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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 3, 483–498

Persistent URL: http://dml.cz/dmlcz/127737

Czechoslovak Mathematical Journal, 52 (127) (2002), 483–498

AN ANALOGUE OF MONTEL'S THEOREM FOR SOME CLASSES OF RATIONAL FUNCTIONS

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(Received June 24, 1999)

Abstract. For sequences of rational functions, analytic in some domain, a theorem of Montel's type is proved. As an application, sequences of rational functions of the best L_p-approximation with an unbounded number of finite poles are considered.

Keywords: normal families, best L_p-approximation

MSC 2000: 30B40, 41A20, 41A50

1. Introduction

Given a domain B in the complex plane $\mathbb{C}$, let $\mathscr{F}$ be a family of functions, analytic and single valued in B ($\mathscr{F} \in \mathscr{A}(B)$). We presume the functions f, $f \in \mathscr{F}$ to be equipped with the sup-(uniform) norm $\|\ldots\|_K$ on compact subsets K of B.

By Montel's classical theorem, if there are two points a and b, $a, b \in \mathbb{C}$, $a \neq b$, such that each function $f \in \mathscr{F}$ takes the values of a and b no more than N times with N being a finite number, then $\mathscr{F}$ is a normal family in B (cf. [7]). If, in addition, the family $\mathscr{F}$ converges uniformly to a function f on a regular compact subset K of B, then it necessarily converges *locally uniformly inside* (l.u.in.) B (uniformly in the sup-norm on compact subsets of B). We remark that in such a case the function f admits an analytic continuation from K into the entire domain B.

Thus, a natural question arises as to what happens if a family of analytic functions converges on a regular continuum K in a domain B and omits merely one finite value in B? As is known, in general only these conditions alone do not guarantee uniform convergence inside the domain itself. This question appears to make sense for sequences of approximating rational functions. To make things clear, we recall a

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* This paper is conducted while visiting the University of Lodz.

well known result by Blatt-Saff-Simkani (cf. [3]). First, for a compact set E, $E \neq 0$, a function f, $f \in C(E)$ and a nonnegative integer n we introduce the notation $P_n(f, E)$, $n = 1, 2, \ldots$, for a polynomial of degree not exceeding n that approximates the function best in the uniform metric on E.

Theorem 1 ([3]). *Let E be a regular compact set in $\mathbb{C}$ with a connected complement. Given a function f, $f \in A(E^\circ) \cap C(E)$ and $f \not\equiv 0$ on each component of E, assume there is a domain U with nonempty intersection with E such that the number of zeros of $P_n(f, E)$ behaves on each arbitrary but fixed compact subset K of U like $o(n)$ as $n \to \infty$. Then the function f admits an analytical continuation into the entire domain U. Even more, under the above conditions the sequence $\{P_n(f, E)\}$ converges l.u.in. the smallest domain E_ϱ, canonical with respect to Green's function of E that contains the domain U (and hence, f is analytically continuable into E_ϱ).*

This result is a consequence of the fundamental theorem of Jentzsch's type concerning the distribution of zeros of polynomials of best uniform approximation. The proof of the last theorem is due to H. P. Blatt, E. B. Saff and M. Simkani (cf. [3]). Another approach, based on Leja's results, is given in [6]. The method used in [3] is extended to rational functions $R_{n,m}(f, E)$ (n—the numerator's degree, m—the denominator's degree) of best uniform approximation with $n \to \infty$ and m fixed. An analogue of the main theorem is proved with an analytic continuation replaced by a meromorphic one with no more than m poles and rational functions converging to f l.u.in. U—(the poles of f (the poles being counted with regard to their multiplicities)). This case was considered independently in [12].

In these statements, the essential role is played by the fact that the denominator's degrees are fixed. It is an open problem whether a result like Blatt-Saff-Simkani's Theorem holds for rational sequences $R_{n,m_n}(f, E)$ of best uniform approximation when $m_n \to \infty$. Nevertheless, in some cases there is uniform convergence, namely when the sequence in question converges "fast" on some compact subset.

In the present paper, sequences of rational functions with unbounded numbers of finite poles are considered. Roughly spoken, we will show that the "fast" convergence on some regular compact subset E of the domain B and only one finite value in Montel's conditions guarantee uniform convergence inside B itself.

Before presenting the results we introduce some notation. Given a compact set K and a function $h(z)$ defined on K, $\nu(h, K)$ will stand for the number of all zeros of $h(z)$ on K. For an infinite sequence $\{f_n\}$ and a domain B, the notation $\{f_n\} \in \mathscr{N}(B)$ means that a) $\{f_n\} \in \mathscr{A}(B)$ and b) $\nu(f_n, K) = o(n)$ as $n \to \infty$ for every compact subset K of B. In the sequel we will express by $\{f_n\} \in \mathscr{L}(B)$ that $\{f_n\}$ converges l.u.in. B. Finally, given a domain D, a regular compact set S, $S \subset D$ and a function f defined on S, we will write $f \in \mathscr{A}_S(D)$ if there is a function $F \in \mathscr{A}(D)$ such that

$F \equiv f$ on S. Analogously, we define $f \in \mathscr{A}_G(D)$ with G being a domain containing a regular subset.

For the sake of clarity we recall that a compact set is said to be *regular*, if its complement $E^c := \overline{\mathbb{C}} - E$ possesses Green's function $G_E(z, \infty)$ with a logarithmic pole at infinity which is continuous in $\overline{E^c} - \infty$. It is known that if a compact set E is regular then its Green capacity $\operatorname{Cap} E$ is positive (cf. [7], Chapter V).

2. Statement of the results

The main result of the present paper is

Theorem 2. *Let $\mathscr{S}$ be a regular continuum in $\mathbb{C}$ and B a domain, $B \supset \mathscr{S}$. Let $\mathscr{F} := \{f_n\}_{n=1,2,\ldots}$ be a sequence of rational functions, $\mathscr{F} \in \mathscr{A}(B)$, with a total number of poles in $\overline{\mathbb{C}}$ of every f_n not exceeding n. Assume there is a function f, $f \not\equiv 0$ on a regular subset of S such that*

$$\limsup_{n\to\infty} \|f_n - f\|_{\mathscr{S}}^{1/n} < 1. \tag{1}$$

If

$$\nu(f_n, K) = o(n) \quad \text{as} \quad n \to \infty$$

for each compact subset K of B, then $\mathscr{F}$ converges locally uniformly inside B; thus, $f \in \mathscr{A}(B)$.

Theorem 2 is applicable to several kinds of approximating sequences. The results known to us will be listed in Section 3.

In the present paper, we apply Theorem 2 to sequences of rational functions of best L_p-approximation.

Let γ be a curve in $\mathbb{C}$, p—a positive number and let f be a function of the class $L_p(\gamma)$. We adopt the notation $\|f\|_{L_p(\gamma)} := \{\int_\gamma |f(t)|^p |\mathrm{d}t|\}^{1/p}$. For any pair (n, m) of nonnegative integers, let $\mathscr{R}_{n,m}^{(\gamma,p)}$ be a rational function of the class $r_{n,m}$ of best L_p-approximation to f on γ.

Theorem 3. *Let (n, m_n) be a sequence of nonnegative pairs, $m_n \leqslant n$, $m_n \leqslant m_{n+1}$, $n \to \infty$. Given a closed analytic rectifiable curve Γ with an interior $\mathscr{G}$, a positive number p and a function f, $f \in L_p(\Gamma)$, assume*

$$\|\mathscr{R}_{n,m_n}^{(\Gamma,p)} - f\|_{L_p(\Gamma)} \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Let U be a domain U, $U \supset \overline{\mathscr{G}}$ such that $\{\mathscr{R}_{n,m_n}^{(\Gamma,p)}\} \in \mathscr{N}(U)$. Then $\{\mathscr{R}_{n,m_n}^{(\Gamma,p)}\} \in \mathscr{L}(U)$ and $f \in \mathscr{A}(U)$.

Remark 1. Given a weight function ω, nonnegative and integrable over the curve Γ, assume now $f \in L_{p,w}(\Gamma)$, that is $\|f\|_{L_{p,w}(\Gamma)} := \{\int_\Gamma |f(z)|^p w(z)|\mathrm{d}z|\}^{1/p} < \infty$; let $\mathscr{R}^{(p,\omega)}_{n,m_n}$ be a rational function of order (n, m_n) of best $L_{p,w}$ aproximation of f on Γ. If $1/w^q$ is, in addition, integrable over Γ for a positive number q then Theorem 3 is extendable to the functions $\{\mathscr{R}^{(p,\omega)}_{n,m_n}(z)\}_{n=1}^\infty$.

As a particular case covered by Theorem 3 we point out rational functions of best L_p-approximation of functions of the class $H_p(\mathscr{T})$, $\mathscr{T}$ being the unit circle, as well as L_p-weighted approximations of functions of the same class. It is a known fact that condition (2) is automatically fulfilled in H_p. Indeed, $\|\mathscr{P}_n - f\|_{L_p(\mathscr{T})} \to 0$ as $n \to \infty$ with $\mathscr{P}_n$ being a trigonometric polynomial of best L_p-approximation of degree n of f on $\mathscr{T}$ (cf. for instance, [20], Chapter I). Statement (2) results now from the minimality property of $\mathscr{R}^{(\mathscr{T},p)}_{n,m_n}$.

Set now $\Delta := [-1, 1]$. Given a weight function $w(x)$, assume the function $f(x)$ to be of the class $L_{p,w}(\Delta)$; $f(x)$ is presumed to be real-valued on Δ.

Given a pair of nonnegative integers (n, m), let $R^{(p,w)}_{n,m}$ be a rational function of order (n, m) of best $L_{p,w}$-approximation of f on Δ. We apply Theorem 2 to establish the validity of

Theorem 4. *Let $w(x)$ be a real-valued weight function, a.e. positive on Δ, integrable over Δ together with $w^{-q}(x)$ for a number $q > 0$ and let $f \in L_{p,w}(\Delta)$; $f(x)$— real-valued on Δ. Let the sequence of pairs (n, m_n) be as in Theorem 3. If there is a domain U such that* a) *$U \supset \Delta$ and* b) *$\{R^{(p,w)}_{n,m_n}\} \in \mathscr{N}(U)$, then $\{R^{(p,w)}_{n,m_n}\} \in \mathscr{L}(U)$ and $f \in \mathscr{A}_\Delta(U)$.*

As before, we observe that

$$\|R^{(p,w)}_{n,m_n} - f\|_{L_{p,w}(\Delta)} \to 0 \quad \text{as} \quad n \to \infty. \tag{3}$$

3. Background

The first result imposing the question to explore the connection between zeros and analytic continuability is the following known Bernstein's theorem:

Theorem 5 ([1]). *Let a function f be real-valued and continuous on an interval Δ. Assume that there is an ellipse $\mathscr{E}$ with foci at ± 1 such that the polynomials $P_n(f, \Delta)$, $n = 1, 2, \ldots$ are, starting with a number n_0, nowhere zero in $\mathscr{E}$. Then $P_n(f, \Delta)$ converge as $n \to \infty$ locally uniformly inside $\mathscr{E}$ (and, hence, $f \in \mathscr{A}_\Delta(\mathscr{E})$).*

In the same paper, S. N. Bernstein pointed out that the theorem holds only for polynomials of best approximation.

We now summarize some results of Montel's type:

Theorem 6 ([2]). *Let* $f(z) = \sum_{n=0}^{\infty} f_n z^n$ *be a power series with a positive radius of convergence,* $\{\pi_n\}_{n=1,2,\ldots}$*—the diagonal sequence in Padé's table associated with* f *and* U*—a disk centred at* $z = 0$ *such that* $\{\pi_n\} \in \mathscr{N}(U)$ *for every* n*. Then* $\{\pi_n\} \in \mathscr{L}(U)$*.*

Originally, Theorem 6 was established under the assumption that $\{\pi_n\} \in \mathscr{U}(U)$. It is easy to show that $\{\pi_n\} \in \mathscr{N}(U)$ leads to uniform boundedness of $\{\pi_n\}$ inside U. Later, in 1982, the statement of Theorem 6 was established by A. A. Gonchar under essentially weaker conditions, involving only that $\{\pi_n\} \in \mathscr{A}(D)$, $n = 1, 2, \ldots$ (cf. [8]).

Further results of Montel's type are obtained in [11], [13] and in [10].

Let now $f(x)$ be a function, real valued and continuous on the interval Δ. The next theorems (Theorem 7 and 8) provide, similarly to Blatt-Saff-Simkani's theorem, a local characterization of the function $f(x)$.

Set, for any pair (n, m) of nonnegative integers,

$$e_{n,m} := e_{n,m}(f) := \|f - R_{n,m}(f, \Delta)\|_{\Delta}.$$

Theorem 7 ([14]). *Given* f *as above, let* (n, m_n) *be a sequence of pairs of nonnegative integers such that* $m_n \leqslant n$*,* $m_n \leqslant m_{n+1} \leqslant 1 + m_n$ *and* $m_n = o(n)$ *as* $n \to \infty$*. Assume*

$$\liminf_{n\to\infty} n\left(\frac{e_{n,m(n)}}{e_{n+1,m(n+1)}} - 1\right) > 0.$$

Then $\{R_{n,m_n}(f, \Delta)\} \in \mathscr{L}(U)$ *for every domain* U *of nonempty intersection with* Δ *for which* $\{R_{n,m_n}\} \in \mathscr{N}(U)$ *(and hence* $f \in \mathscr{A}_{\Delta\cap U}(U)$*).*

Theorem 8 ([14]). *Under the same conditions as in Theorem 7 on the function* $f(x)$*, the domain* U *and the sequence* $\{R_{n,n}(f, \Delta)\}$ *(namely,* $U \cap \Delta \neq 0$ *and* $\{R_{n,n}(f, \Delta)\} \in \mathscr{N}(U)$*), assume there is a domain* W *of* Δ *such that the numbers* $\gamma_n = \gamma_n(W)$ *of those poles of* $R_{n,n}(f, \Delta)$ *which lie outside* W *satisfy*

(i) $$\liminf_{n\to\infty} \frac{\gamma_n}{n} > 0.$$

If

(ii) $$\liminf_{n\to\infty} n\left(\frac{e_n}{e_{n+1}} - 1\right) > 0$$

with $e_n := e_{n,n}$*, then* $\{R_{n,n}(f, \Delta)\} \in \mathscr{L}(U)$ *(and thus* $f \in \mathscr{A}_{\Delta\cap U}(U)$*).*

As an illustration of Theorem 8, consider the function $f(x) = |x|$. It was proved in [4] that both the poles and the zeros of the rational function $R_{n,n}$ of best uniform approximation lie interlacing each other on the imaginary axis and go, as $n \to \infty$, to infinity. Hence, condition (i) holds. Further, it was shown in [19] that $\lim_{n\to\infty} \mathrm{e}^{\pi\sqrt{n}} e_n = 8$. This implies (ii).

In view of Theorem 10, $f(x) = |x|$ would be analytically continuable in any domain intersecting the segment $[-1, 1]$ and nonintersecting the imaginary axis (as $|x| = z$, if $\operatorname{Re} z > 0$, and $|x| = -z$ otherwise).

This is a known fact established in [4]; it was shown there that $R_n(z) \to z$ uniformly inside the right half-plane $\{z, \operatorname{Re} z > 0\}$ and to $-z$ inside the left half-plane $\{z, \operatorname{Re} z < 0\}$.

Other functions which fulfil conditions (i) and (ii) are the exponential function (cf. [5]) and the function $f(x) := |x|^\alpha$, $\alpha > 0$, $\alpha \neq 0, 2, 4, \ldots$ (cf. [18]). For the former,

$$e_n = \frac{2^{-2n}(n!)^2}{(2n)!(2n+1)!}(1 + o(1)),$$

hence (ii) is true. On the other hand, both the poles and the zeros of $f = \mathrm{e}^z$ tend to infinity. For the latter,

$$|\sin((\pi\alpha/2)| \exp(-\pi\sqrt{n\alpha}) \leqslant e_n \leqslant \exp(-\pi\sqrt{n\alpha})$$

and all poles and zeros cluster to the axis $\{z, \operatorname{Re} z = 0\}$. On the other hand, it is shown in [18] that $\{R_{n,n}(z)\}$ converges to z^α in $\{z, \operatorname{Re} z > 0\}$ and to $(-z)^\alpha$ in $\{z, \operatorname{Re} z < 0\}$.

4. Preliminaries

Let $f, g \in L_p[a, b]$. We first recall the basic fact that

$$\|f + g\|^p_{L_p[a,b]} \leqslant C(\|f\|^p_{L_p[a,b]} + \|g\|^p_{L_p[a,b]})$$

with $C = \max[1, 2^{p-1}]$ (cf. [21]). If $p \geqslant 1$ then Minkowski's inequality is valid, i.e.

$$\|f + g\|_{L_p[a,b]} \leqslant \|f\|_{L_p[a,b]} + \|g\|_{L_p[a,b]}.$$

Given a set e of positive Lebesgue measure (we write $m(e) > 0$), let functions $\{f_n\}_{n=1}^\infty$ be defined on e. The sequence is said to converge in measure on e, if for every positive ε and δ $m\{z, z \in e, |f_n(z) - f_m(z)| \geqslant \varepsilon\} < \delta$ holds for all n, m large enough (cf. [16]). By a theorem of Natanson, if a sequence $\{f_n\}_{n=1}^\infty$ converges to a function f in $L_p[a, b]$, then it converges in measure on $[a, b]$, too (cf. [16]).

A function f is said to belong to the class H_p if a) f is analytic in the unit disk T and b) $\sup\limits_{\varrho\to 1}\int_0^{2\pi}|f(\varrho\exp(\mathrm{i}\tau))|^p\,\mathrm{d}\tau$ is bounded. (If $p=\infty$, then $\sup\limits_{\varrho\to 1}|f(\varrho\exp(\mathrm{i}\tau))|$ should be bounded, respectively.)

If $f\in H_p$, then the nontangential limits $\lim\limits_{z'\to z, z'\in T} f(z')$ exist for almost all z, $z\in\mathscr{T}$ ($\mathscr{T}:=$ the unit circle) (cf. [17]). One can define $f(\exp(\mathrm{i}\tau))$ as the limit of $f(\varrho\exp(\mathrm{i}\tau))$ as $\varrho\to 1$, $\tau\in[0,2\pi]$. It is customary to write $f(\exp(\mathrm{i}\tau))$ instead of $\lim\limits_{\varrho\to 1} f(\varrho\exp(\mathrm{i}\tau))$. Recall that the nontangential limit function $f(\exp(\mathrm{i}\tau))\in L_p(\mathscr{T})$ (cf. [17]). If $f,g\in H_p$ and $f=g$ for $z\in E$, E being a subset of $\mathscr{T}$ of positive measure, then $f\equiv g$ (Privalov's uniqueness theorem for H_p cf. [17]). We recall that the uniqueness theorem preserves its validity under the same condition (namely, $f=g$ on a subset of $\mathscr{T}$ of positive measure) also for functions analytic and single valued in T—the theorem of Privalov-Luzin (cf. [17]). Further, according to Ostrowski-Khinchine's theorem, if a sequence $\{f_n\}$ with $f_n\in H_p$ and $\|f_n\|_{L_p(\mathscr{T})}\leqslant C$, $n=1,2,\ldots$ converges on $\mathscr{T}$ in measure to a function f, then $\{f_n\}\in\mathscr{L}_F(T)$ with a limit function F coinciding with f a.e. on $\mathscr{T}$ and being an element of H_p.

In the sequel, C_n, $n=1,2,\ldots$, denote positive constants which do not depend on the integer n and are different at different occurrences.

Finally, given a sequence $\{\ldots\}$, we will use the notation $\{\ldots\}\in\mathscr{U}_B(B)$ to express that $\{\ldots\}$ is uniformly bounded inside B (in the sup-norm on compact subsets), and $\{\ldots\}\in\mathscr{L}_f(B)$—that $\{\ldots\}$ converges l.u.in. B to the function f.

The proofs will be preceded by a few lemmas.

Let $\mathscr{F}:=\{F_n\}_{n=1,2,\ldots}$ be functions locally single-valued and analytic in a domain B except perhaps for branch points, and let each $|F_n|$ starting with a number n_0 be single-valued there. We say that a harmonic function v is *a harmonic majorant* for $\mathscr{F}$ in B, if for every compact subset M of B the inequality

$$\limsup_{n\to\infty}\|F_n\|_M\leqslant\exp\|v\|_M$$

is valid.

Lemma 1 ([22]). *Let B be a domain in $\mathbb{C}$, $\mathscr{F}:=\{F_n\}_{n=1}^{\infty}$ a sequence as above and let v be a harmonic majorant for $\mathscr{F}$ in B.*

If there is a regular compact set M, $M\subset B$, where the strict inequality holds, i.e. if

$$\limsup_{n\to\infty}\|F_n\|_M<\exp\|v\|_M,$$

then the strict inequality holds on every compact subset of B.

Also the next lemma will be of importance for the coming considerations.

Lemma 2. *Let γ be a closed analytic curve in $\mathbb{C}$ and p a positive number. Denote by D the finite domain bounded by γ. Let $g \in \mathscr{A}(D) \cap C(\overline{D})$. Then for each compact subset F of D there exists a constant $C_1 = C_1(F)$ such that*

$$\|g\|_F \leqslant C_1 \|g\|_{L_{p(\gamma)}}.$$

The case when γ coincides with the unit circle $\mathscr{T}$ was considered by J. L. Walsh (cf. [21], Chapter V). The proof for the general case proceeds along the same line of reasoning, after mapping conformally $\overline{D}$ onto $\overline{T}$, so we may omit it.

Lemma 3 ([9])**.** *Let E be a regular compact set in $\mathbb{C}$ and D a domain, $D \supset E$. Given a sequence $\{R_n\}$ of rational functions with a total number of poles in $\overline{\mathbb{C}} \leqslant n$, assume $R_n \in \mathscr{A}(D)$ for every n. Then for every compact set K in $D - E$ there is a constant λ_1, $\lambda_1 = \lambda(K)$, $\lambda_1 > 1$ such that for every n*

$$\|R_n\|_K \leqslant \lambda_1^n \|R_n\|_E.$$

The constant $\lambda_1(K)$ is given by $\lambda_1(K) = \sup_{z' \in K, z'' \in D^c} G_E(z', z'')$.

A similar result holds also for $L_{p,w}$-norms. To be precise, we present

Lemma 4. *Let E be a regular continuum and $w(x)$—a weight function, a.e. positive and integrable over ∂E together with $w(x)^{-q}$ for some positive q. Then, under the conditions of Lemma 4, for every $p > 0$ and for every compact set K, $K \subset D - E$ there is a positive constant $\lambda_2 = \lambda_2(K, p)$, $\lambda_2 > 1$, such that*

$$\|R_n\|_K \leqslant C \lambda_2^n \|R_n\|_{L_{p,w}(\partial E)}.$$

For the particular case when ∂E is an analytic closed curve, Lemma 3 was announced and proved by J. L. Walsh (cf. [21], Chapter V). The proof of the form presented here follows the main idea of Walsh.

Analyzing the proofs of Lemma 3 and Lemma 4 we arrive at

Remark 2. Let $\{K_n\}$ be a family of compact sets, $\ldots \supset K_n \supset K_{n+1} \supset \ldots$, $E = \bigcap K_i$. Then $\lambda_i(K_n, p) \to 1^+$ as $n \to \infty$, $i = 1, 2$.

In the forthcoming proofs we shall need inequalities of Nikolsky's type estimating from above the uniform norm of a polynomial on the interval Δ by its L_p-norm.

Lemma 5 ([15]). *For an a.e. positive weight function $\omega(x)$ satisfying*

$$\int_{-1}^{1} \omega(x)\,\mathrm{d}x = 1$$

set $\varphi(\omega,\varepsilon) = \inf\{\int_A \omega(x)\,\mathrm{d}x\colon\ A \subset [-1,1], \mu(A) \geqslant \varepsilon\}$, $0 < \varepsilon \leqslant \pi$, *where* $\mu(A)$ *is the Chebyshev measure of* A. *Let* $\varepsilon_n(\omega)$ *be the unique solution of* $\varphi(\omega,\varepsilon) = \exp(-n\varepsilon)$. *Then for every* p, $0 < p \leqslant \infty$ *and for every polynomial* p_n *of degree* n *we have*

$$\|p_n\|_{\Delta} \leqslant \exp(cn\varepsilon_n(\omega))\|p_n\|_{L_p(\Delta)},$$

where $c > 0$ *depends only on* p *and* ω.

We note that $\varepsilon_n(\omega) \to 0$ as $n \to \infty$ (cf. [15]).

5. Proofs

P r o o f of Theorem 2. Under the conditions of the theorem and by virtue of Lemma 3, there is a compact set F, $F^\circ \neq \emptyset$, $S \subset F \subset B$ such that

$$\limsup_{n\to\infty} \|f_{n+1} - f_n\|_F^{1/n} < 1.$$

Hence, $f_n \to g$ uniformly on F as $n \to \infty$ for some function g, $g \in \mathscr{A}(F)$. In view of the conditions of the theorem, $g \not\equiv 0$ on F and $f \equiv g$ there. Let F' be a regular subset of F with nonempty interior such that $g \neq 0$ there. By means of classical Hurwitz's theorem,

$$|f_n|^{1/n} \to 1, \quad \text{as} \quad n \to \infty \tag{5}$$

uniformly on F'.

Select now a simply connected domain W satisfying $F \subset W \subset B$. For every n, $n > n_1$, let $\pi_n(z) := \prod_{i=1}^{k_n} (z - \chi_{i,n})$ be the monic polynomial with zeros at all zeros of f_n on $\overline{W}$. (If $k_n = 0$, then $\pi_n(z) \equiv 1$.) In view of the hypothesis of the theorem,

$$k_n := \deg \pi_n = o(n) \quad \text{as} \quad n \to \infty. \tag{6}$$

Observe that $\pi_n(z) \neq 0$ for $z \in F'$, as well as for z in a neighbourhood of F'. Therefore, by (5) and (6),

$$|\pi_n(z)|^{1/n} \to 1 \quad \text{as} \quad n \to \infty \tag{7}$$

uniformly on F'.

Fixing an arbitrary point b in F', we take into consideration the family $\chi := \{\chi_n\}_{n=n_1}^{\infty}$ with $\chi_n := \chi_n = \{\pi_n^{-1} \cdot f_n\}^{1/n}$ and χ_n being that regular branch in W for which $|\arg \chi_n(b)| < 1/n$. Obviously, $\chi \in \mathscr{A}(W)$.

We now claim that

$$\chi_n(z) \to 1 \quad \text{as} \quad n \to \infty \tag{8}$$

l.u.in. W.

Indeed, regarding (5) and (7) as well as the choice of the regular branches, we see that (8) is valid uniformly on F'. Hence, by Lemma 3, $\{\chi_n\} \in \mathscr{U}(W)$. Statement (8) follows now by applying subsequently the compactness principle and Vitali's theorem.

Moreover, from (8) and (6) we get

$$\limsup_{n\to\infty} \|f\|_K^{1/n} \leqslant 1$$

and

$$\limsup_{n\to\infty} \|f_{n+1} - f_n\|_K^{1/n} \leqslant 1$$

with K being any compact set in W. By (1), for $K = S$ the inequality is strict. Hence, owing to Lemma 1, the strict inequality holds on each compact subset of W. Consequently, $\{f_n\} \in \mathscr{L}(W)$, and by (1), $\{f_n\} \in \mathscr{L}_f(W)$. Letting W tend to B we arrive at the statement of Theorem 2. □

P r o o f of Theorem 3. Set $\mathscr{R}_{n,m_n}^{(\Gamma,p)} := \mathscr{R}_n$, $n = 1, 2, \ldots$

Fixing an arbitrary point z_0 in $\mathscr{G}$, let φ be the unique univalent function which maps $\mathscr{G}$ onto $\{w, |w| < 1\}$ in a way that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$. The function φ maps Γ onto $\{w, |w| = 1\}$ in a one-to-one way. Further, both functions φ and φ' are analytic in $\mathscr{G}$ and continuous on $\overline{\mathscr{G}}$. We remark that (in the case considered) φ' is nonzero in $\overline{\mathscr{G}}$ (theorems of Caratheodory and of Lindelöf, cf. [7]).

Let $\psi(w)$ be the inverse of $\varphi(z)$; we recall that $\psi' \in \mathscr{A}(T) \cap C(\overline{T})$ and $\psi'(w) \neq 0$ for $|w| \leqslant 1$. Let $(\psi')^{1/p}$ be that regular branch for which $|\arg(\psi'(0))^{1/p}| < 1/n$.

Set $r_n := (\mathscr{R}_n \circ \psi)(\psi')^{1/p}$; apparently, $r_n \in H_p$, $n = 1, 2, \ldots$

By means of Natanson's theorem, $\{r_n\}$ converges in measure on $\mathscr{T}$ as $n \to \infty$. With $\tilde{F}$ being the limit function, we note that $\tilde{F}(w) = (f \circ \psi(w))\psi'(w)^{1/p}$ a.e. on $\mathscr{T}$.

From (2), we get

$$\|r_n\|_{L_p(\mathscr{T})} \leqslant C_3 \tag{9}$$

for all n large enough, say $n \geqslant n_2$. Consequently, regarding Ostrowski-Khinchine's theorem, $\{r_n\} \in \mathscr{L}(T)$ and the limit function r, being an element of H_p, coincides with $\tilde{F}$ a.e. on $\mathscr{T}$; hence $r(w) = (f \circ \psi)(w)\psi'(w)^{1/p}$ a.e. on $\mathscr{T}$. Recall that $r(\exp \mathrm{i}\tau)$ should be regarded as $\lim\limits_{\varrho \to 1} r(\varrho \exp \mathrm{i}\tau)$.

Setting $r \circ \varphi := F$, we get

$$\lim_{n\to\infty} \mathscr{R}_n(z) = F \tag{10}$$

l.u.in. $\mathscr{G}$; $F \in \mathscr{A}(\mathscr{G})$. We remark that the nontangential limits $F(z)$ exist for almost all $z \in \Gamma$ and $f = F$ a.e. there.

Note that by virtue of Privalov's uniqueness theorem, $F \not\equiv 0$ in $\mathscr{G}$.

Using the same argument as in the proof of Theorem 2, we see that

$$\lim_{n\to\infty} \|\mathscr{R}_n\|_K^{1/n} \leqslant 1 \tag{11}$$

for each compact set K in W.

We are now going to prove that

$$\limsup_{n\to\infty} \|f - \mathscr{R}_n\|_{L_p(\Gamma)}^{1/n} < 1. \tag{12}$$

For this purpose, we introduce, for a given number ϱ, $\varrho > 1$, the level curve $\Gamma_\varrho := \{z, G_{\mathscr{G}}(z,\infty) = \ln \varrho\}$ (that is, the level curve associated with the number ϱ and canonical with respect to Green's function).

Select a number $\mu, 1 < \mu < \sup\{\varrho, \Gamma_\varrho \subset U\}$. Let $\Omega = \{\omega_n\}_{n=1,2,\dots}$ be a sequence of monic polynomials, nonzero in $\mathscr{G}^c$ and satisfying, for every $\varrho > 1$,

$$\lim_{n\to\infty} \left\| \frac{\omega_n(u)}{\omega_n(v)} \right\|_{u\in\Gamma,\ v\in\Gamma_\varrho}^{1/n} = \frac{1}{\varrho}.$$

For each n, let W_n be the polynomial of degree n which interpolates the rational function $\mathscr{R}_{n+1}(z)$ at all zeros of $\omega_{n+1}(z)$. The application of Hermite-Lagrange's interpolation formula yields, for each $z \in \overline{T}$,

$$\mathscr{R}_{n+1}(z) - W_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\mu} \frac{\omega_{n+1}(z)}{\omega_{n+1}(t)} \frac{\mathscr{R}_{n+1}(t)}{t - z}\, \mathrm{d}t.$$

Select now a positive number θ_1 such that $\exp \theta_1 < \mu$. For all n sufficiently large, say $n > n_3$, we may write, after taking into account (11),

$$\|\mathscr{R}_{n+1} - W_n\|_{\overline{\mathscr{G}}} \leqslant C_5 (\exp \theta_1/\mu)^n. \tag{13}$$

On the other hand, we have an obvious inequality

$$\|\mathscr{R}_{n+1} - W_n\|_{L_p(\Gamma)} \leqslant C_6 \|\mathscr{R}_{n+1} - W_n\|_{\overline{\mathscr{G}}}. \tag{14}$$

Consider first the case when $p < 1$. By the minimality property, we have

$$\|f - \mathscr{R}_n\|^p_{L_p(\Gamma)} \leqslant \|f - W_n\|^p_{L_p(\Gamma)} \leqslant \|f - \mathscr{R}_{n+1}\|^p_{L_p(\Gamma)} + \|\mathscr{R}_{n+1} - W_n\|^p_{L_p(\Gamma)}.$$

After taking into account (13) and (14), we obtain

$$\|f - \mathscr{R}_n\|^p_{L_p(\Gamma)} - \|f - \mathscr{R}_{n+1}\|^p_{L_p(\Gamma)} \leqslant C_7\{\exp\theta_1/\mu\}^{pn}. \tag{15'}$$

For $p \geqslant 1$, after handling $\|f - \mathscr{R}_n\|_{L_p(\Gamma)}$ similarly and applying Minkowski's inequality, we get

$$\|f - \mathscr{R}_n\|_{L_p(\Gamma)} - \|f - \mathscr{R}_{n+1}\|_{L_p(\Gamma)} \leqslant C_7\{\exp\theta_1/\mu\}^{n}. \tag{15''}$$

In view of the conditions of the theorem, for every n the inequality $\|f - \mathscr{R}_n\|_{L_p(\Gamma)} \geqslant \|f - \mathscr{R}_{n+1}\|_{L_p(\Gamma)}$ holds. Consequently, (12) follows now from inequalities (15) and from (2), after passing to the limit.

Let now E be a regular continuum in $\mathscr{G}$. The application of Lemma 2 leads, thanks to (12), to the inequality

$$\limsup_{n\to\infty} \|\mathscr{R}_{n+1} - \mathscr{R}_n\|_E^{1/n} < 1.$$

Hence, regarding (11), we get

$$\limsup_{n\to\infty} \|\mathscr{R}_n - F\|_E^{1/n} < 1.$$

Thus, all conditions of Theorem 2 are fulfilled with respect to the sequence $\{\mathscr{R}_n\}$. This proves Theorem 3. □

P r o o f of Remark 1. Recall that ω is nonnegative on Γ and integrable together with ω^{-q} for a $q > 0$. Following [21], Chapter V, we have, by Hölder's inequality,

$$\int_\Gamma |f(t) - \mathscr{R}_n(t)|^{pq/(1+q)}|\mathrm{d}t| \leqslant \left(\int_\Gamma \frac{1}{\omega(t)^q}|\mathrm{d}t|\right)^{1/(1+q)} \times \left(\int_\Gamma \omega(t)|f(t) - \mathscr{R}_n(t)|^p|\mathrm{d}t|\right)^{q/(1+q)}.$$

By virtue of (12), we are able to prove analogues of Lemmas 2 and 3 for weighted L_p-approximation. With these notations, the proof in question follows the main idea of the proof of Theorem 3. □

P r o o f of Theorem 4. Set, as before, $R_n := R_{n,m_n}^{(p,w)}$. We presume f not to vanish identically on a subinterval of Δ^* of positive length. We assume that $\Delta^* \equiv \Delta$ Additionally, we assume that $\int_{-1}^{1} \omega(x)\,\mathrm{d}x = 1$ and that U is a bounded domain.

Thanks to (3), we have

$$\|R_n\|_{L_{p,w}(\Delta)} \leqslant C_8 \tag{9'}$$

with C_8 being an appropriate positive constant and $n \geqslant n_4$. We remark that (9′) preserves its validity for any subinterval Δ'. Taking into account (9′), (3) and Remark 2, we obtain

$$\lim_{n\to\infty} \|R_n\|_{\Delta'}^{1/n} = 1 \tag{16}$$

for any regular subinterval Δ' of Δ.

Set $R_n = P_n/Q_n$, $Q_n(z) = \prod(z-\eta'_{n,i})^* \prod(1-z/\eta''_{n,i})^*$, where η'_n and η''_n are those zeros of Q_n which are situated on the disk $D := \{z, |z| \leqslant 2\,\mathrm{diam}(U)\}$ and outside, respectively.

Let now W be a domain such that $\Delta \subset W \subset \overline{W} \subset U$. As above, let π_n, $n = n_5, \ldots$ be the monic polynomial of degree $k_n = \nu(R_n, \overline{W})$, the zeros of which coincide with all zeros of R_n on $\overline{W}$. Recall that

$$k_n = o(n) \quad \text{as} \quad n \to \infty.$$

Hence, for every regular compact set K in W,

$$\limsup_{n\to\infty} \|\pi_n\|_K^{1/n} \leqslant 1.$$

On the other hand, we have (cf. [7], Chapter V)

$$\liminf_{n\to\infty} \|\pi_n\|_K^{1/k_n} \geqslant \mathrm{Cap}\, K. \tag{17}$$

Combing both inequalities, we get

$$\lim_{n\to\infty} \|\pi_n\|_K^{1/n} \to 1. \tag{18}$$

For any n, $n \geqslant n_5$, select a number a_n, $a_n \in \Delta$ such that $R_n(a_n) \neq 0$ and introduce $\chi_n := \{R_n \pi_n^{-1}\}^{1/n}$ with $|\arg \chi_n(a_n)| \leqslant 1/n$. Apparently, the rational functions χ_n do not vanish on $\overline{W}$ and $\{\chi_n\} \in \mathscr{A}(W)$.

Writing $P_n = \pi_n p_n$ we observe that $\chi_n := (p_n/Q_n)^{1/n}$. We claim that

$$\chi_n \to 1 \quad \text{as} \quad n \to \infty \tag{19}$$

l.u.in. W.

First, we show that $\{\chi_n\} \in \mathscr{U}(W)$.

Indeed,

$$\|R_n\|_{L_p(\Delta)} = \frac{1}{|Q_n(\xi_n)|}\|P_n\|_{L_p(\Delta)}$$

for an appropriate point ξ_n, $\xi_n \in \Delta$; this equality together with $(9')$ implies

$$\|P_n\|_{L_p(\Delta)} \leqslant C_8\|Q_n\|_\Delta.$$

By means of Lemma 5 we obtain

$$\|P_n\|_\Delta \leqslant C_9^n\|Q_n\|_\Delta,$$

where C_9 stands for $\exp c$ and $n \geqslant n_6$ is sufficiently large and such that $\varepsilon_n(\omega) < 1$.

Select now a number R, $R > 1$, in the way that the interior of the ellipse $\mathscr{E}_R$ with foci at ± 1 and a ratio major axes/minor axes $= (R^2+1)/(R^2-1)$ contains $\overline{U}$. Estimating now $\|P_n\|_{\mathscr{E}_R}$ by Bernstein-Walsh's lemma, we come to

$$\|P_n\|_{\mathscr{E}_R} \leqslant \|P_n\|_\Delta R^n \leqslant \|Q_n\|_{(\Delta)} C_9^n R^n,$$

which implies

$$\|P_n\|_{\mathscr{E}_R} \leqslant C_{10}^n R^n.$$

From here we get

$$\|p_n\|_{\mathscr{E}_R} \leqslant \frac{C_{10}^n R^n}{C_{11}^{k_n}}$$

with $C_{11} := \min\{|\pi_n(z)|, z \in \mathscr{E}_R\}$. Recalling that $k_n = o(n)$ as $n \to \infty$, we arrive at

$$\|p_n\|_{\mathscr{E}_R} \leqslant C_{12}^n$$

with C_{12} depending on R and W but not on n. Given a compact subset K of W, we have

$$\begin{aligned}\|\chi_n\|_K \leqslant \{\|p_n\|_K / \min\{|Q_n(z)|, z \in K\}\}^{1/n} &\leqslant \{\|p_n\|_{\mathscr{E}_R} / \min\{|Q_n(z)|, z \in W\}\}^{1/n} \\ &\leqslant C_{13}.\end{aligned}$$

Thus, $\{\chi_n\} \in \mathscr{U}(W)$.

Let now Δ' be an arbitrary regular subinterval of Δ. Thanks to

$$|\mathscr{R}(z)|^{1/n} = |\pi_n(z)|^{1/n}|\chi_n(z)| \tag{21}$$

and with regard to (16) and (18), we may write

$$\liminf_{n\to\infty} \|\chi_n\|_{\Delta'} \geqslant 1.$$

Hence, there exists a sequence $\{\kappa_n\}$, $\kappa_n \in \Delta'$ such that $\liminf_{n\to\infty} |\chi_n(\kappa_n)| \geqslant 1$. On the other hand, regarding again (21) and taking into account (17), we obtain that

$$\limsup_{n\to\infty} |\chi_n(\tau_n)| \leqslant 1$$

for an appropriate sequence $\{\tau_n\}$, $\tau_n \in \Delta'$, $|\pi_n(\tau_n)| = \|\pi_n\|_{\Delta'}$.

Let now $\tilde{\mathrm{X}}$ be any limit function of the sequence $\{\chi_n\}$, that is $\tilde{\mathrm{X}} = \lim_{n\in\Lambda} \chi_n$ (locally uniformly inside W) for an infinite sequence Λ. Having in mind the two last relations, we see that $|\tilde{\mathrm{X}}|$ takes the unitvalue on each arbitrary regular subinterval Δ'. Hence, $|\tilde{\mathrm{X}}| \equiv 1$ on Δ. Statement (19) results immediately from the symmetry of the functions R_n, $n = 1, 2, \ldots$ with respect to the real axis, after keeping track of the choice of the regular branches χ_n.

Thanks to the arbitrariness of W, the convergence takes place everywhere (on compact subsets) in the entire domain U.

Coming back to the functions R_n, we get

$$\limsup \|R_n\|_K^{1/n} \leqslant 1 \quad \text{as} \quad n \to \infty$$

for every compact subset K in U.

In the same way as in the previous proofs, we show that

$$\limsup \|R_{n+1} - R_n\|_{L_{p,w}(\Delta)}^{1/n} < 1$$

and

$$\limsup \|R_{n+1} - R_n\|_E^{1/n} < 1$$

for an appropriate regular compact set E with nonempty interior with $\Delta \subset E \subset U$. Thus, $\{R_n\}$ converges uniformly on E to a function, say $\Phi(z)$, where $\Phi(z) \in \mathscr{A}(E)$, $\Phi(z) \equiv f$ a.e. on Δ and $\limsup \|f - R_n\|_E^{1/n} < 1$. Hence all conditions of Theorem 2 are fulfilled with respect to the sequence $\{R_{n,m_n}\}$ and Theorem 2 is applicable.

Theorem 4 is proved. □

References

[1] *S. N. Bernstein*: On the distribution of zeros of polynomials to a continuous function positive on a real interval. Complex Works, Vol. I. Acad. Nauk UdSSR, 1952, pp. 443–451. (In Russian.)

[2] *G. Baker, Jr., and P. Graves-Morris*: Padé Approximants. Encyclopedia of Mathematics and Its Applications, Part I, Volume 13, 14. Addison-Wesley Publishing Company, Massachusetts, 1981.

[3] *H.-P. Blatt, E. B. Saff and M. Simkani*: Jentzsch-Szegő type theorems for the zeros of best approximants. J. London Math. Soc. *38* (1988), 307–316.

[4] *H.-P. Blatt, P. Isenles and E. B. Saff*: Remarks on behavior of zeros of polynomials of best approximating polynomials and rational functions. Algorithmus for Approximation (J. C. Mason, U. C. Cox, eds.). Inst. Math. Conf. Ser., New Ser. 10, Oxford Press, Oxford, 1987, pp. 437–445.

[5] *D. Braess*: On a conjecture of Meinardus on rational approximation of e^x, II. J. Approx. Theory *40* (1984), 375–379.

[6] *J. Gilewicz and W. Pleśniak*: Distribution of zeros of sequences of polynomials. Ann. Polon. Math *30* (1993), 165–177.

[7] *G. M. Golusin*: Geometric Theory of Functions of a Complex Variable. Nauka, Moscow, 1966. (In Russian.)

[8] *A. A. Gonchar*: On uniform convergence of diagonal Padé approximant; English transl. in Math.-UdSSR Sbornik *46* (1983); Matem. Sb. *118(160)* (1982), 535–563.

[9] *A. A. Gonchar*: A local conjecture for the uniqueness of analytic functions. Matem. Sbornik *89* (1972), 148–164. (In Russian.)

[10] *H. Gonska and R. K. Kovacheva*: An extension of Montel's theorems to some rational approximating sequences. Bull. Soc. Lettres Lodz *21* (1996), 73–86.

[11] *R. Grothmann and E. B. Saff*: On the behavior of zeros and poles of best uniform polynomial and rational approximation. Nonlinear Numerical Methods and Rational Approximations. D. Reidel Publ. Co., Dordrecht, 1988, pp. 57–77.

[12] *R. K. Kovacheva*: On the behaviour of Chebyshev rational approximants with a fixed number of poles. Math. Balkanica *3* (1989), 244–256.

[13] *R. K. Kovacheva*: Diagonal rational Chebyshev approximants and holomorphic continuation of functions. Analysis *10* (1990), 147–161.

[14] *R. K. Kovacheva*: An analogue of Montel's theorem to rational approximating sequences. Comp. Ren. Acad. Bulg. Scien. *50* (1997), 9–12.

[15] *A. Kroo and J. Swetits*: On density of interpolation points, a Kadec type theorem and Saff's principle of contamination in L_p-approximation. Constr. Approx. *8* (1992), 87–103.

[16] *I. P. Natanson*: The Theory of Analytic Functions. Nauka, Moscow, 1950. (In Russian.)

[17] *I. I. Privalov*: Boundary Oroperties of Analytic Functions. Nauka, Moscow, 1950. (In Russian.)

[18] *E. B. Saff and H. Stahl*: Ray sequences of best rational approximants for $|x|^\alpha$. Canad. J. Math. *49* (1997), 1034–1065.

[19] *H. Stahl*: Best uniform rational approximation of $|x|$ on $[-1, 1]$. Mat. Sb. *8* (1983), 85–118.

[20] *A. F. Timan*: Approximation of Real-Vaued Functions. GIFMAT, 1960. (In Russian.)

[21] *J. L. Walsh*: Interpolation and approximation by rational functions in the complex plane. AMS Colloquium Publications, Volume XX, 1960; Translation in Russian. IL, Moscow, 1969.

[22] *J. L. Walsh*: Overconvergence, degree of convergence and zeros of sequences of analytic functions. Duke Math. J. *13* (1946), 195–235.

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