Czechoslovak Mathematical Journal

C. K. Fong

A continous version of Orlicz-Pettis theorem via vector-valued Henstock-Kurzweil integrals

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 3, 531–536

Persistent URL: http://dml.cz/dmlcz/127740

Czechoslovak Mathematical Journal, 52 (127) (2002), 531–536

A CONTINOUS VERSION OF ORLICZ-PETTIS THEOREM VIA VECTOR-VALUED HENSTOCK-KURZWEIL INTEGRALS

C. K. Fong, Ottawa

(Received July 23, 1998)

Abstract. We show that a Pettis integrable function from a closed interval to a Banach space is Henstock-Kurzweil integrable. This result can be considered as a continuous version of the celebrated Orlicz-Pettis theorem concerning series in Banach spaces.

Keywords: Pettis integrability, HK-integrals, Saks-Henstock's property

MSC 2000: 28E50, 28A75

The Orlicz-Pettis theorem (see [2], p. 60, [3], Chapter IV or [4], p. 22]) in Banach space theory says that every weakly subseries convergent series is strongly convergent. It is natural to ask if there is a continuous version of this theorem. The weakly subseries convergence for series is naturally replaced by Pettis integrability for functions. An appropriate substitute for norm convergence for series is, as we will show, the Henstock-Kurzweil integrable functions on a closed interval with values in a Banach space, which will be defined below.

The notation and the terminology used in the present paper are standard. Throughout the paper, we fix a closed interval $[a,b]$ and a Banach space E with its dual denoted by E^*. For results about series and sequences in Banach spaces, we refer to Day [2] and Diestel [3]. Papers on vector measures and integration of vector-valued functions are numerous; see Diestel and Uhl [4] and the reference there. In contrast, papers about vector-valued Henstock-Kurzweil integrals on $[a,b]$ are few and the one we need for the present paper is Š. Schwabik [8]. On the contrary, it is easier to find a source for scalar-valued Henstock-Kurzweil integrals, e.g. Henstock [5], Pfeffer [7] or the illuminating article by Bartle [1].

Let f be a function from the closed interval $[a,b]$ into the Banach space E. We say that f is Pettis integrable if f is measurable (see [4], p. 41) and, for every Borel set A in $[a,b]$, there exists an element v_A in E such that, for all e^* in E^*, we have $e^*f \in L^1[a,b]$ and $e^*v_A = \int_A e^*f(x)\,\mathrm{d}x$; (see [4], p. 52). Here v_A, which is uniquely determined by f and A, will be denoted by $\int_A f(x)\,\mathrm{d}x$ or simply $\int_A f$. A consequence of the Orlicz-Pettis theorem mentioned at the beginning is that, for a Pettis integrable function $f\colon [a,b] \to E$, the map μ from the Borel family of $[a,b]$ into E defined by $\mu(A) = \int_A f$ is a vector measure, that is, for a sequence $\{A_n\}$ of pairwise disjoint Borel sets in $[a,b]$, the series $\sum\limits_{n=1}^{\infty} \mu(A_n)$ converges in norm to $\mu\Bigl(\bigcup\limits_{n=1}^{\infty} A_n\Bigr)$ and, as a consequence, the series $\sum\limits_{n=1}^{\infty} \mu(A_n)$ is unconditionally convergent; see [4], Corollary 6 on p. 54.

Now we define the Henstock-Kurzweil integral for a function f from $[a,b]$ to E. By a *tagged partition* of $[a,b]$ we mean a finite set $T = \{t_j\}_{j=1}^N$ of points (called *tags*) in $[a,b]$, together with a collection (called a *partition*) of the same number of non-overlapping closed intervals $\mathscr{P} = \{I_j\}_{j=1}^N$ such that $t_j \in I_j$ for each j and $\bigcup\limits_{j=1}^{N} I_j = [a,b]$. We denote this tagged partition by $\mathscr{P}^T$ or $\{I_j, t_j\}_{j=1}^N$, depending on which one is more convenient. When the requirement $\bigcup\limits_{j=1}^{N} I_j = [a,b]$ is dropped, we call $\mathscr{P}^T$ a *tagged figure*. Given a function $f\colon [a,b] \to E$ and a tagged partition (or a tagged figure) $\mathscr{P}^T = \{I_j, t_j\}_{j=1}^N$, the *Riemann sum* $S(f, \mathscr{P}^T)$ is the element in E given by

$$S(f, \mathscr{P}^T) = \sum_{j=1}^{N} f(t_j)|I_j|,$$

where $|I_j|$ stands for the length of the interval I_j. For a subset A of $[a,b]$, by a *gauge* on A we mean a real-valued function γ on A such that $\gamma(t) > 0$ for all $t \in A$. Given a gauge γ on $[a,b]$, we say that a tagged partition or a tagged figure $\{I_j,\ t_j\}_{j=1}^N$ is *γ-fine* if $I_j \subseteq (t_j - \gamma(t_j), t_j + \gamma(t_j))$ for each j. (It is easy to show by a compactness argument that, given any gauge γ on $[a,b]$, there is a γ-fine tagged partition of $[a,b]$.) For each gauge γ on $[a,b]$, we denote by $R(f,\gamma)$ the set of all Riemann sums $S(f, \mathcal{P}^T)$ with γ-fine tagged partitions $\mathcal{P}^T$:

$$R(f,\gamma) = \{S(f, \mathscr{P}^T) \mid \mathscr{P}^T \text{ is a } \gamma\text{-fine tagged partition}\}.$$

For $A \subseteq [a,b]$, denote by ΓA the set of all gauges on A. Notice that $\Gamma[a,b]$ is directed downward: for all γ_1 and γ_2 in $\Gamma[a,b]$, there exists $\gamma \in \Gamma[a,b]$ such that $\gamma_1 \geqslant \gamma$ and $\gamma_2 \geqslant \gamma$. Denote by $\mathcal{HK}(f)$ the collection of all sets of the form $R(f,\gamma)$ with $\gamma \in \Gamma[a,b]$:

$$\mathcal{HK}(f) = \{R(f,\gamma) \mid \gamma \in \Gamma[a,b]\}.$$

Then $\mathcal{HK}(f)$ is a *directed filter base* in E in the sense that

$$\gamma_1 \geqslant \gamma_2 \Rightarrow R(f,\gamma_1) \supseteq R(f,\gamma_2).$$

We say that f is *Henstock-Kurzweil integrable*, or simply HK-integrable, if this directed filter base is convergent, that is, for every $\varepsilon > 0$, there exists some $\gamma \in \Gamma[a,b]$ such that $\operatorname{diam} R(f,\gamma) < \varepsilon$. (Here we denote by $\operatorname{diam} S$ the diameter of a nonempty set S in E: $\operatorname{diam} S = \sup\limits_{u,v\in S} \|u-v\|$.) Thus, f is HK-integrable if there is a (unique) element e in E such that, given $\varepsilon > 0$, there is $\gamma \in \Gamma[a,b]$ such that the set $R(f,\gamma)$ is within ε of e, that is, $R(f,\gamma) \subseteq B(e,\varepsilon)\colon = \{x \in E \mid \|x-e\| < \varepsilon\}$. We call this unique element e the Henstock-Kurzweil integral of f and denote it by $\int_a^b f(x)\,\mathrm{d}x$ or simply by $\int_a^b f$. When E is $\mathbb{R}$ or $\mathbb{C}$, the definition of Henstock-Kurzweil integrals given here agrees with the usual one; see [1], [5] or [7]. If the directed set $\Gamma[a,b]$ is replaced by its subset consisting of all positive constants, we obtain a less fine (directed) filter base

$$\mathscr{R}(f)\colon = \{R(f,\delta) \mid \delta \in (0,\infty)\}$$

and its limit, if it exists, is the usual Riemann integral of f. For basic properties of vector-valued HK-integrals used in the present paper, we refer the reader to Schwabik [8]. Here we mention a crucial one, the Saks-Henstock Lemma: *let* $f\colon [a,b] \to E$ *be* HK-*integrable and* A *be an arbitrary set in* $[a,b]$. *Then, given* $\varepsilon > 0$*, there exists a gauge* γ *on* A *such that for each* γ*-fine partial tagged-partition* $\{I_j,t_j\}_{j=1}^N$ *we have*

$$\Big\|\sum_{j=1}^N \Big\{ f(t_j)|I_j| - \int_{I_j} f \Big\}\Big\| < \varepsilon.$$

We say that a vector-valued function $f\colon [a,b] \to E$ has the Saks-Henstock property if f is HK-integrable and, for each $\varepsilon > 0$, there is a gauge γ on $[a,b]$ such that, whenever $\{I_j,t_j\}_{j=1}$ is a γ-fine tagged partition of $[a,b]$, we have $\sum\limits_{j=1}^N \big\|f(t_j)|I_j| - \int_{I_k} f\big\| < \varepsilon$. We need the following result from [8] in our paper:

Lemma. *If* $f\colon [a,b] \to E$ *is Bochner integrable, then* f *has the Saks-Henstock property and its* HK-*integral over* $[a,b]$ *coincides with its Bochner integral.*

Now we are going to establish the main result of this paper: Pettis integrable functions on $[a,b]$ into E are Henstock-Kurzweil integrable. In order to make our argument running smoothly, we first state some known facts about Pettis integrals, which are largely taken from Diestel and Uhl [4], so that we can use them freely in our proof.

Let $f\colon [a,b] \to E$ be a Pettis integrable function and denote by F its indefinite integral: $F(x) = \int_a^x f$ with $a \leqslant x \leqslant b$. Recall that the Pettis integrability for f means that f is measurable and, for each Borel set A in $[a,b]$, there is an element v_A in E such that $e^*f \in L^1[a,b]$ and $e^*v_A = \int_A e^*f$ for all $e^* \in E^*$. We denote this element v_A by $\int_A f$ and, in case A is an interval, say $A = [c,d]$, we also denote it by $\int_c^d f$. If follows immediately from the Orlicz-Pettis theorem that $\mu(A) = \int_A f$ defines a vector-valued measure μ: for every sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint Borel sets, the series $\sum\limits_{n=1}^{\infty} \mu(A_n)$ converges in norm to $\mu\Bigl(\bigcup\limits_{n=1}^{\infty} A_n\Bigr)$; see [4], p. 53, Theorem 5. Clearly $\mu(A) = 0$ for each Borel set A with zero Lebesgue measure. It follows that, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $\|\mu(A)\| < \varepsilon$ for every Borel set A with $\lambda(A) < \delta$; see [4], p. 10, Theorem 1 (Pettis). With no loss of generality, we shall assume f Borel measurable.

Let $A_0 = \{x \in [a,b] \mid f(x) = 0\}$ and $A_n = \{x \in [a,b] \mid n-1 < \|f(x)\| \leqslant n\}$ for positive integers n. Then A_n $(n \geqslant 0)$ are mutually disjoint and $[a,b] = \bigcup\limits_{n=0}^{\infty} A_n$. Hence the series $\sum\limits_{n=0}^{\infty} \int_{A_n} f \equiv \sum\limits_{n=0}^{\infty} \int_a^b \chi_{A_n} f$ converges (unconditionally) to $\int_a^b f$; here χ_{A_n} is the characteristic function of A_n.

Now let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\|\mu(A)\| < \varepsilon$ for all Borel sets A in $[a,b]$ with $\lambda(A) < \delta$. Take a large enough positive integer n_0 such that $\sum\limits_{n>n_0} \lambda(A_n) \equiv \lambda\Bigl(\bigcup\limits_{n>n_0} A_n\Bigr) < \delta$. For each positive integer n, since $\chi_{A_n} f$ is bounded and measurable, by [4], p. 45, Theorem 2, it is Bochner integrable. Hence, by Lemma above, it is also Henstock-Kurzweil integrable and it has the Saks-Henstock property. Thus there is a gauge γ_n on $[a,b]$ such that, for every γ_n-fine tagged partition $\{I_j, t_j\}_{j=1}^{N}$ of $[a,b]$, we have

$$\sum_{j=1}^{N} \Bigl\| \chi_{A_n}(t_j) f(t_j) |I_j| - \int_{I_j} \chi_{A_n} f \Bigr\| < \frac{\varepsilon}{2^n}, \tag{1}$$

which also gives $\Bigl\| \sum\limits_{j=1}^{N} \chi_{A_n}(t_j) f(t_j) |I_j| - \int_a^b \chi_{A_n} f \Bigr\| < \varepsilon/2^n$. Notice that (1) is trivial when $n = 0$. For convenience, set $\gamma_0(t) \equiv 1$. Define a gauge γ on $[a,b]$ by putting

$$\gamma(t) = \begin{cases} \min\{\gamma_0(t), \gamma_1(t), \dots, \gamma_{n_0}(t)\} & \text{if } t \in A_n \text{ for some } n \leqslant n_0, \\ \min\{\gamma_0(t), \gamma_1(t), \dots, \gamma_n(t)\} & \text{if } t \in A_n \text{ for some } n > n_0. \end{cases}$$

Let $\mathscr{P}^T = \{I_j, t_j\}_{j=1}^{N}$ be a γ-fine tagged partition of $[a,b]$. Let $S_n = \{j \mid t_j \in A_n\}$

for $n \geqslant 0$. Then

$$(2) \qquad \Big\| \sum_{j=1}^{N} f(t_j)|I_j| - \int_a^b f \Big\|$$
$$= \Big\| \sum_{n=0}^{n_0} \Big\{ \sum_{j \in S_n} f(t_j)|I_j| - \int_{A_n} f \Big\} + \sum_{n>n_0} \sum_{j \in S_n} f(t_j)|I_j| - \int_{B_0} f \Big\|,$$

where $B_0 = \bigcup_{n>n_0} A_n$. For $n \leqslant n_0$, the tagged partition $\mathscr{P}^T$ is γ_n-fine (because $\gamma(t) \leqslant \gamma_n(t)$ for all t) and hence

$$(3) \qquad \Big\| \sum_{j \in S_n} f(t_j)|I_j| - \int_{A_n} f \Big\| = \Big\| \sum_{j=1}^{N} \chi_{A_n}(t_j) f(t_j)|I_j| - \int_a^b \chi_{A_n} f \Big\| < \frac{\varepsilon}{2^n}$$

in view of the inequality following (1). For $n > n_0$, since $\{I_j, t_j\}_{j \in S_n}$ is a γ_n-fine partial tagged-partition, it is not hard to derive from (1) that

$$(4) \qquad \sum_{j \in S_n} \Big\| \chi_{A_n}(t_j) f(t_j)|I_j| - \int_{I_j} \chi_{A_n} f \Big\| \equiv \sum_{j \in S_n} \Big\| f(t_j)|I_j| - \int_{I_j \cap A_n} f \Big\| < \frac{\varepsilon}{2^n}.$$

Notice that $A_0 := \bigcup_{n>n_0} \bigcup_{j \in S_n} I_j \cap A_n$ is a subset of $B_0 \equiv \bigcup_{n>n_0} A_n$ and hence its Lebesgue measure is less than δ. Thus $\|\mu(A_0)\| = \Big\| \sum_{n>n_0} \sum_{j \in S_n} \int_{I_j \cap A_n} f \Big\| < \varepsilon$. It is clear from (4) and the last inequality that

$$(5) \qquad \Big\| \sum_{n>n_0} \sum_{j \in S_n} f(t_j)|I_j| \Big\| < 2\varepsilon.$$

Since $\lambda(B_0) = \lambda\big(\bigcup_{n>n_0} A_n \big) < \delta$, we have $\big\| \int_{B_0} f \big\| < \varepsilon$. This inequality, together with (2), (3) and (5), gives $\Big\| \sum_{j=1}^{N} f(t_j)|I_j| - \int_a^b f \Big\| < 5\varepsilon$. Now we can conclude

Theorem. *Pettis integrable functions on* $[a, b]$ *are Henstock-Kurzweil integrable.*

It is natural to ask if a Pettis integrable function has the Henstock property, or more generally if a HK-integrable function has the Saks-Henstock property. It would be of great interest if the answer turned out to depend on the geometry of the Banach space E. But at present we do not have any evidence for such speculations. We leave the investigation of the Henstock property to our future work.

References

[1] *R. G. Bartle*: Return to the Riemann integral. Amer. Math. Monthly *103* (1996), 625–632.
[2] *M. M. Day*: Normed Linear Spaces. Academic Press Inc., New York, 1962.
[3] *J. Diestel*: Sequences and Series in Banach Spaces. Springer-Verlag, New York, 1984.
[4] *J. Diestel and J. J. Uhl, Jr.*: Vector Measures. Mathematical Surveys, No. 15. Amer. Math. Soc., Providence, 1997.
[5] *R. Henstock*: The General Theory of Integration. Clarendon Press, Oxford, 1991.
[6] *W. F. Pfeffer*: The Riemann Approach to Integration. Cambridge Tracts in Mathematics, No. 109. Cambridge University Press, Cambridge, 1993.
[7] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, 1970.
[8] *Š. Schwabik*: Abstract Bochner and McShane Integrals. Ann. Math. Sil. *1564(10)* (1996), 21–56.

Author's address: School of Mathematics and Statistics, Carleton University, KIS 5B6 Ottawa, Ontario, Canada, e-mail: `ckfong@math.carleton.ca`.