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UPPER BOUND FOR THE NON-MAXIMAL EIGENVALUES OF IRREDUCIBLE NONNEGATIVE MATRICES

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Abstract. We present a lower and an upper bound for the second smallest eigenvalue of Laplacian matrices in terms of the averaged minimal cut of weighted graphs. This is used to obtain an upper bound for the real parts of the non-maximal eigenvalues of irreducible nonnegative matrices. The result can be applied to Markov chains.

Keywords: eigenvalue, irreducible nonnegative matrix, averaged minimal cut

MSC 2000: 15A42, 05C50

1. Introduction

The matrices in this paper are real and square. The eigenvalues of an $n \times n$ matrix A are arranged in the non-increasing order with respect to their real parts:

$$\operatorname{Re}\lambda_1(A) \geqslant \operatorname{Re}\lambda_2(A) \geqslant \ldots \geqslant \operatorname{Re}\lambda_n(A). \tag{1}$$

Given n real numbers $a_1, a_2, \ldots, a_n$, denote by $\overline{a} = \max\{a_i\colon 1 \leqslant i \leqslant n\}$ and $\underline{a} = \min\{a_i\colon 1 \leqslant i \leqslant n\}$.

For a given $n \times n$ symmetric nonnegative matrix $C = (c_{ij})$, we associate a weighted graph $G_c = (V, E)$ with $V = \{1, 2, \ldots, n\}$, $(i, j) \in E$ if and only if $c_{ij} > 0$ and $i \neq j$, and the weight of the edge (i, j) is c_{ij}. Let r_i be the i-th row sum of C, $i = 1, 2, \ldots, n$. Then

$$L(G_c) = \operatorname{diag}(r_1, r_2, \ldots, r_n) - C \tag{2}$$

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is the Laplacian matrix of the weighted graph G_c [5] (if C is a $(0,1)$-matrix, then $L(G_c) = L(G)$ is the Laplacian matrix of G). It is easily seen that $L(G_c)$ is a singular, positive semidefinite matrix. Moreover, if C is irreducible, then $\lambda_{n-1}(L(G_c)) > \lambda_n(L(G_c) = 0$.

Let G_c be a weighted graph. The edge-density [6], [7] of a subset M of the vertex set V is defined to be

$$\varrho_c(M) = \sum_{i\in M,\, j\notin M} \frac{c_{ij}}{|M|(n-|M|)}, \tag{3}$$

and, the averaged minimal cut [2], [6] of G_c is defined to be

$$\gamma(G_c) = \min\{\varrho_c(M)\colon\ 0 < |M| < n\}, \tag{4}$$

where $|M|$ is the cardinality of the set M. Since $\gamma(G_c) = 0$ if and only if C is reducible, it is also called the averaged measure [2] of irreducibility of C.

In Section 2 we use $\gamma(G_c)$ to obtain a lower and an upper bound for $\lambda_{n-1}(L(G_c))$ (i.e., the algebraic connectivity of G_c [5], [6]). This, in turn, will be applied to obtain, in Section 3, an upper bound for real parts of the non-maximal eigenvalues of irreducible nonnegative matrices. This has applications to Markov chains in Section 4.

2. Laplacian matrices

In order to prove our results, we first give the following inequality which may be of independent interest.

Lemma 2.1. *If n positive numbers $d_1, d_2, \ldots, d_n$ and n real numbers $x_1, x_2, \ldots, x_n$ satisfy the condition $\sum\limits_{i=1}^{n} x_i/d_i = 0$, then*

$$\sum_{i=1}^{n-1} i(n-i)(x_i - x_{i+1})^2 \geqslant 2\underline{d} \sum_{i=1}^{n} \frac{x_i^2}{d_i}. \tag{5}$$

P r o o f. Let the $n \times n$ matrix $S = (s_{ij})$ correspond to the quadratic form of the left-hand side in (5). It is easily seen that S is a symmetric positive semidefinite matrix with the eigenvectors $e = (1, 1, \ldots, 1)^T$ and $f = (n-1, n-3, n-5, \ldots, -n+1)^T$ corresponding to the eigenvalues $\lambda_n(S) = 0$ and $\lambda_{n-1}(S) = 2$, respectively (cf. [2]). Thus $S - 2I_n$ has only one negative eigenvalue, where I_n is the identity matrix.

Denote $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Since $D^{\frac12}(S - 2I_n)D^{\frac12}$ is congruent to $S - 2I_n$, $D^{\frac12}(S - 2I_n)D^{\frac12}$ and $S - 2I_n$ have the same numbers of positive, negative and zero eigenvalues. Therefore $\lambda_{n-1}\big(D^{\frac12}(S - 2I_n)D^{\frac12}\big) = 0$. Thus by [4, p. 242],

$$0 \leqslant \lambda_{n-1}(D^{\frac12}SD^{\frac12}) + \lambda_1(-2D) = \lambda_{n-1}(D^{\frac12}SD^{\frac12}) - 2\underline{d}.$$

Hence, by the Courant-Fischer Theorem and in view of the identity $\sum\limits_{i=1}^{n} x_i/d_i = 0$, we have that,

$$\begin{aligned} 2\underline{d} \leqslant \lambda_{n-1}(D^{\frac12}SD^{\frac12}) &= \min_{y^T D^{-\frac12} e = 0} \frac{y^T D^{\frac12} S D^{\frac12} y}{y^T y} \\ &= \min_{z^T D^{-1} e = 0} \frac{z^T S z}{z^T D^{-1} z} \leqslant \frac{x^T S x}{x^T D^{-1} x}, \end{aligned}$$

where $z = D^{\frac12} y$ and $x = (x_1, x_2, \ldots, x_n)^T$. Therefore (5) holds. $\square$

Theorem 2.2. *Let G_c be a weighted connected graph (i.e., $C = (c_{ij})$ is irreducible) with n vertices. Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be a positive diagonal matrix and $\Omega = DL(G_c)$. Then*

$$2\underline{d}\gamma(G_c) \leqslant \lambda_{n-1}(\Omega) \leqslant n\overline{d}\gamma(G_c). \tag{6}$$

Proof. Since Ω is similar to $D^{\frac12}L(G_c)D^{\frac12}$, all of the eigenvalues of Ω are real and $\lambda_n(\Omega) = 0$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the real eigenvector of Ω corresponding to the eigenvalue $\lambda_{n-1}(\Omega)$, i.e.,

$$DL(G_c)x = \lambda_{n-1}(\Omega)x. \tag{7}$$

Without loss of generality, we assume that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$ and $L(G_c) = (l_{ij})$. So $\sum\limits_{j=1}^{n} l_{ij} = \sum\limits_{i=1}^{n} l_{ij} = 0$. Hence by (7),

$$\begin{aligned} \sum_{i=1}^{m} \lambda_{n-1}(\Omega)\frac{x_i}{d_i} &= \sum_{i=1}^{m}\sum_{j=1}^{n} l_{ij}x_j = \sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}x_j + \sum_{i=1}^{m}\sum_{j=m+1}^{n} l_{ij}x_j \\ &= \sum_{j=1}^{m}\Big(-\sum_{i=m+1}^{n} l_{ij}\Big)x_j + \sum_{i=1}^{m}\sum_{j=m+1}^{n} l_{ij}x_j \\ &= \sum_{j=1}^{m}\sum_{i=m+1}^{n} (-l_{ij}x_j) + \sum_{i=1}^{m}\sum_{j=m+1}^{n} l_{ij}x_j \\ &= \sum_{i=1}^{m}\sum_{j=m+1}^{n} (-l_{ij}(x_i - x_j)) \geqslant \sum_{i=1}^{m}\sum_{j=m+1}^{n} -l_{ij}(x_m - x_{m+1}) \\ &\geqslant \gamma(G_c)m(n-m)(x_m - x_{m+1}). \end{aligned}$$

Multiplying the above inequality by $x_m - x_{m+1}$ and summing them for $m = 1, 2, \ldots, n-1$, we have

$$\lambda_{n-1}(\Omega)\sum_{i=1}^{n}\frac{x_i^2}{d_i} \geqslant \gamma(G_c)\sum_{i=1}^{n-1} i(n-i)(x_i - x_{i+1})^2, \tag{8}$$

since $\lambda_{n-1}(\Omega)\sum\limits_{i=1}^{n}\frac{x_i}{d_i} = e^T D^{-1}\Omega x = 0$ by (7), where $e = (1, 1, \ldots, 1)^T$. Combining Lemma 2.1 and (8), we obtain the left inequality in (6).

Let M_0 be a proper subset of the vertex set V such that $\gamma(G_c) = \varrho_c(M_0)$. Define an n-dimensional vector $y = (y_1, y_2, \ldots, y_n)^T$ where $y_i = \frac{a}{\sqrt{d_i}}$ if $i \in M_0$, and $y_i = -\frac{b}{\sqrt{d_i}}$ if $i \notin M_0$ where $a = \sum\limits_{i \notin M_0}\frac{1}{d_i}$, $b = \sum\limits_{i \in M_0} 1/d_i$. It is easily seen that $y^T D^{-\frac{1}{2}} e = 0$. Hence by the Courant-Fischer Theorem,

$$\begin{aligned}\lambda_{n-1}(\Omega) = \lambda_{n-1}(D^{\frac{1}{2}}L(G_c)D^{\frac{1}{2}}) &= \min_{zD^{-\frac{1}{2}}e=0}\frac{z^T D^{\frac{1}{2}}L(G_c)D^{\frac{1}{2}}z}{z^T z} \leqslant \frac{y^T D^{\frac{1}{2}}L(G_c)D^{\frac{1}{2}}}{y^T y}\\ &= \gamma(G_c)\Big(\frac{1}{a}+\frac{1}{b}\Big)|M_0|(n-|M_0|) \leqslant n\bar{d}\gamma(G_c).\end{aligned}$$

□

Corollary 2.3 ([2], [7]). *Let G be a simple connected graph with n vertices. Then*

$$2\gamma(G) \leqslant \lambda_{n-1}(L(G)) \leqslant n\gamma(G). \tag{9}$$

Proof. It follows from (6) and $d_1 = d_2 = \ldots = d_n = 1$. □

Corollary 2.4. *Let G be a simple graph with n vertices. Let A be the adjacency matrix and Δ, δ be the maximum and the minimum vertex degree of G, respectively. Then*

$$\delta - n\gamma(G) \leqslant \lambda_2(A) \leqslant \Delta - 2\gamma(G). \tag{10}$$

Proof. Since $\delta I_n - L(G) = A - (\mathrm{diag}(r_1, r_2, \ldots, r_n) - \delta I_n)$, we have that $\lambda_2(\delta I_n - L(G)) \leqslant \lambda_2(A)$. Hence, by (6), the left inequality in (10) holds. In a similar way, the right inequality in (10) is also obtained. □

For an $n \times n$ nonnegative matrix A and positive vectors x and y in $\mathbb{R}^n$, define

$$\eta(A,x,y) = \min \frac{\sum\limits_{i\in M,\, j\notin M} (a_{ij}x_jy_i + a_{ji}x_iy_j)}{2|M|(n-|M|)}, \tag{11}$$

where the minimum is taken over all nonempty subsets M of $\{1,2,\dots,n\}$. If $y = x$, we denote $\eta(A,x,y)$ by $\eta(A,x)$.

Lemma 3.1. *Let A be an $n \times n$ irreducible symmetric nonnegative matrix and let $Au = \lambda_1(A)u$, $u = (u_1,u_2,\dots,u_n) > 0$. Then*

$$\lambda_1(A) - \frac{n}{\underline{u}^2}\eta(A,u) \leqslant \lambda_2(A) \leqslant \lambda_1(A) - \frac{2}{\overline{u}^2}\eta(A,u). \tag{12}$$

Proof. Let $U = \operatorname{diag}(u_1,u_2,\dots,u_n)$ and $C = UAU$. Define G_c to be the weighted graph associated with C. Then $L(G_c) = U(\lambda_1(A)I_n - A)U$. Now choosing $D = U^{-2}$ and $\Omega = U^{-1}(\lambda_1(A)I_n - A)U$, it follows that $\lambda_{n-1}(\Omega) = \lambda_1(A) - \lambda_2(A)$. Since A is symmetric, $Au = \lambda_1(A)u$, $u^T = \lambda_1(A)u^T$ and

$$\gamma(G_c) = \min \frac{\sum\limits_{i\in M,\, j\notin M} a_{ij}u_iu_j}{|M|(n-|M|)} = \eta(A,u).$$

Thus, by Theorem 2.2, (12) holds. □

Theorem 3.2. *Let A be an $n \times n$ irreducible nonnegative matrix. Let $Au = \lambda_1(A)u$, $u > 0$, $v^TA = \lambda_1(A)v^T$, $v > 0$, $w_i = u_iv_i$. Then*

$$\operatorname{Re}\lambda_2(A) \leqslant \lambda_1(A) - \frac{2}{\overline{w}}\eta(A,u,v). \tag{13}$$

Proof. Let $d_i = \sqrt{v_i/u_i}$, $D = \operatorname{diag}(d_1,d_2,\dots,d_n)$ and $B = \frac{1}{2}(DAD^{-1} + (DAD^{-1})^T)$. Then $D^2u = v$ and

$$B(Du) = \frac{DAu + D^{-1}AD^2u}{2} = \frac{D\lambda_1(A)u + D^{-1}\lambda_1(A)v}{2} = \lambda_1(A)(Du).$$

Moreover, it is easily seen that $\eta(B,Du) = \eta(A,u,v)$ and $(Du)_i = \sqrt{u_iv_i} = \sqrt{w_i}$. On the other hand, it follows from [6, p. 237] that $\lambda_1(A) + \lambda_2(B) = \lambda_1(B) + \lambda_2(B) \geqslant \lambda_1(A) + \operatorname{Re}\lambda_2(A)$. Thus (13) follows from (12). □

Corollary 3.3 ([2]). *Let A be an $n \times n$ doubly stochastic matrix. Then*

$$|1 - \lambda_2(A)| \geqslant 2\gamma(G_A). \tag{14}$$

P r o o f. If A is reducible, then $\gamma(G_A) = 0$ and (14) holds. We now assume that A is irreducible. Since A is a doubly stochastic matrix, we have that $u = v = (1, 1, \ldots, 1)^T$ and $\eta(A, u, v) = \gamma(G_A)$. Therefore, by (13),

$$|1 - \lambda_2(A)| \geqslant |1 - \operatorname{Re} \lambda_2(A)| \geqslant 2\gamma(G_A).$$

□

4. Applications to Markov chains

Markov chains techniques are often used to model the behavior of large irreducible nearly uncoupled evolutionary systems in which the states naturally divide into k-clusters such that the states within each cluster are strongly coupled, but the clusters themselves are only weakly coupled to each other. We may use a stochastic matrix P to describe the states of such a chain. In [3], Hartfiel and Meyer defined the uncoupling measure of P as following:

$$\sigma(P) = \min \Bigg(\sum_{i \in M_1,\, j \notin M_1} p_{ij} + \sum_{i \in M_2,\, j \notin M_2} p_{ij} \Bigg), \tag{15}$$

where the minimum is taken over all nonempty proper subsets M_1, M_2 of $\{1, 2, \ldots, n\}$ with $M_1 \cap M_2 = \emptyset$.

The following theorem provides the relation between $\sigma(P)$ and $\lambda_2(P)$.

Theorem 4.1. *Let P be an $n \times n$ irreducible stochastic matrix and $v = (v_1, v_2, \ldots, v_n)^T$ be the stationary distribution vector of P. Denote*

$$\mu = \max\left\{ \frac{v_i}{v_j} \colon\ 1 \leqslant i, j \leqslant n \right\}.$$

Then

$$\sigma(P) \leqslant \frac{2n^2 + (-1)^n - 1}{8}\, \mu |1 - \lambda_2(P)|. \tag{16}$$

P r o o f. Since P is a stochastic matrix, we have that $Pe = e$, $e = (1, 1, \ldots, 1)^T$ and

$$\begin{aligned}\eta(P, e, v) &= \min \frac{\sum\limits_{i\in M, j\notin M} (p_{ij}v_i + p_{ji}v_j)}{2|M|(n - |M|)} \geqslant \underline{v} \min \frac{\sum\limits_{i\in M, j\notin M} (p_{ij} + p_{ji})}{2|M|(n - |M|)} \\ &\geqslant \frac{4\underline{v}}{2n^2 + (-1)^n - 1} \min \sum_{i\in M, j\notin M} (p_{ij} + p_{ji}) = \frac{4\underline{v}}{2n^2 + (-1)^n - 1} \sigma(P).\end{aligned}$$

On the other hand, by Theorem 3.2,

$$|1 - \lambda_2(P)| \geqslant |1 - \operatorname{Re} \lambda_2(P)| \geqslant \frac{2\eta(P, e, v)}{\overline{v}}.$$

Hence

$$\sigma(P) \leqslant \frac{2n^2 + (-1)^n - 1}{4\underline{v}} \eta(P, e, v) \leqslant \frac{2n^2 + (-1)^n - 1}{8} \mu|1 - \lambda_2(P)|.$$

□

Corollary 4.2. *Let P be a doubly stochastic matrix. Then*

$$\sigma(P) \leqslant \frac{2n^2 + (-1)^n - 1}{8} |1 - \lambda_2(P)|. \tag{17}$$

P r o o f. If P is reducible, then $\sigma(P) = 0$ and (17) holds. If P is irreducible, then it follows from (16) and $\mu = 1$. □

Corollary 4.3. *Let P be an irreducible stochastic matrix and*

$$p = \min\{p_{ij} \colon\ p_{ij} \neq 0, i \neq j\}.$$

Then

$$\sigma(P) \leqslant \frac{n^2}{4p^{n-1}} |1 - \lambda_2(P)|. \tag{18}$$

P r o o f. It follows from (4.2) and $\mu \leqslant (1/p)^{n-1}$ by [8]. □

Remark 4.4. Theorem 4.1, Corollary 4.2 and 4.3 partly answer the Hartfiel and Meyer's Conjecture [3].

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