j\in J} P_j$ *is a generator for the category* $\operatorname{mod}\Lambda$ *(but not necessarily a finite length module).*
2. *The modules* P_j, $j \in J$, *are quasiprogenerators as studied in* [5] *and induce category equivalences*

$$\operatorname{Mod}\Gamma_j \;\underset{G_j=\operatorname{Hom}_\Lambda(P_j,-)}{\overset{F_j=-\otimes_{\Gamma_j}P_j}{\rightleftarrows}}\; \operatorname{Gen} P_j.$$

3. *The functors in* 2. *induce the equivalence*

$$\coprod_{j\in J} \operatorname{mod}\Gamma_j \;\underset{(G_j)}{\overset{(F_j)}{\rightleftarrows}}\; \operatorname{mod}\Lambda$$

(but not necessarily an equivalence $\operatorname{Mod}\prod\Gamma_j \simeq \operatorname{Mod}\Lambda$*).*

Let now A be an elementary k-algebra over a finite field $k = \mathbb{F}_{p^n}$. We construct a nonartinian noetherian ring Λ such that $\operatorname{mod} A \cong \operatorname{mod}\Lambda$.

Example 1. Let K be the algebraic closure of $k = \mathbb{F}_{p^n}$ and $\varphi \in \operatorname{Aut} K$ the Frobenius automorphism $a \mapsto a^{(p^n)}$. According to Cozzens [3], the ring $\Sigma = K[X, X^{-1}; \varphi]$ is a noetherian V-ring with only one simple injective module $E = \Sigma/(X-1) = K \circlearrowleft \varphi$, which has the endomorphism ring $\operatorname{End} E_\Sigma = \operatorname{Fix}\varphi = k$. We obtain from Theorem A that the categories $\operatorname{mod} A$ and $\operatorname{mod} A \otimes_k \Sigma$ are equivalent.

Moreover, we can obtain twisted versions Λ' of algebras Λ of wild representation type such that Λ' has finite (tame) representation type, in the sense that $\operatorname{mod}\Lambda'$ contains only finitely many indecomposable modules, up to isomorphism (the isoclasses of indecomposable modules in $\operatorname{mod}\Lambda'$ can be classified). Recall that the algebras

$$K[X, X^{-1}, Y]/(Y)^2 \quad \text{and} \quad K[X, X^{-1}, Y, Z]/(Y, Z)^2$$

are of wild representation type; indeed, their finite dimensional factors modulo the ideal generated by $(X-1)^3$ are wild [9].

Example 2. Let K be an algebraically closed field of positive characteristic p, $\varphi\colon a \mapsto a^{(p^n)}$ the Frobenius automorphism and $k = \operatorname{Fix}\varphi$ as in Example 1. We obtain from the theorem that the categories of finite length modules over

$$K[X, X^{-1}; \varphi][Y]/(Y)^2 \quad \text{and} \quad K[X, X^{-1}; \varphi][Y, Z]/(Y, Z)^2$$

are equivalent to $\operatorname{mod} k[Y]/(Y)^2$ (finite type) and to $\operatorname{mod} k[Y, Z]/(Y, Z)^2$ (tame type, Kronecker), respectively.

Example 3. The following example is based on the V-ring constructed by Osofsky [8]. Let $k = \mathbb{F}_p(T)$ for p an odd prime number and let K be a two field over k as defined in [8]. For $\varphi\colon a \mapsto a^p$ the Frobenius map, the ring $\Sigma = K[X, X^{-1}; \varphi]$ is a noetherian V-ring with infinitely many pairwise nonisomorphic simple modules. We assume that the simple modules over this skew polynomial ring can be classified, up to isomorphism, and choose a set of representatives $\{E_j \colon j \in J\}$. In this setup, an infinite set of representatives for the isomorphism classes of indecomposable finite length modules over

$$K[X, X^{-1}; \varphi][Y]/(Y)^2 \cong k[Y]/(Y)^2 \otimes_k \Sigma$$

is given by $\{E_j \colon j \in J\} \cup \{P_j \colon j \in J\}$.

The equivalence $\coprod \operatorname{mod}\Gamma_j \to \operatorname{mod}\Lambda$ in Theorem A of course preserves almost split sequences in the category of finite length modules (see [2], [1] or [11] for the notion of almost split maps). But note that a sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{mod}\Gamma_j$ which is almost split in $\operatorname{Mod}\Gamma_j$ does not necessarily give rise to an almost split sequence in $\operatorname{Mod}\Lambda$. For the rings constructed from Cozzens' domain we have the following more precise result.

Proposition B. *Let Σ and k be as in Example* 1*, let A be an elementary k-algebra, and put $\Lambda = A \otimes_k \Sigma$.*

1. *For each indecomposable $M \in \operatorname{mod}\Lambda$ there exists a left and a right minimal almost split morphism in the category $\operatorname{mod}\Lambda$.*
2. *Among the morphisms in* 1.*, only the left minimal almost split morphisms starting from the indecomposable injective Λ-modules of finite length are almost split in $\operatorname{Mod}\Lambda$.*

Organization of the article. In the first section we investigate the quasiprogenerators P_j and prove Theorem A. For certain rings Λ we describe in the second section the almost split sequences in the category $\operatorname{mod}\Lambda$.

Notation. We assume the notation above Theorem A also in the sequel. For "$\otimes_k$" we write "$\otimes$".

1. Projective generators in the category of finite length modules

We start by giving a description of the finite length Λ-modules in terms of Σ-modules and Σ-homomorphisms.

Suppose that the quiver Q is given by a finite set Q_0 of points and a finite set Q_1 of arrows $\alpha\colon s(\alpha) \to t(\alpha)$. Since $\Lambda = kQ/I \otimes \Sigma = \Sigma Q/\Sigma I$ we obtain that a Λ-module consists of Σ-modules M_i for $i \in Q_0$ and Σ-linear maps $f_\alpha\colon M_{s(\alpha)} \to M_{t(\alpha)}$ for $\alpha \in Q_1$ such that the ΣQ-module given by the M_i and the f_α is annihilated by I. Moreover, the Λ-homomorphisms from (M_i, f_α) to (N_i, g_α) are just the tuples $(h_i\colon M_i \to N_i)_{i\in Q_0}$ of Σ-homomorphisms which satisfy $g_\alpha h_{s(\alpha)} = h_{t(\alpha)} f_\alpha$ for all $\alpha \in Q_1$.

Since the ideal $(\operatorname{Rad} A) \otimes \Sigma$ in $\Lambda = A \otimes \Sigma$ is nilpotent, each simple Λ-module is a module over

$$(A/\operatorname{Rad} A) \otimes \Sigma = \prod_{i\in Q_0} e_i \otimes \Sigma.$$

From a set of representatives for the isomorphism classes of the simple Σ-modules, $\{E_j\colon j \in J\}$, we hence obtain a set of representatives for the isomorphism classes of the simple Λ-modules, namely $\{e_i \otimes E_j\colon i \in Q_0,\ j \in J\}$.

In particular, each simple Λ-module is a simple Σ-module. Hence, the finite length Λ-modules are precisely the Λ-modules of finite Σ-length.

Note that the modules E_j are projective objects in the category $\operatorname{mod}\Sigma$, and their direct sum $\bigoplus_{j\in J} E_j$ is a projective generator for $\operatorname{mod}\Sigma$. We have the following result for $\operatorname{mod}\Lambda$.

Lemma 1.1.

1. *The modules $P_j = A \otimes_k E_j$, $j \in J$, are projective objects in $\operatorname{mod}\Lambda$.*
2. *For $j \in J$ and X_A an injective module, the module $X \otimes_k E_j$ is an injective Λ-module.*
3. *Each module $P_{j\,\Lambda}$ has length $\dim A_k$, and every submodule of P_j is generated by P_j.*
4. *The endomorphism ring of P_j is $\Gamma_j = \operatorname{End} P_j \cong A \otimes_k \operatorname{End} E_j$, and P_j is a free module over Γ_j of rank $\dim_{\operatorname{End} E_j} E_j$.*

5. *The direct sum* $P = \bigoplus_{j \in J} P_j$ *is a generator for* $\operatorname{mod} \Lambda$.

P r o o f. 1. A straightforward argument using our description of the $kQ \otimes_k \Sigma$-modules and homomorphisms shows that the $kQ \otimes \Sigma$-modules $P'_{i,j} = e_i kQ \otimes E_j$ have the factorization property of a projective object in the category of all those $kQ \otimes \Sigma$-modules which have finite Σ-length. Hence the module $P_j = P'_j / P'_j I$, where $P'_j = \bigoplus_{i \in Q_0} P'_{i,j}$, is a projective object in the category $\operatorname{mod} \Lambda$.

2. The injectivity of $X \otimes_k E_j$ in the category of all Λ-modules can be verified in a similar way.

3. Each composition factor of $P_{j\,\Lambda}$ is of type $e_i \otimes E_j$ for some i, so P_j has length $\dim A_k$. Moreover, since each such factor is an epimorphic image of P_j it follows from 1. that each submodule of P_j is an epimorphic image of some sum P_j^n.

4. Since A_k has finite dimension, it follows from [7, II.2] that the endomorphism ring of $(A \otimes_k E_j)_{A \otimes \Sigma}$ is $\operatorname{End} A_A \otimes_k \operatorname{End} E_{j\,\Sigma}$. The second assertion is clear.

5. Each simple Λ-module is of type $e_i \otimes E_j$ for some $i \in Q_0$ and $j \in J$, so a similar argument as for 3. yields the last assertion. □

Notation. For a right Λ-module M denote by $\operatorname{Gen} M$ and $\operatorname{gen} M$ the full subcategories of $\operatorname{Mod} \Lambda$ and $\operatorname{mod} \Lambda$ consisting of all modules generated by M, respectively. The Grothendieck category of all modules subgenerated by M is denoted by $\sigma[M]$; for an investigation of $\sigma[M]$ see [10].

For the proof that the functors $- \otimes_{\Gamma_j} P_j$ and $\operatorname{Hom}_\Lambda(P_j, -)$ induce an equivalence $\operatorname{Gen} P_j \simeq \operatorname{Mod} \Gamma_j$ we first need a lemma.

Lemma 1.2. *For* $j \in J$ *the following assertions hold.*

1. *The class* $\operatorname{Gen} P_j$ *is closed under submodules, that is* $\operatorname{Gen} P_j = \sigma[P_j]$.
2. *The module* P_j *is a projective object in* $\operatorname{Gen} P_j$.

P r o o f. 1. Any cyclic submodule of the sum $P_j^{(\lambda)}$ has finite length, and so is in $\operatorname{gen} P_j$ by Lemma 1.1.

2. Let $g\colon Y \to Z$ be an epimorphism in $\operatorname{Gen} P_j$, $f\colon P_j \to Z$ a homomorphism, write $P_j = p\Lambda$ and choose $y \in Y$ such that $g(y) = f(p)$. Since the restriction $g\colon y\Lambda \to g(y)\Lambda$ is an epimorphism of finite length modules, and since P_j is projective in the category of finite length modules, f factors over g. □

Hence P_j is a *quasiprogenerator*, that is a finitely generated Λ-module which is a projective generator in $\sigma[P_j]$. This fact can also be obtained from [6, Corollary 2.4] by observing that $E_{j\,\Sigma}$ is a quasiprogenerator which generates P_j as right Σ-module. The following result is a consequence of [5, Theorem 2.6].

Proposition 1.3. *The modules P_j induce the equivalences of categories*

$$\operatorname{Mod}\Gamma_j \quad \underset{G_j=\operatorname{Hom}_\Lambda(P_j,-)}{\overset{F_j=-\otimes_{\Gamma_j}P_j}{\rightleftarrows}} \quad \operatorname{Gen} P_j.$$

We are now ready to show Theorem A.

Proof. The first assertion is a consequence of Lemma 1.1, the second assertion has been shown in Proposition 1.3.

For the third assertion note that the generator P of $\operatorname{mod}\Lambda$ is a direct sum of isotypic Σ-modules P_j. Hence every indecomposable Λ-module is isotypic when considered as a Σ-module. Moreover, a module $M \in \operatorname{mod}\Lambda$ is in $\operatorname{gen} P_j$ if and only if all its Σ-composition factors are isomorphic to E_j. Hence $\operatorname{mod}\Lambda = \coprod_{j\in J} \operatorname{gen} P_j$ and the second assertion of the theorem follows from Proposition 1.3.

Examples for the negative assertions in brackets are given by Cozzens' domain and by Osofsky's domain, see Examples 1 and 3. □

2. Almost split sequences for finite length modules

In this section we investigate the minimal almost split morphisms for indecomposable finite length Λ-modules. Using the functor F_j from Theorem A we obtain almost split sequences in the category $\operatorname{mod}\Lambda$ from the almost split sequences in the category $\operatorname{mod}\Gamma_j$. In Proposition 2.1 we will see that these sequences are not necessarily almost split in the category $\operatorname{Mod}\Lambda$, i.e. the factorization property of an almost split sequence may fail for a test module which is not of finite length. In this case, the almost split sequences in $\operatorname{mod}\Lambda$ will not coincide with the almost split sequences in $\operatorname{Mod}\Lambda$, which can be constructed by means of the dual and the transpose (see e.g. [1]), and the latter sequences cannot consist of finite length modules.

In [11] and [12] Zimmermann constructed a differential polynomial ring Λ for which the indecomposable finite length modules do admit almost split sequences in $\operatorname{mod}\Lambda$ which are not almost split in $\operatorname{Mod}\Lambda$. We will obtain another family of rings with this property.

Let therefore $\Lambda = A \otimes_k \Sigma$ and $\Gamma_j = A \otimes_k \operatorname{End} E_j$ be as above. It follows from Theorem A that if $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is an almost split sequence in $\operatorname{mod}\Gamma_j$ then

$$\mathcal{E}\colon\ 0 \longrightarrow X \otimes_{\Gamma_j} P_j \xrightarrow{u\otimes 1} Y \otimes_{\Gamma_j} P_j \xrightarrow{v\otimes 1} Z \otimes_{\Gamma_j} P_j \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod}\Lambda$.

Proposition B is a consequence of the following result, which implies that under certain assumptions the sequences $\mathcal{E}$ are not almost split in $\operatorname{Mod}\Lambda$.

Proposition 2.1. *Let Σ be a V-ring and E_Σ a simple injective nonprojective module such that* $\operatorname{End} E_\Sigma = k$ *is a subfield of the centre of* Σ. *Let A be an elementary k-algebra and put $\Lambda = A \otimes \Sigma$.*

1. *Let $0 \to X \to Y \to Z \to 0$ be an almost split sequence in* $\operatorname{mod} A$. *Then*

$$0 \longrightarrow X \otimes E \xrightarrow{f} Y \otimes E \xrightarrow{g} Z \otimes E \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod}\Lambda$ *which is not an almost split sequence in* $\operatorname{Mod}\Lambda$. *In particular, the homomorphisms*

$$p = 1 \otimes \pi\colon\ Z \otimes \Sigma \longrightarrow Z \otimes E \quad \text{and}$$
$$t = (f, 0)\colon\ X \otimes E \longrightarrow \{(u, v) \in (Y \otimes E) \oplus (Z \otimes \Sigma)\colon\ g(u) = p(v)\}$$

are nonsplit and will not factor over g and f, respectively.

2. *For Z_A a projective indecomposable module, the map*

$$g = \iota \otimes 1\colon\ (\operatorname{Rad} Z) \otimes E \longrightarrow Z \otimes E$$

is right minimal almost split in $\operatorname{mod}\Lambda$, *but not right minimal almost split in* $\operatorname{Mod}\Lambda$. *In particular, $p\colon\ Z \otimes \Sigma \to Z \otimes E$ will not factor over g.*

3. *For X_A an injective indecomposable module, the map*

$$f = \pi \otimes 1\colon\ X \otimes E \longrightarrow (X/\operatorname{Soc} X) \otimes E$$

is left minimal almost split in $\operatorname{mod}\Lambda$ *and in* $\operatorname{Mod}\Lambda$.

Proof. Since $- \otimes E\colon\ \operatorname{mod} A \to \operatorname{mod}\Lambda$ is an equivalence of categories, the maps f and g are left and right minimal almost split morphisms in $\operatorname{mod}\Lambda$, respectively.

The map $p = 1 \otimes \pi\colon\ Z \otimes \Sigma \to Z \otimes E$ is not a split epimorphism in $\operatorname{Mod}\Lambda$, since it is not a split epimorphism of Σ-modules. We show that p does not factor over $g = g_1 \otimes 1$. Assume there is h such that $g \circ h = p$. Since $\dim Z_k$, $\dim Y_k < \infty$, there exist $h_1 \in \operatorname{Hom}_A(Z, Y)$, $h_2 \in \operatorname{Hom}_\Sigma(\Sigma, E)$ such that $h = h_1 \otimes h_2$ [7, II.2]. From $1 \otimes \pi = (g_1 \circ h_1) \otimes h_2$ it follows that there is a unit $a \in k$ such that $g_1 \circ h_1 = a \cdot 1$. Hence g_1 is a split epimorphism—a contradiction. Thus p is not a right almost split morphism in $\operatorname{Mod}\Lambda$.

It is well known that a monomorphism f is a left minimal almost split map if and only if the cokernel g of f is a right minimal almost split map. The proof of this result yields the construction of t as a map into the fibre product of g and p.

From the equivalence $\operatorname{mod} A \simeq \operatorname{mod}\Lambda$ we obtain that for X_A an indecomposable injective module also the module $(X \otimes_k E)_\Lambda$ is indecomposable, has a simple socle, and the map $\pi \otimes 1\colon\ X \otimes E \to (X/\operatorname{Soc} X) \otimes E$ is the projection modulo the socle. Since $X \otimes E$ is an injective Λ-module (Lemma 1.1), it follows that $\pi \otimes 1$ is a left minimal almost split map in $\operatorname{Mod}\Lambda$. □

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