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ULTRA *LI*-IDEALS IN LATTICE IMPLICATION ALGEBRAS

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Abstract. We define an ultra *LI*-ideal of a lattice implication algebra and give equivalent conditions for an *LI*-ideal to be ultra. We show that every subset of a lattice implication algebra which has the finite additive property can be extended to an ultra *LI*-ideal.

Keywords: lattice implication algebra, (ultra) *LI*-ideal, finite additive property

MSC 2000: 03G10, 06B10, 54E15

Introduction

In order to research a logical system whose propositional value is given in a lattice, Y. Xu [5] proposed the concept of lattice implication algebras, and discussed some of their properties. Also, in [4], Y. Xu discussed the homomorphisms between lattice implication algebras. Y. Xu and K. Y. Qin [6] introduced the notion of filters in a lattice implication algebra, and investigated their properties. In [1], Y. B. Jun et al. proposed the concept of an *LI*-ideal of a lattice implication algebra and discussed the relationship between filters and *LI*-ideals, and studied how to generate an *LI*-ideal by a set. This paper is devoted to the discussion of ultra *LI*-ideals of lattice implication algebras. We give equivalent conditions for an *LI*-ideal to be ultra. We show that every subset of a lattice implication algebra which has the finite additive property can be extened to an ultra *LI*-ideal.

By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\rightarrow$" satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$

for all $x, y, z \in L$.

In the sequel the binary operation "$\rightarrow$" will be denoted by juxtaposition. We can define a partial ordering "$\leqslant$" on a lattice implication algebra L by $x \leqslant y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following relations hold (see [5]):

(1) $0x = 1$, $1x = x$ and $x1 = 1$,
(2) $x' = x0$,
(3) $xy \leqslant (yz)(xz)$,
(4) $x \vee y = (xy)y$,
(5) $x \leqslant y$ implies $yz \leqslant xz$ and $zx \leqslant zy$.

In a lattice implication algebra L, if we denote $(xy')'$ by $x \times y$ and $x'y$ by $x + y$, then the following relations are easily proved:

(6) $x + y = y + x$,
(7) $(x + y) + z = x + (y + z)$,
(8) $x + y \geqslant x \vee y$,
(9) $x \times y = y \times x$,
(10) $(x \times y) \times z = x \times (y \times z)$,
(11) $x \times y \leqslant x \wedge y$.

A subset A of a lattice implication algebra L is called an *LI-ideal* of L (see [1]) if it satisfies

(LI1) $0 \in A$,
(LI2) $(xy)' \in A$ and $y \in A$ imply $x \in A$ for all $x, y \in L$.

An *LI*-ideal A of a lattice implication algebra L is said to be *proper* if $A \neq L$.

Theorem 2.1. ([1, Theorem 2.2]) *Let A be an LI-ideal of a lattice implication algebra L and let $x \in A$. If $y \leqslant x$, then $y \in A$ for all $y \in L$.*

Let A be a subset of a lattice implication algebra L. Then the least *LI*-ideal containing A is called the *LI-ideal generated* by A, denoted by $\langle A \rangle$.

The next statement gives a description of the elements of $\langle A\rangle$.

Theorem 2.2. ([1, Theorem 2.9]) *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A\rangle = \{x\in L \mid a_n'(\ldots(a_1'x')\ldots)=1 \text{ for some } a_1,\ldots,a_n\in A\}.$$

ULTRA LI-IDEALS

We start by providing a characterization of LI-ideals.

Proposition 3.1. *Let A be a subset of a lattice implication algebra L. Then A is an LI-ideal of L if and only if the following implications hold:*

(i) *$x\in A$ and $y\leqslant x$ imply $y\in A$,*

(ii) *$x\in A$ and $y\in A$ imply $x+y\in A$.*

Proof. If A is an LI-ideal of L, then (i) holds by Theorem 2.1. Let $x,y\in A$. Then

$$((x+y)y)' = ((x'y)y)' = (x'\vee y)' = x\wedge y'\leqslant x.$$

It follows from Theorem 2.1 that $((x+y)y)'\in A$ and hence $x+y\in A$ by (LI2). Conversely, let A be a subset of L satisfying the conditions (i) and (ii). Since $0\leqslant x$ for all $x\in L$ and hence for all $x\in A$, it follows from (i) that $0\in A$. Suppose $(xy)'\in A$ and $y\in A$. Then $(xy)'+y\in A$ by (ii), and

$$(xy)'+y = ((xy)')'y = (xy)y = x\vee y\geqslant x.$$

Using (i), we have $x\in A$ which proves (LI2), completing the proof. □

Theorem 3.2. *If A is a subset of a lattice implication algebra L, then*

$$\langle A\rangle = \{x\in L \mid x\leqslant a_1+a_2+\ldots+a_n \text{ for some } a_1,\ldots,a_n\in A\}.$$

Proof. By Theorem 2.2 it is sufficient to show that

$$x\leqslant a_1+a_2+\ldots+a_n \iff a_n'(\ldots(a_1'x')\ldots)=1. \tag{3.1}$$

We will prove (3.1) by induction on n. If $n=1$, then

$$x\leqslant a_1 \iff xa_1=1 \iff a_1'x'=1;$$

hence (3.1) holds for $n = 1$. Suppose (3.1) is true for $n = k$, i.e.,

$$x \leqslant a_1 + a_2 + \ldots + a_k \iff a_k'(\ldots(a_1'x')\ldots) = 1.$$

Then

$$\begin{aligned} & x \leqslant a_1 + a_2 + \ldots + a_k + a_{k+1} = a_{k+1} + a_1 + a_2 + \ldots + a_k \\ & \iff x \leqslant a_{k+1}'(a_1 + a_2 + \ldots + a_k) = (a_1 + a_2 + \ldots + a_k)'a_{k+1} \\ & \iff (a_1 + a_2 + \ldots + a_k)' \leqslant xa_{k+1} = a_{k+1}'x' \\ & \iff (a_{k+1}'x')' \leqslant a_1 + a_2 + \ldots + a_k \\ & \iff a_k'(a_{k-1}'(\ldots(a_1'(a_{k+1}'x'))\ldots)) = 1 \\ & \iff a_{k+1}'(a_k'(\ldots(a_2'(a_1'x'))\ldots)) = 1, \end{aligned}$$

which shows that (3.1) holds for $n = k + 1$. This completes the proof. □

Definition 3.3. A subset A of a lattice implication algebra L is said to have the *finite additive property* if $a_1 + a_2 + \ldots + a_n \neq 1$ for any finite members $a_1, a_2, \ldots, a_n$ of A.

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.4. *For a subset A of a lattice implication algebra L, $\langle A\rangle$ is a proper LI-ideal of L if and only if A has the finite additive property.*

Definition 3.5. An LI-ideal A of a lattice implication algebra L is said to be *ultra* if for every $x \in L$, the following equivalence holds:

$$x \in A \iff x' \notin A. \tag{3.2}$$

Theorem 3.6. *Let A be a subset of a lattice implication algebra L. Then A is an ultra LI-ideal of L if and only if A is a maximal proper LI-ideal of L.*

Proof. Suppose that A is an ultra LI-ideal of L. Since $0 \in A$, we have $1 = 0' \notin A$, and hence A is proper. If B is an LI-ideal of L and $A \subsetneq B$, then there exists $x \in L$ such that $x \in B$ and $x \notin A$. By (3.2) we have $x' \in A \subsetneq B$, and so $1 = x + x' \in B$. It follows that $B = L$ and B is not proper. Therefore A is a maximal proper LI-ideal of L.

Conversely, assume that A is a maximal proper LI-ideal of L. For each $x \in L$, we claim that (3.2) is true. Assume $x' \notin A$ and let $B = A \cup \{x\}$. Then B has the finite additive property. In fact, suppose $y_1, \ldots, y_n \in B$. If $y_1, \ldots, y_n \in A$, then $y_1 + \ldots + y_n \neq 1$ because A is proper. Now if there exists $i \leqslant n$ such that $y_i = x$, then

$$y_1 + \ldots + y_n = x + y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n.$$

If $y_1 + \ldots + y_n = 1$ then $x'(y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n) = 1$, i.e., $x' \leqslant y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n$. Thus $x' \in A$ by Theorem 2.1, a contradiction. This proves that B has the finite additive property. Using Corollary 3.4, $\langle B \rangle$ is a proper LI-ideal of L. Since $A \subseteq \langle B \rangle$ and A is a maximal proper LI-ideal, we have $\langle B \rangle = A$ and hence $x \in \langle B \rangle = A$. Suppose $x \in A$. If $x' \in A$, then $1 = x + x' \in A$; hence A is not a proper LI-ideal. This is a contradiction. Therefore $x' \notin A$ and the proof is complete. $\square$

Theorem 3.7. *Let A be a subset of a lattice implication algebra L. If A has the finite additive property, then there exists an ultra LI-ideal B of L containing A.*

Proof. Let

$$\Omega = \{B \mid B \text{ is a proper } LI\text{-ideal of } L \text{ containing } A\}.$$

Then $\langle A \rangle \in \Omega$ and hence $\Omega \neq \emptyset$. Suppose $B_1 \subseteq B_2 \subseteq \ldots$ is a chain of elements of Ω and let $C = \bigcup\limits_i B_i$. Then (i) $A \subseteq C$, (ii) $1 \notin C$ (because $1 \notin B_i$ for all i), (iii) $0 \in C$, and (iv) if $(xy)'$, $y \in C$ then there exists i such that $(xy)', y \in B_i$ and so $x \in B_i \subseteq C$. This shows that C is a proper LI-ideal of L containing A so that $C \in \Omega$. By Zorn's lemma, Ω has a maximal element, say D, which is the desired ultra LI-ideal of L. $\square$

Since every proper LI-ideal has the finite additive property, we have the following corollary.

Corollary 3.8. *Every proper LI-ideal of a lattice implication algebra can be extended to an ultra LI-ideal.*

Theorem 3.9. *Let A be a proper LI-ideal of a lattice implication algebra L. Then A is ultra if and only if for every $a, b \in L$, whenever $a \times b \in A$ then $a \in A$ or $b \in A$.*

Proof. Suppose A is ultra and let $a, b \in L$. If $a \times b \in A$, then $(a \times b)' \notin A$. Since $(a \times b)' = ((ab')')' = ab' = a' + b'$, it follows that $a' \notin A$ or $b' \notin A$ so that $a \in A$ or $b \in A$. Conversely, assume that for every $a, b \in L$, $a \in A$ or $b \in A$ whenever $a \times b \in A$. Then for each $x \in L$, if $x' \notin A$ then $x' \times x = (x'x')' = 1' = 0 \in A$, which implies that $x \in A$. Clearly if $x \in A$, then $x' \notin A$. This completes the proof. $\square$

Theorem 3.10. *Let $f\colon L \to M$ be an implication homomorphism of lattice implication algebras satisfying $f(0) = 0$.*

(i) *If B is an ultra LI-ideal of M, then $f^{-1}(B)$ is an ultra LI-ideal of L.*

(ii) *If f is an isomorphism and if A is an ultra LI-ideal of L, then $f(A)$ is an ultra LI-ideal of M.*

Proof. (i) Clearly $0 \in f^{-1}(B)$. Let $x, y \in L$ be such that $(xy)' \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(y) \in B$ and $(f(x)f(y))' = (f(xy))' = f((xy)') \in B$. Since B is an LI-ideal of M, it follows from (LI2) that $f(x) \in B$ so that $x \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an LI-ideal of L. For each $x \in L$, we have

$$x \in f^{-1}(B) \Longleftrightarrow f(x) \in B \Longleftrightarrow f(x') = (f(x))' \notin B \Longleftrightarrow x' \notin f^{-1}(B).$$

Hence $f^{-1}(B)$ is an ultra LI-ideal of L.

(ii) Note that $0 = f(0) \in f(A)$. Let $x, y \in M$ be such that $(xy)' \in f(A)$ and $y \in f(A)$. Then there exist $u \in L$ and $v \in A$ such that $f(u) = x$ and $f(v) = y$. It follows that

$$f((uv)') = (f(uv))' = (f(u)f(v))' = (xy)' \in f(A)$$

so that $(uv)' \in A$. Using $v \in A$, we know that $u \in A$ and so $x = f(u) \in f(A)$. Thus $f(A)$ is an LI-ideal of M. For each $y \in M$, let $x \in L$ be such that $f(x) = y$. Then

$$y \in f(A) \Leftrightarrow x = f^{-1}(y) \in A \Leftrightarrow x' \notin A \Leftrightarrow y' = (f(x))' = f(x') \notin f(A).$$

Therefore $f(A)$ is an ultra LI-ideal of M. This completes the proof. $\square$

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