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TORSION CLASSES OF SPECKER LATTICE ORDERED GROUPS

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Abstract. In this paper we investigate the relations between torsion classes of Specker lattice ordered groups and torsion classes of generalized Boolean algebras.

Keywords: Specker lattice ordered group, generalized Boolean algebra, torsion class

MSC 2000: 06F15, 06E99

Specker lattice ordered groups have been investigated by Conrad and Darnel [3], [4], [5], and by Conrad and Martinez [7]. Below we write "Specker group" instead of "Specker lattice ordered group".

The notion of a torsion class of lattice ordered groups was introduced by Martinez [13].

In [5] it was proved that the class $\mathcal{S}_G$ of all Specker groups is a torsion class of lattice ordered groups.

Radical classes of generalized Boolean algebras were studied in [11]. Let Y be a radical class of generalized Boolean algebras; we define Y to be a torsion class of generalized Boolean algebras if it is closed with respect to homomorphic images.

We denote by $\mathcal{T}^s$ the collection of all torsion classes X of lattice ordered groups such that $X \subseteq \mathcal{S}_G$. Further, let $\mathcal{T}^b$ be the collection of all torsion classes of generalized Boolean algebras.

In the present paper we show that there exists a one-to-one mapping φ_0 of $\mathcal{T}^s$ onto $\mathcal{T}^b$ such that, whenever $X_1, X_2 \in \mathcal{T}^s$, then

$$X_1 \subseteq X_2 \Leftrightarrow \varphi_0(X_1) \subseteq \varphi_0(X_2).$$

Further, we prove that $\mathcal{T}^s$ is a large collection (in the sense that there exists an injective mapping of the class of all infinite cardinals into $\mathcal{T}^s$).

1. Preliminaries

For the sake of completeness we recall some relevant definitions.

We denote by $\mathcal{G}$ the class of all lattice ordered groups. Let $G \in \mathcal{G}$. An element $0 < s \in G$ is called singular if $x \wedge (s - x) = 0$ whenever $0 \leqslant x \leqslant s$. The set of all singular elements of G is denoted by $S(G)$; further, we put $S_0(G) = S(G) \cup \{0\}$. Then $S_0(G)$ is a sublattice of the lattice G^+.

A lattice ordered group G is a Specker group if G is generated as a group by the set $S(G)$. (Cf. [6].)

For each $G \in \mathcal{G}$ let $C(G)$ be the system of all convex ℓ-subgroups of G; this system is partially ordered by the set-theoretical inclusion. Then $C(G)$ is a complete lattice.

A torsion class of lattice ordered groups is defined to be a nonempty subclass X of $\mathcal{G}$ such that

(i) X is closed with respect to homomorphisms;
(ii) if $G_1 \in X$ and $G_2 \in C(G_1)$, then $G_2 \in X$;
(iii) if $G \in \mathcal{G}$ and $\{G_i\}_{i \in I} \subseteq C(G) \cap X$, then $\bigvee\limits_{i \in I} G_i \in X$.

If X is a nonempty subclass of $\mathcal{G}$ which is closed with respect to isomorphisms and satisfies the conditions (ii), (iii), then X is called a radical class of lattice ordered groups (cf. [10]).

A lattice L is a generalized Boolean algebra if it has the least element 0 and if for each $x \in L$, the interval $[0, x]$ of L is a Boolean algebra.

Let $\mathcal{B}$ be the class of all generalized Boolean algebras. For each $B \in \mathcal{B}$, the system $J(B)$ of all ideals of B (partially ordered by the set-theoretical inclusion) is a complete lattice.

The torsion class of generalized Boolean algebras is defined by conditions which are analogous to the conditions (i), (ii), (iii) above with the distinction that G and $C(G_1)$ are replaced by B and $J(B_1)$.

A nonempty subclass Y of $\mathcal{B}$ which is closed with respect to isomorphisms and satisfies the conditions analogous to (ii) and (iii) (in the above specified sense) is called a radical class of generalized Boolean algebras.

2. Auxiliary results

For lattice ordered groups we apply the notation as in Birkhoff [1] and Conrad [2].

It is well-known that an element $0 < s$ of a lattice ordered group G belongs to $S(G)$ if and only if the interval $[0, s]$ of G is a Boolean algebra.

The following lemma is easy to verify (cf. also [5]).

Lemma 2.1. *Let G be a Specker group. Then $S_0(G)$ is a generalized Boolean algebra.*

The following result is known (cf. [4], Proposition 2.6). Let us remark that a simple alternative proof of 2.2 can be performed by applying Carathéodory functions (for this notion, cf. Gofman [8] and the author [9], [12]).

Lemma 2.2. *Let B be a generalized Boolean algebra. There exists a Specker group G such that $B = S_0(G)$.*

Let $G \in \mathcal{G}$. An indexed system $(a_i)_{i\in I}$ of elements of G^+ is called disjoint if $a_{i(1)} \wedge a_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

Let Z be the additive group of all integers. If $G \in \mathcal{G}$, $x \in G$ and if 0 is the neutral element of Z, then we define $0x$ to be the neutral element of G. (We do not distinguish typographically the neutral element of G and the neutral element of Z; from the context it will be clear which of these elements is taken into consideration.)

From [5], Proposition 1.2 we obtain

Lemma 2.3. *The following conditions for G are equivalent:*

(i) *G is a Specker group.*

(ii) *For each $0 \neq x \in G$ there exist a disjoint system (x_i) $(i = 1, 2, \ldots, n)$ of elements of $S(G)$ and integers α_i $(i = 1, 2, \ldots, n)$ such that*

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n. \tag{1}$$

Under the notation as in 2.3 we say that (1) is a representation of the element x.

A simple calculation yields

Lemma 2.4. *Let (1) be a representation of an element x of a Specker group G. Then $x > 0$ if and only if $\alpha_i > 0$ for $i = 1, 2, \ldots, n$. Further, $x \in S(G)$ if and only if $\alpha_i \in \{0, 1\}$ for each $i = 1, 2, \ldots, n$ and if there is $i \in \{1, 2, \ldots, n\}$ with $\alpha_i \neq 0$.*

Let us extend the definition of the representation so that for $x = 0$ we consider (1) to be a representation of x if $\alpha_i = 0$ $(i = 1, 2, \ldots, n)$. Moreover, for any $x \in G$ the relation $x_i = 0$ for some $i \in \{1, 2, \ldots, n\}$ will be also allowed.

Lemma 2.5. *Let x and y be elements of a Specker group G. Then there exist $t_1, t_2, \ldots, t_k \in S(G)$ and integers $\gamma_1, \gamma_2, \ldots, \gamma_k$, $\gamma'_1, \gamma'_2, \ldots, \gamma'_k$ such that x and y have representations*

$$x = \gamma_1 t_1 + \gamma_2 t_2 + \ldots + \gamma_k t_k,$$
$$y = \gamma'_1 t_1 + \gamma'_2 t_2 + \ldots + \gamma'_k t_k.$$

P r o o f. In view of 2.3 there exist representations

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n,$$
$$y = \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_m y_m.$$

Denote

$$v = \Big(\bigvee x_i\Big) \vee \Big(\bigvee y_j\Big)$$

with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Hence v belongs to $S_0(G)$. Thus $[0, v]$ is a Boolean algebra.

Let x_{n+1} be the complement of the element $x_1 \vee x_2 \vee \ldots \vee x_n$ in the Boolean algebra $[0, v]$ and let y_{m+1} be defined analogously. We put

$$I = \{1, 2, \ldots, n+1\}, \quad J = \{1, 2, \ldots, m+1\}, \quad \alpha_{n+1} = 0 = \beta_{m+1}.$$

Hence we have representations

$$x = \sum_{i \in I} \alpha_i x_i, \quad y = \sum_{j \in J} \beta_j y_j.$$

Denote $x_i \wedge y_j = z_{ij}$ for $i \in I$ and $j \in J$. We obtain

$$v = \bigvee_{i \in I} x_i = \bigvee_{j \in J} y_j,$$

whence for each $i \in I$ we have

$$x_i = x_i \wedge v = x_i \wedge \Big(\bigvee_{j \in J} y_j\Big) = \bigvee_{j \in J} (x_i \wedge y_j)$$
$$= \bigvee_{j \in J} z_{ij} = \sum_{j \in J} z_{ij},$$

since the indexed system $(z_{ij})_{j \in J}$ is disjoint. Analogously,

$$y_j = \sum_{i \in I} z_{ij}.$$

Therefore we get

(2) $$x = \sum_{i \in I, j \in J} \alpha_i z_{ij},$$

(3) $$y = \sum_{i \in I, j \in J} \beta_i z_{ij},$$

and (2), (3) are representations of x and y, respectively. This completes the proof. □

Let B be a generalized Boolean algebra and let A be an ideal of B. In view of 2.2 there exists a Specker group G such that $S_0(G) = B$. We denote by G_1 the set of all $x \in G$ for which there exists a representation (1) such that $x_i \in A$ for $i = 1, 2, \ldots, n$.

Lemma 2.6. *G_1 is a convex ℓ-subgroup of G and $S_0(G_1) = A$.*

P r o o f. a) Let $0 < x \in G_1$ and $y \in G$, $0 < y \leqslant x$. We can assume that (1) is a representation of x and that $x_i \in A$ for $i = 1, 2, \ldots, n$. Further, in view of 2.4, $\alpha_i \geqslant 0$ for $i = 1, 2, \ldots, n$.

There exists a representation of the element y having the form

$$y = \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_m y_m$$

with $y_j \in S(G)$ and $\beta_j > 0$ for $j = 1, 2, \ldots, m$.

Consider the element y_1. Put

$$\begin{aligned} x_0 &= x_1 + x_2 + \ldots + x_n = \vee x_i \ (i = 1, 2, \ldots, n), \\ k &= \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}. \end{aligned}$$

We have $y_1 \leqslant x$, whence $y_1 \leqslant k x_0$. Further, $x_0 \in A$.

In view of the Riesz decomposition property (cf., e.g., Conrad [2], p. 0,19) there are $y_{11}, y_{12}, \ldots, y_{1k} \in G$ such that $0 \leqslant y_{1t} \leqslant x_0$ for $t = 1, 2, \ldots, k$ and

$$y_1 = y_{11} + y_{12} + \ldots + y_{1k}.$$

The relation $y_1 \in S(G)$ yields that the system (y_{1t}) $(t = 1, 2, \ldots, k)$ is disjoint. Thus

$$y_1 = y_{11} \vee y_{12} \vee \ldots \vee y_{1k} \leqslant x_0.$$

Hence $y_1 \in A$. Similarly, $y_2, \ldots, y_m \in A$. We conclude that $y \in G_1$.

b) If $x \in G$ and if x has a representation (1), then $-x$ has the representation

$$-x = (-\alpha_1)x_1 + \ldots + (-\alpha_n)x_n.$$

Hence if x belongs to G_1, then $-x$ belongs to G_1 as well.

c) Let x and y be elements of G_1. We apply the same notation as in the proof of 2.5 and we can assume that all x_i and all y_j belong to A. Then all z_{ij} belong to A.

From (2) and (3) we conclude that $x + y$ has the representation

$$x + y = \sum_{i \in I, j \in J} (\alpha_i + \beta_j) z_{ij}.$$

Thus $x + y \in G_1$.

Let $i \in I$ and $j \in J$. Put

$$\gamma_{ij} = \min\{\alpha_i, \beta_j\}, \quad \delta_{ij} = \max\{\alpha_i, \beta_j\}.$$

Then

$$x \wedge y = \sum_{i \in I, j \in J} \gamma_{ij} z_{ij},$$
$$x \vee y = \sum_{i \in I, j \in J} \delta_{ij} z_{ij},$$

whence $x \wedge y$, $x \vee y \in G_1$. Therefore in view of b), G_1 is an ℓ-subgroup of G. This fact and a) yield that G_1 is a convex ℓ-subgroup of G.

d) The relation $A \subseteq S_0(G_1)$ is obviously valid. Let $0 < x \in S_0(G_1)$. We apply the notation as above. We can assume that $x_i > 0$ and $\alpha_i > 0$ for $i = 1, 2, \ldots, n$.

Since x is singular, the system $(\alpha_i x_i)$ $(i = 1, 2, \ldots, n)$ is disjoint and all $\alpha_i x_i$ belong to $\mathcal{S}(G)$. In view of 2.4, $\alpha_i = 1$ for $i = 1, 2, \ldots, n$. Hence

$$x = x_1 + x_2 + \ldots + x_n = x_1 \vee x_2 \vee \ldots \vee x_n \in A.$$

□

The following result is well-known.

Lemma 2.7. *Let $G \in \mathcal{G}$ and let $\{G_i\}_{i \in I}$ be a nonempty system of elements of $C(G)$. Put $H = \bigvee_{i \in I} G_i$. Then H is the set of all elements $h \in G$ which can be expressed in the form*

$$h = h_1 + h_2 + \ldots + h_n,$$

where $h_j \in \bigcup_{i \in I} G_i$ for each $j \in \{1, 2, \ldots, n\}$. If $h > 0$, then there are h_i with the mentioned property such that $h_j > 0$ for $j = 1, 2, \ldots, n$.

Lemma 2.8. *Let $B \in \mathcal{B}$ and let $\{A_i\}_{i \in I}$ be a nonempty system of elements of $J(B)$. Put $A = \bigvee_{i \in I} A_i$. Then A is the set of all elements $a \in B$ which can be expressed in the form*

$$a = a_1 \vee a_2 \vee \ldots \vee a_n,$$

where $a_j \in \bigcup_{i \in I} A_i$ for each $j \in \{1, 2, \ldots, n\}$.

The proof is simple and will be omitted.

Lemma 2.9. *Let $G \in \mathcal{G}$, $G_i \in C(G)$ $(i \in I)$, $\bigvee_{i\in I} G_i = H$, $B_i = S_0(G_i)$. Then $S_0(H) = \bigvee_{i\in I} B_i$, where $\bigvee_{i\in I} B_i$ is taken with respect to the lattice $J(S_0(G))$.*

P r o o f. Let $i \in I$. We have $G_i \in C(G)$. From this relation we infer that

$$S_0(G_i) = S_0(H) \cap G_i,$$

whence $B_i \subseteq S_0(H)$. From this and from the fact that $S_0(H)$ is an ideal of $S_0(G)$ we obtain

$$\bigvee_{i\in I} B_i \subseteq S_0(H).$$

Let $0 < h \in S_0(H)$. Then in view of 2.7 there are $h_1, h_2, \ldots, h_k \in \bigcup_{i\in I} G_i$ such that $0 < h_t$ $(t = 1, 2, \ldots, k)$ and $h = h_1 + h_2 + \ldots + h_k$. Since h is singular in G all $h_1, h_2, \ldots, h_k$ are singular in G, hence for each $t \in \{1, 2, \ldots, k\}$ there is $i(t) \in I$ such that $h_t \in S_0(G_{i(t)}) = B_{i(t)}$. Moreover, the system $(h_t)_{t=1,2,\ldots,k}$ is disjoint. Thus

$$h = h_1 \vee h_2 \vee \ldots \vee h_k.$$

Therefore $h \in \bigvee_{i\in I} B_i$. □

3. The mapping φ

For each Specker group G we put

$$\varphi(G) = S_0(G).$$

From 2.3 we conclude

Lemma 3.1. *If G_1 and G_2 are Specker groups such that $\varphi(G_1)$ is isomorphic to $\varphi(G_2)$, then G_1 and G_2 are isomorphic.*

Let K^s and K^b be the collection of all nonempty classes of Specker groups or of generalized Boolean algebras, respectively. For each $X \in K^s$ we put

(1) $$\varphi(X) = \{\varphi(G) \colon G \in X\} = Y.$$

Hence φ is a mapping of K^s into K^b.

Lemma 3.1.1. *φ is a one-to-one mapping of K^s onto K^b such that, if $X_1, X_2 \in K^s$, then*

$$X_1 \subseteq X_2 \Leftrightarrow \varphi(X_1) \subseteq \varphi(X_2). \tag{2}$$

P r o o f. In view of 2.2, φ is an epimorphism, and according to 3.1, φ is a monomorphism. The validity of (2) is then obvious. □

Let X and Y be as in (1). If X satisfies the condition (ii) from Section 1 then we say that it is closed with respect to convex ℓ-subgroups. Under the analogous assumption on Y we say that Y is closed with respect to ideals.

If the condition (iii) from Section 1 is fulfilled for X then X is said to be closed under joins; the same term will be applied for Y under the analogous assumption.

Lemma 3.2. *Let $X \in K^s$, $Y = \varphi(X)$. Then the following conditions are equivalent:*

(i) *X is closed with respect to convex ℓ-subgroups;*
(ii) *Y is closed with respect to ideals.*

P r o o f. a) Assume that (i) is valid. Let $B \in Y$ and let A be an ideal of B. There exists $G \in X$ with $B = S_0(G)$. Let G_1 be as in 2.6. Then $G_1 \in X$, hence $S_0(G_1) \in Y$. In view of 2.6, $S_0(G_1) = A$. Therefore (ii) holds.

b) Suppose that the condition (ii) is satisfied. Let $G \in X$ and $G_1 \in C(X)$. Then

$$S_0(G_1) = S_0(G) \cap G_1,$$

whence $S_0(G_1)$ is an ideal of $S_0(G)$. Thus $S_0(G_1)$ belongs to Y. Therefore G_1 is an element of X. This yields that the condition (i) is valid. □

The following assertions 3.3 and 3.4 slightly sharpen some results of [12], Section 3; in fact, several steps in the proofs are the same.

Lemma 3.3. *Let $X \in K^s$ be closed with respect to convex ℓ-subgroups, $Y = \varphi(X)$. Then the following conditions are equivalent:*

(i) *X is closed under joins;*
(ii) *Y is closed under joins.*

P r o o f. a) Let (i) be valid. Let B be a generalized Boolean algebra and let $\{B_i\}_{i\in I}$ be a nonempty subset of $J(B)$ such that all B_i belong to Y.

In view of 2.2 there is a Specker group G such that $S_0(G) = B$. Further, according to 2.9, for each B_i there is $G_i \in C(G)$ with $S_0(B_i) = G_i$. Hence $G_i \in X$ for each

$i \in I$. The condition (i) yields that the lattice ordered group $H = \bigvee_{i\in I} G_i$ belongs to X. Now from 2.9 we conclude that $\bigvee_{i\in I} B_i$ is an element of Y. Hence (ii) holds.

b) Assume that (ii) is satisfied. Let G be a lattice ordered group and let $\emptyset \neq \{G_i\}_{i\in I} \subseteq C(G)$ such that $G_i \in X$ for each $i \in I$. Then $B_i = S_0(G_i) \in Y$ for each $i \in I$. Put $B = S_0(G)$. We have $\{B_i\}_{i\in I} \subseteq J(B)$. In view of (ii), $\bigvee_{i\in I} B_i \in Y$. Thus 2.9 yields that $\bigvee_{i\in I} G_i$ belongs to X. Hence (i) holds. □

Corollary 3.4. *Let X and Y be as in 3.2. The following conditions are equivalent:*

(i) *X is a radical class of lattice ordered groups;*

(ii) *Y is a radical class of generalized Boolean algebras.*

We denote by $\mathcal{R}^s$ the collection of all radical classes X of lattice ordered groups such that $X \subseteq \mathcal{S}_G$. Further, let $\mathcal{R}^b$ be the collection of all radical classes of generalized Boolean algebras. Let φ_0 be the mapping φ reduced to the collection $\mathcal{R}^s$.

Lemma 3.5. *φ_0 is a one-to-one mapping of $\mathcal{R}^s$ onto $\mathcal{R}^b$ such that if $X_1, X_2 \in \mathcal{R}^s$, then*

$$X_1 \subseteq X_2 \Leftrightarrow \varphi_0(X_1) \subseteq \varphi_0(X_2).$$

P r o o f. This is a consequence of 3.1.1 and 3.4. □

4. Torsion classes

Let G be a Specker group. Put $B = S_0(G)$. Let A be an ideal in B and let G_1 be as in 2.6. In view of 2.6 we have $S_0(G_1) = A$. If G_2 is another convex ℓ-subgroup of G and if $S_0(G_2) = A$, then $G_2 = G_1$. In fact, all elements of G_2 are linear combinations with integral coeficients of elements of $A \subseteq G_1$, whence $G_2 \subseteq G_1$; similarly, $G_1 \subseteq G_2$. Hence there is a one-to-one correspondence ψ between the elements of $C(G)$ and the elements of $J(B)$; this correspondence is given by

$$\psi(H) = S_0(H),$$

where H runs over $C(G)$.

Since G is abelian, each element $H \in C(G)$ is an ℓ-ideal of G and thus it is a kernel of a congruence ϱ_1 on G, and each congruence on G can be constructed in this way.

Similarly, each ideal A of B is kernel of a congruence on the generalized Boolean algebra B and in this way we obtain all congruences on B.

Let G_1 and A be as above. Let us construct the factor lattice ordered group $\overline{G} = G/G_1$ and the factor generalized Boolean algebra $\overline{B} = B(A)$.

For $g \in G$ and $b \in B$ we put

$$\overline{g} = g + G_1 = \{g_1 \in G\colon\ g_1 \varrho_1 g\},$$
$$\widetilde{b} = \{b_1 \in B\colon\ b_1 \varrho_2 b\},$$

where ϱ_1 (and ϱ_2) is the congruence relation on G generated by G_1 (or the congruence relation on B generated by A, respectively).

Lemma 4.1. *Let $b, b_1 \in B$. Then $b\varrho_2 b_1$ if and only if $b\varrho_1 b_1$.*

P r o o f. Let $b\varrho_2 b_1$. Denote

$$u = b \wedge b_1, \quad v = b \vee b_1.$$

Then $u\varrho_2 v$. Let u' be the complement of u in the interval $[0, v]$. Thus

$$u' = u' \wedge v, \quad 0 = u' \wedge u,$$

whence $0\varrho_2 u'$. It is well-known that the kernel of ϱ_2 is the ideal A of B. Thus $u' \in A$ and so $0\varrho_1 u'$. Since

$$u = 0 \vee u, \quad v = u' \vee u$$

we obtain $u\varrho_1 v$ and this yields $b\varrho_1 b_1$.

Conversely, suppose that $b\varrho_1 b_1$. Then by analogous steps as above (with ϱ_1 and ϱ_2 interchanged) we conclude that $b\varrho_2 b_1$. □

Lemma 4.2. *Let $x \in G$. The following conditions are equivalent:*

(i) $\overline{x} \in S_0(\overline{G})$;

(ii) *there exists $x_1 \in \overline{x}$ such that $x_1 \in S_0(G)$.*

P r o o f. The implication (ii)⇒(i) is obvious. Assume that (i) is valid.

The case $\overline{x} = \overline{0}$ is trivial. Suppose that $\overline{x} > \overline{0}$. Then without loss of generality we can assume that $x > 0$. Hence there exist $s_1, s_2, \ldots, s_k \in S(G)$ and positive integers $n_1, n_2, \ldots, n_k$ such that

$$x = n_1 s_1 + n_2 s_2 + \ldots + n_k s_k \tag{1}$$

and, moreover, the set $\{s_1, s_2, \ldots, s_k\}$ is disjoint.

If all s_i belong to G_1 then $\overline{x} = \overline{0}$, which is a contradiction. Then the elements s_i belonging to G_1 can be omitted in (1); assume that $s_1, s_2, \ldots, s_m \notin G_1$ and $s_{m+1}, \ldots, s_k \in G_1$. Put

$$x_1 = n_1 s_1 + n_2 s_2 + \ldots + n_m s_m.$$

We obtain $x_1 \in G_1$ and $0 < x_1 \in \overline{x}$.

Assume that $s_i \notin G_1$ and $n_i \geqslant 2$ for some $i \in 1, 2, \ldots, m$. Then $\overline{0} < 2\overline{s}_i \leqslant \overline{x}$, whence $2\overline{s}_i \in S_0(\overline{G})$, but $\overline{0} < \overline{s}_i = \overline{s}_i \wedge \overline{s}_i = \overline{s}_i \wedge (2\overline{s}_i - \overline{s}_i)$, hence $2\overline{s}_i$ fails to be singular. Then $2\overline{s}_i \notin S_0(\overline{G})$, which is a contradiction. Thus $n_i = 1$ and hence x_1 can be written in the form

$$x_1 = s_1 \vee s_2 \vee \ldots \vee s_m$$

with $s_1, s_2, \ldots, s_m \in S_0(G)$. Therefore $x_1 \in S_0(G)$. □

Let f be a mapping of $S_0(\overline{G})$ into $B/A = \overline{B}$ which is defined as follows. For $\overline{x} \in S_0(\overline{G})$ we put

$$f(\overline{x}) = \widetilde{x}_1,$$

where x_1 is as in 4.2.

If $x_1, x_2 \in B$ and both x_1 and x_2 satisfy the condition (ii) from 4.2, then $\overline{x}_1 = \overline{x} = \overline{x}_2$. Thus in view of 4.1 we obtain $\widetilde{x}_1 = \widetilde{x}_2$. Hence the mapping f is correctly defined.

Let $\widetilde{z} \in \widetilde{B}$. Then $f(\overline{z}) = \widetilde{z}$, hence f is an epimorphism. Suppose that $\overline{x}, \overline{y} \in S_0(\overline{G})$ and $f(\overline{x}) = f(\overline{y})$. In other words, we have $f(\overline{x}) = \widetilde{x}_1$, $f(\overline{y}) = \widetilde{y}_1$ and $\widetilde{x}_1 = \widetilde{y}_1$. Then 4.1 yields that $\overline{x}_1 = \overline{y}_1$. Since $\overline{x}_1 = \overline{x}$ and $\overline{y}_1 = \overline{y}$ we get $\overline{x} = \overline{y}$. Therefore f is a monomorphism.

Further, in view of 4.1 we conclude that the mapping f is regular with respect to the lattice operations (i.e., if $x, y \in S_0(\overline{G})$, then $f(x \vee y) = f(x) \vee f(y)$, and similarly for the operation $\wedge$). Thus we have

Lemma 4.3. *f is an isomorphism of the generalized Boolean algebra $S_0(\overline{G})$ onto the generalized Boolean algebra $\overline{B}$.*

Now let X be a nonempty class of Specker groups and $Y = \varphi(X)$, where φ is as in Section 3.

Lemma 4.4. *The following conditions are equivalent:*

(i) *X is closed with respect to homomorphic images.*
(ii) *Y is closed with respect to homomorphic images.*

P r o o f. a) Assume that (i) is valid. Let $B \in Y$ and let A be an element of $J(B)$. We have to verify that $\overline{B} = B/A$ belongs to Y.

There exists $G \in X$ with $\varphi(G) = B$. Let G_1 be as in 2.6. In view of (i) we have $G/G_1 = \overline{G} \in X$. Hence $S_0(\overline{G}) \in Y$. According to 4.3, $S_0(\overline{G})$ is isomorphic to $\overline{B}$. Therefore $\overline{B} \in Y$.

b) Conversely, suppose that (ii) holds. Let $G \in X$ and $G_1 \in C(G)$. We have to verify that $\overline{G} = G/G_1$ belongs to X.

Denote $B = S_0(G)$. Thus $\varphi(G) = S_0(G) = B \in Y$. According to 4.3, $S_0(\overline{G})$ is a homomorphic image of B and hence, in view of (ii), $S_0(\overline{G})$ belongs to Y. Since

$$\varphi(\overline{G}) = S_0(\overline{G})$$

we obtain that $\overline{G}$ must belong to X. □

Let $\mathcal{T}^s$ and $\mathcal{T}^b$ be the collection of all torsion classes of Specker groups or the collection of all torsion classes of generalized Boolean algebras, respectively.

Further, let φ_1 be the mapping φ reduced to the collection $\mathcal{T}^s$.

From 3.5 and 4.4 we conclude

Theorem 4.5. *φ_1 is a one-to-one mapping of $\mathcal{T}^s$ onto $\mathcal{T}^b$ such that for $X_1, X_2 \in \mathcal{T}^s$ we have*

$$X_1 \subseteq X_2 \Leftrightarrow \varphi_1(X_1) \subseteq \varphi_1(X_2).$$

Let K be the class of all infinite cardinals. For each $\alpha \in K$ we denote by $\mathcal{A}(\alpha)$ the class of all generalized Boolean algebras B such that, whenever $[x, y]$ is an interval of B, then

$$\operatorname{card}[x, y] \leqslant \alpha.$$

It is obvious that in the definition of $\mathcal{A}(\alpha)$ it suffices to take into account the intervals $[x, y]$ with $x = 0$.

Lemma 4.6. *Let $\alpha \in K$. Then $\mathcal{A}(\alpha)$ is a radical class of generalized Boolean algebras.*

P r o o f. In view of the definition, $\mathcal{A}(\alpha)$ is closed with respect to ideals. It remains to verify that it is closed with respect to joins. Let $B \in \mathcal{B}$ and $\{A_i\}_{i \in I}$ be as in 2.8. Suppose that all A_i belong to $\mathcal{A}(\alpha)$. We apply the notation as in 2.8. We have to show that A belongs to $\mathcal{A}(\alpha)$.

Let $a \in B$. In ivew of 2.8 and according to Lemma 3.1 from [11] we infer that the element a can be written in the form

$$a = y_1 \vee y_2 \vee \ldots \vee y_n$$

such that $y_j \in \bigcup A_i$ $(i \in I)$ for each $j = 1, 2, \ldots, n$, and $y_{j(1)} \wedge y_{j(2)} = 0$ whenever $j(1)$ and $j(2)$ are distinct elements of the set $\{1, 2, \ldots, n\}$.

Then Lemma 3.2 of [11] yields that

$$[0, a] \simeq [0, y_1] \times [0, y_2] \times \ldots \times [0, y_n],$$

whence $\operatorname{card}[0, a] \leqslant \alpha$ and thus $A \in \mathcal{A}(\alpha)$. □

Lemma 4.7. *Let $\alpha \in K$, $B \in \mathcal{B}$ and let $[x, y]$ be an interval of B with $\operatorname{card}[x, y] \leqslant \alpha$. Let A be an ideal of B; put $\overline{B} = B/A$. For $z \in B$ let $\overline{z}$ be the class in B/A containing the element z. Then $\operatorname{card}[\overline{x}, \overline{y}] \leqslant \alpha$.*

P r o o f. Consider the mapping $f = [x, y] \to [\overline{x}, \overline{y}]$ defined by $f(z) = \overline{z}$ for each $z \in [x, y]$. Let $t \in B$, $\overline{t} \in [\overline{x}, \overline{y}]$. Put $t_1 = (t \vee x) \wedge y$. Then $t_1 \in [x, y]$ and $\overline{t}_1 = \overline{t}$, whence $f(t_1) = \overline{t}$. Therefore the mapping f is surjective. Thus $\operatorname{card}[\overline{x}, \overline{y}] \leqslant \operatorname{card}[x, y] \leqslant \alpha$. □

From 4.6 and 4.7 we conclude

Proposition 4.8. *Let $\alpha \in K$. Then $\mathcal{A}(\alpha)$ is a torsion class of generalized Boolean algebras.*

For $\alpha \in K$ let B_α be the free Boolean algebra with α free generators. Then

(i) $B_\alpha \in \mathcal{A}(\alpha)$;

(ii) if $\beta \in K$ and $\beta > \alpha$, then $B_\beta \notin \mathcal{A}(\alpha)$.

We put $f_1(\alpha) = \mathcal{A}(\alpha)$ for each $\alpha \in K$. In view of 4.8, (i) and (ii) we have

Lemma 4.9. *f_1 is an injective mapping of the class K into $\mathcal{T}^b$.*

Let φ_1 be as in 4.5. For each $\alpha \in K$ we set

$$f_2(\alpha) = \varphi_1^{-1}(f_1(\alpha)).$$

From 4.5 and 4.9 we obtain

Theorem 4.10. *f_2 is an injective mapping of the class K into $\mathcal{T}^s$.*

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