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HOMOMORPHISMS OF HETEROGENEOUS ALGEBRAS

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Abstract. A construction of all homomorphisms of a heterogeneous algebra into an algebra of the same type is presented. A relational structure is assigned to any heterogeneous algebra, and homomorphisms between these relational structures make it possible to construct homomorphisms between heterogeneous algebras. Homomorphisms of relational structures can be constructed using homomorphisms of algebras that are described in [11].

Keywords: heterogeneous algebra, relational structure, homomorphism, decomposable mapping, category of heterogeneous algebras

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1. Introduction

The author has published some papers in which he presented methods of constructing homomorphisms of algebraic structures using the construction of homomorphisms of mono-unary algebras. This means that the construction of homomorphisms of general structures may be reduced to the construction of homomorphisms of mono-unary algebras. In this way, constructions of homomorphisms of relational structures and of mono-n-ary algebras were published in [9] and [10], constructions of homomorphisms of general algebras were published in [11].

In the present paper we prove that a construction of homomorphisms of a heterogeneous algebra into another algebra of the same type may be reduced to a construction of the so called strong homomorphisms of suitable relational structures. Since the last construction is known for relational structures with a finite set of relations (see [9]), we are able to construct homomorphisms of heterogeneous algebras with a finite number of operations.

Let $I \neq \emptyset$ be a set, A_i a set for any $i \in I$. Let o be an element such that $o \notin \bigcup\limits_{i\in I} A_i$; it will be regarded as *zero*. For any $i_0 \in I$ and $a \in A_{i_0}$ put $s(a) = (x_i)_{i\in I}$ where $x_{i_0} = a$ and $x_i = o$ for any $i \in I$ with $i \neq i_0$. This means that we replace an element in A_{i_0} by a sequence. Put $\mathbf{S}_{i\in I} A_i = \{s(a);\ a \in \bigcup\limits_{i\in I} A_i\}$. Hence an element $a \in \bigcup\limits_{i\in I} A_i$ may be replaced by several sequences; the number of these sequences equals the number of indices $i \in I$ such that $a \in A_i$.

Suppose that $t = (t_i)_{i\in I} \in \mathbf{S}_{i\in I} A_i$. Then there exists exactly one $i_0 \in I$ such that $t_{i_0} \in A_{i_0}$ while $t_i = o$ for any $i \in I$ with $i \neq i_0$. We put $j(t) = i_0$, $c(t) = t_{i_0} = t_{j(t)}$. Hence the meaning of the mapping c is the following: The sequence $t \in \mathbf{S}_{i\in I} A_i$ is replaced by its only nonzero coordinate. It is easy to see that the following holds.

Lemma 1. *Let $I \neq \emptyset$ be a set, A_i a set for any $i \in I$. Then the following assertions hold.*

(i) *If $i \in I$ and $a \in A_i$ then $c(s(a)) = a$.*

(ii) *If $t \in \mathbf{S}_{i\in I} A_i$ then $s(c(t)) = t$.*

Let $I \neq \emptyset$ be a set, A_i, A'_i sets for any $i \in I$, let o, o' be elements such that $o \notin \bigcup\limits_{i\in I} A_i$, $o' \notin \bigcup\limits_{i\in I} A'_i$. We construct the sets $\mathbf{S}_{i\in I} A_i$, $\mathbf{S}_{i\in I} A'_i$ where o' plays the same role in the construction of $\mathbf{S}_{i\in I} A'_i$ as o has played in the construction of $\mathbf{S}_{i\in I} A_i$. A mapping h of the set $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A'_i$ is called *decomposable* if there exists a mapping h_i of the set A_i into A'_i for any $i \in I$ such that $h(t) = s(h_{j(t)}(c(t)))$ holds for any $t \in \mathbf{S}_{i\in I} A_i$. These mappings h_i are called *components* of h, the system $(h_i)_{i\in I}$ is referred to as the *system of components* of h.

Lemma 2. *Let $I \neq \emptyset$ be a set, $(A_i)_{i\in I}$, $(A'_i)_{i\in I}$ systems of sets, h a decomposable mapping of the set $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A'_i$ whose system of components is $(h_i)_{i\in I}$. Then $j(h(t)) = j(t)$, $c(h(t)) = h_{j(t)}(c(t))$ for any $t \in \mathbf{S}_{i\in I} A_i$.*

Proof. This follows directly by definitions. □

This lemma describes the way of constructing a decomposable mapping h if its system of components $(h_i)_{i\in I}$ is given. For any $t \in \mathbf{S}_{i\in I} A_i$ we find the only index $j(t) \in I$ such that the only coordinate $c(t)$ of t different from o is in $A_{j(t)}$. Then we use the mapping $h_{j(t)}$ to obtain the corresponding element in $A'_{j(t)}$ and we construct $s(h_{j(t)}(c(t)))$ by defining the ith coordinate of $h(t)$ as o' for any $i \neq j(t)$.

On the other hand, if $(h_i)_{i\in I}$ is a sequence of mappings such that h_i maps A_i into A'_i for any $i \in I$, then for any $t \in \mathbf{S}_{i\in I} A_i$ the objects $j(t)$ and $c(t)$ are correctly defined. Hence, we put $h(t) = s(h_{j(t)}(c(t)))$ and, clearly, h is a decomposable mapping of the set $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A'_i$.

In what follows we need a criterion of decomposability.

Lemma 3. *Let $I \neq \emptyset$ be a set, $(A_i)_{i\in I}$, $(A'_i)_{i\in I}$ systems of sets. A mapping h of the set $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A'_i$ is decomposable if and only if $j(h(t)) = j(t)$ holds for any $t \in \mathbf{S}_{i\in I} A_i$.*

P r o o f. If h is decomposable, then $j(h(t)) = j(t)$ holds for any $t \in \mathbf{S}_{i\in I} A_i$ by Lemma 2.

Suppose that $j(h(t)) = j(t)$ is satisfied for any $t \in \mathbf{S}_{i\in I} A_i$.

Let $i \in I$, $x \in A_i$ be arbitrary. Since $j(s(x)) = i = j(h(s(x)))$, we obtain $c(h(s(x))) \in A'_i$. Thus the mapping $c \circ h \circ s$ restricted to A_i is a mapping h_i of the set A_i into A'_i. Hence, for any $t \in \mathbf{S}_{i\in I} A_i$ we obtain $s(h_{j(t)}(c(t))) = (s \circ (c \circ h \circ s) \circ c)(t) = (s \circ c)(h((s \circ c)(t))) = (s \circ c)(h(t)) = h(t)$ by Lemma 1. This means that h is decomposable. $\square$

Example 1. Put $I = \{1, 2\}$, $A_1 = \{a, b\}$, $A_2 = \{a, c\}$, $A'_1 = \{a'\}$, $A'_2 = \{a', c'\}$. Then $\mathbf{S}_{i\in I} A_i = \{ao, bo, oa, oc\}$, $\mathbf{S}_{i\in I} A'_i = \{a'o', o'a', o'c'\}$. Let h be a mapping of $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A'_i$ such that $h(ao) = a'o'$, $h(bo) = a'o'$, $h(oa) = o'a'$, $h(oc) = o'a'$. It is easy to see that $j(h(t)) = j(t)$ holds for any $t \in \mathbf{S}_{i\in I} A_i$. It follows by Lemma 3 that h is a decomposable mapping of the set $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A'_i$ where $h_1(a) = a'$, $h_1(b) = a'$, $h_2(a) = a'$, $h_2(c) = a'$.

3. Relational structures

Let A be a set, $n \geqslant 1$ an integer. We denote by A^n the cartesian product $A \times \ldots \times A$ where A appears n times. An element of A^n will be denoted by $(x_1, \ldots, x_n)$ where $x_i \in A$ for any i with the property $1 \leqslant i \leqslant n$. In particular, if $x \in A$ then (x) is an element in A^1.

A subset $r \subseteq A^n$ is called an *n-ary relation* on A.

Suppose that $K \neq \emptyset$ is a set and that an integer $e(k) \geqslant 1$ is assigned to any $k \in K$. Let r_k be an $e(k)$-ary relation on the set A for any $k \in K$; the number $e(k)$ is called the *arity* of r_k. Then the set A provided with all relations of the system $(r_k)_{k\in K}$ is called a *relational structure* and is denoted by $(A, (r_k)_{k\in K})$. Two relational structures $(A, (r_k)_{k\in K})$, $(A', (r'_k)_{k\in K})$ are said to be *similar* if the arity of r'_k equals arity of r_k for every $k \in K$.

Let $(A, (r_k)_{k\in K})$, $(A', (r'_k)_{k\in K})$ be similar relational structures.

A mapping h of the set A into A' is called a *homomorphism* of the relational structure $(A, (r_k)_{k\in K})$ into $(A', (r'_k)_{k\in K})$ if for any $k \in K$ and any elements $x_1, \ldots, x_{e(k)}$ in A with the property $(x_1, \ldots, x_{e(k)}) \in r_k$ the condition $(h(x_1), \ldots, h(x_{e(k)})) \in r'_k$ is satisfied. In particular, for $e(k) = 1$, the condition $(x) \in r_k$ implies $(h(x)) \in r'_k$.

A mapping h of the set A into A' is called a *strong homomorphism* of the relational structure $(A,(r_k)_{k\in K})$ into $(A',(r'_k)_{k\in K})$ if it has the following properties.

(i) For any $k\in K$ with the property $e(k)=1$ and any $x'\in A'$ the condition $(x')\in r'_k$ holds if and only if there exists $x\in A$ such that $(x)\in r_k$ and $h(x)=x'$.

(ii) For any $k\in K$ with the property $e(k)>1$, for any elements $x_1,\dots,x_{e(k)-1}$ in A, and for any element $x'_{e(k)}$ in A' the condition $(h(x_1),\dots,h(x_{e(k)-1}),x'_{e(k)})\in r'_k$ is satisfied if and only if there exists an element $x_{e(k)}\in A$ such that $(x_1,\dots,x_{e(k)-1},x_{e(k)})\in r_k$ and $h(x_{e(k)})=x'_{e(k)}$.

It is easy to see that the condition (i) can be considered to be an extension of (ii) to the case $e(k)=1$. Furthermore, any strong homomorphism of the relational structure $(A,(r_k)_{k\in K})$ into $(A',(r'_k)_{k\in K})$ is a homomorphism. Cf. [9].

4. Heterogeneous algebras

Let $I\neq\emptyset$ be a set and A_i a set for any $i\in I$. Suppose that $K\neq\emptyset$ is a set and that for any $k\in K$ an integer $a(k)\geqslant 0$ and a finite sequence $i(k)$ of elements in I of length $a(k)+1$ are defined where $i(k)=i(0,k)$ if $a(k)=0$ and $i(k)=i(0,k)i(1,k)\dots i(a(k),k)$ if $a(k)\geqslant 1$; here $i(0,k),\dots,i(a(k),k)$ are supposed to be elements of I. Furthermore, for any $k\in K$ an *operation* f_k is defined as follows.

(i) If $a(k)=0$, then $f_k\in A_{i(0,k)}$.

(ii) If $a(k)\geqslant 1$, then f_k is a mapping of the set $A_{i(1,k)}\times\dots\times A_{i(a(k),k)}$ into $A_{i(0,k)}$.

Then the ordered pair $((A_i)_{i\in I},(f_k)_{k\in K})$ is called a *heterogeneous algebra* (cf. [1], [3], [4], [5]). The number $a(k)$ is called the *arity* of the operation f_k, the finite sequence $i(k)$ its *schema*.

Heterogeneous algebras $((A_i)_{i\in I},(f_k)_{k\in K})$, $((A'_i)_{i\in I},(f'_k)_{k\in K})$ are said to be *similar* if for any $k\in K$ the operations f_k, f'_k have the same arity and the same schema.

Let $((A_i)_{i\in I},(f_k)_{k\in K})$, $((A'_i)_{i\in I},(f'_k)_{k\in K})$ be similar heterogeneous algebras. Suppose that h_i is a mapping of the set A_i into A'_i for any $i\in I$. The system $(h_i)_{i\in I}$ of mappings is called a *homomorphism* of the algebra $((A_i)_{i\in I},(f_k)_{k\in K})$ into $((A'_i)_{i\in I},(f'_k)_{k\in K})$ if the following conditions are satisfied.

(α) If $k\in K$ and $a(k)=0$ hold, then $h_{i(0,k)}(f_k)=f'_k$.

(β) If $k\in K$, $a(k)\geqslant 1$, $i(k)=i(0,k)i(1,k)\dots i(a(k),k)$, and $x_1\in A_{i(1,k)},\dots,x_{a(k)}\in A_{i(a(k),k)}$ hold, then $f'_k(h_{i(1,k)}(x_1),\dots,h_{i(a(k),k)}(x_{a(k)}))=h_{i(0,k)}(f_k(x_1,\dots,x_{a(k)}))$.

Example 2. Let $I=\{1,2\}$, A_1, A_2, A'_1, A'_2 be the same as in Example 1. Put $K=\{1,2\}$; let the operations f_1, f_2 be such that their arities and schemas are as follows: $a(1)=1$, $i(1)=21$, $a(2)=0$, $i(2)=1$. Suppose that the values of these operations are defined in the following way: $f_1(a)=a$, $f_1(b)=c$, $f_2=a$.

Let the operations f'_1, f'_2 have the same arities and schemas as f_1, f_2, respectively; their values are defined in the following way: $f'_1(a')=a'$, $f'_2=a'$.

Then $((A_i)_{i\in I},(f_k)_{k\in K})$, $((A'_i)_{i\in I},(f'_k)_{k\in K})$ are similar heterogeneous algebras.

5. Relations of heterogeneous algebras to relational structures

Let $((A_i)_{i\in I},(f_k)_{k\in K})$ be a heterogeneous algebra. As above, $a(k)$ is the arity and $i(k)$ is the schema of the operation f_k for any $k\in K$. We assign a relational structure on the set $\mathbf{S}_{i\in I}A_i$ to this algebra by defining the relations r_k for any $k\in K$ as follows.

(a) If $k\in K$ and $a(k)=0$, then $r_k=\{(t_k)\}$ where $j(t_k)=i(0,k)$, $c(t_k)=f_k$.

(b) If $k\in K$ and $a(k)\geqslant 1$, then $r_k=\{(x_1,\dots,x_{a(k)},x_{a(k)+1});\ c(x_1)\in A_{i(1,k)},\dots,$ $c(x_{a(k)})\in A_{i(a(k),k)},\ c(x_{a(k)+1})=f_k(c(x_1),\dots,c(x_{a(k)}))\in A_{i(0,k)}\}$.

Hence, $(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$ is a relational structure where $a(k)+1$ is the arity of r_k for any $k\in K$. We put

$$Fo((A_i)_{i\in I},(f_k)_{k\in K})=(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$$

where $(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$ has been constructed in the just described way.

Example 3. Let $((A_i)_{i\in I},(f_k)_{k\in K})$, $((A'_i)_{i\in I},(f'_k)_{k\in K})$ be the same heterogeneous algebras as in Example 2. Then $Fo((A_i)_{i\in I},(f_k)_{k\in K})=(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$ where $\mathbf{S}_{i\in I}A_i$ was constructed in Example 1 and $r_1=\{(ao,oa),(bo,oc)\}$, $r_2=\{(ao)\}$. Similarly, $\mathbf{S}_{i\in I}A'_i$ was defined in Example 1 and $r'_1=\{(a'o',o'a')\}$, $r'_2=\{(a'o')\}$.

Lemma 4. *Let $((A_i)_{i\in I},(f_k)_{k\in K})$, $((A'_i)_{i\in I},(f'_k)_{k\in K})$ be similar heterogeneous algebras, h a decomposable homomorphism of the relational structure $Fo((A_i)_{i\in I},(f_k)_{k\in K})$ into $Fo((A'_i)_{i\in I},(f'_k)_{k\in K})$. Then h is strong.*

Proof. Put $Fo((A_i)_{i\in I},(f_k)_{k\in K})=(\mathbf{S}_{i\in I},(r_k)_{k\in K})$, $Fo((A'_i)_{i\in I},(f'_k)_{k\in K})=(\mathbf{S}_{i\in I}A'_i,(r'_k)_{k\in K})$ where the relations r_k, r'_k are defined by conditions (a), (b). Let $(h_i)_{i\in I}$ be the system of components of the homomorphism h.

Let $k\in K$, $a(k)=0$, $i(k)=i(a,0)$, $x'\in\mathbf{S}_{i\in I}A'_i$ hold. By the definition of r_k, r'_k, we obtain $r_k=\{(s(f_k))\}$, $r'_k=\{(s(f'_k))\}$ using Lemma 1. Since h is a homomorphism, $(h(s(f_k)))\in r'_k$ holds, which entails $s(f'_k)=h(s(f_k))$. Hence, $(x')\in r'_k$ means $x'=s(f'_k)$. Putting $x=s(f_k)$ we obtain $(x)\in r_k$ and $h(x)=h(s(f_k))=s(f'_k)=x'$. Thus (i) in the definition of a strong homomorphism is satisfied.

Suppose $k \in K$, $a(k) > 0$, $i(k) = i(0,k)i(1,k)\dots i(a(k),k)$, let the elements $x_1,\dots,x_{a(k)}$ in $\mathbf{S}_{i\in I}A_i$, $x' \in \mathbf{S}_{i\in I}A'_i$ be such that $(h(x_1),\dots,h(x_{a(k)}),x') \in r'_k$. By (b) we obtain $c(h(x_1)) \in A'_{i(1,k)},\dots,c(h(x_{a(k)})) \in A'_{i(a(k),k)}$, $c(x') = f'_k(c(h(x_1)),\dots,c(h(x_{a(k)})))$. Since h is a decomposable mapping with the system of components $(h_i)_{i\in I}$ we have $j(h(x_1)) = i(1,k) = j(x_1),\dots,j(h(x_{a(k)})) = i(a(k),k) = j(x_{a(k)})$ by Lemma 3, which implies $c(x_1) \in A_{j(x_1)} = A_{i(1,k)}$, $c(x_{a(k)}) \in A_{j(x_{a(k)})} = A_{i(a(k),k)}$. It follows that the value $f_k(c(x_1),\dots,c(x_{a(k)}))$ is defined and is in the set $A_{i(0,k)}$. Let $x \in \mathbf{S}_{i\in I}A_i$ be such that $c(x) = f_k(c(x_1),\dots,c(x_{a(k)}))$. It follows that $(x_1,\dots,x_{a(k)},x) \in r_k$ and the fact that h is a homomorphism implies $(h(x_1),\dots,h(x_{a(k)}),h(x)) \in r'_k$. By the definition of r'_k we obtain $c(h(x)) = f'_k(c(h(x_1)),\dots,c(h(x_{a(k)}))) = c(x')$. By Lemma 1 it follows that $h(x) = s(c(h(x))) = s(c(x')) = x'$.

Hence, condition (ii) from the definition of strong homomorphisms is satisfied. $\square$

Theorem 1. *Let $((A_i)_{i\in I},(f_k)_{k\in K})$, $((A'_i)_{i\in I},(f'_k)_{k\in K})$ be similar heterogeneous algebras, h_i a mapping of the set A_i into A'_i for any $i \in I$. Then the following assertions are equivalent.*

(i) *The system of mappings $(h_i)_{i\in I}$ is a homomorphism of the heterogeneous algebra $((A_i)_{i\in I},(f_k)_{k\in K})$ into $((A'_i)_{i\in I}(f'_k)_{k\in K})$.*

(ii) *The decomposable mapping h of the set $\mathbf{S}_{i\in I}A_i$ into $\mathbf{S}_{i\in I}A'_i$ whose components form the system $(h_i)_{i\in I}$ is a homomorphism of the relational structure $Fo((A_i)_{i\in I},(f_k)_{k\in K})$ into $Fo((A'_i)_{i\in I},(f'_k)_{k\in K})$.*

P r o o f. (1) Let (i) hold.

We have $Fo((A_i)_{i\in I},(f_k)_{k\in K}) = (\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$, $Fo((A'_i)_{i\in I},(f'_k)_{k\in K}) = (\mathbf{S}_{i\in I}A'_i,(r'_k)_{k\in K})$ and the system of mappings $(h_i)_{i\in I}$ is a homomorphism of the first algebra into the other. Furthermore, $(h_i)_{i\in I}$ is a system of components of a decomposable mapping h that maps the set $\mathbf{S}_{i\in I}A_i$ into $\mathbf{S}_{i\in I}A'_i$. By the definition of the operators c, j, s and of the mapping h we obtain $c(t) \in A_{j(t)}$ and $h(t) = s(h_{j(t)}(c(t)))$ for any $t \in \mathbf{S}_{i\in I}A_i$.

Let $k \in K$ be arbitrary. Two cases may occur.

(A) $a(k) = 0$.

If $(x) \in r_k$ holds, then $x = s(f_k)$, $f_k \in A_{i(0,k)}$ by (a) and, hence, $h(x) = h(s(f_k)) = s(h_{j(s(f_k))}(c(s(f_k)))) = s(h_{i(0,k)}(f_k)) = s(f'_k)$ by the definition of a decomposable mapping and by Lemma 1. Since $r'_k = \{(s(f'_k))\}$ by (a), we obtain $(h(x)) \in r'_k$.

(B) $a(k) \geqslant 1$.

Let $x_1,\dots,x_{a(k)},x_{a(k)+1}$ be arbitrary elements in the set $\mathbf{S}_{i\in I}A_i$ and suppose $(x_1,\dots,x_{a(k)},x_{a(k)+1}) \in r_k$. Then $c(x_1) \in A_{i(1,k)},\dots,c(x_{a(k)}) \in A_{i(a(k),k)}$,

$c(x_{a(k)+1}) = f_k(c(x_1), \ldots, c(x_{a(k)})) \in A_{i(0,k)}$ by (b). It follows that

$$c(h(x_l)) = h_{j(x_l)}(c(x_l)) = h_{i(l,k)}(c(x_l))$$

for any integer l with $1 \leqslant l \leqslant a(k)$ and

$$c(h(x_{a(k)+1})) = h_{j(x_{a(k)+1})}(c(x_{a(k)+1})) = h_{i(0,k)}(c(x_{a(k)+1}))$$

by Lemma 2. Thus,

$$\begin{aligned} f'_k(h_{i(1,k)}(c(x_1)), \ldots, h_{i(a(k),k)}(c(x_{a(k)}))) &= h_{i(0,k)}(f_k(c(x_1), \ldots, c(x_{a(k)}))) \\ &= h_{i(0,k)}(c(x_{a(k)+1})). \end{aligned}$$

By (b), it follows that

$$(s(h_{i(1,k)}(c(x_1))), \ldots, s(h_{i(a(k),k)}(c(x_{a(k)}))), s(h_{i(0,k)}(c(x_{a(k)+1})))) \in r'_k$$

by virtue of Lemma 1. By the definition of a decomposable mapping we obtain

$$(h(x_1), \ldots, h(x_{a(k)}), h(x_{a(k)+1})) \in r'_k.$$

Thus, (i) implies (ii).

(2) Let (ii) hold.

Suppose that $(h_i)_{i \in I}$ is the system of components of a decomposable mapping h of the set $\mathbf{S}_{i \in I} A_i$ into $\mathbf{S}_{i \in I} A'_i$. Let h be a homomorphism of the relational structure $Fo((A_i)_{i \in I}, (f_k)_{k \in K}) = (\mathbf{S}_{i \in I} A_i, (r_k)_{k \in K})$ into $Fo((A'_i)_{i \in I}, (f'_k)_{k \in K}) = (\mathbf{S}_{i \in I} A'_i, (r'_k)_{k \in K})$.

Let $k \in K$ be arbitrary. Two cases are possible.

(A) $a(k) = 0$.

Then $f_k \in A_{i(0,k)}$ and $r_k = \{(t_k)\}$ where $j(t_k) = i(0,k)$, $c(t_k) = f_k$ by (a). Since h is a homomorphism, we obtain $(h(t_k)) \in r'_k$, i.e. $h(t_k) = t'_k$ for some $t'_k \in \mathbf{S}_{i \in I} A'_i$. By Lemma 2 we have $c(t'_k) = c(h(t_k)) = h_{i(0,k)}(c(t_k)) = h_{i(0,k)}(f_k)$. By (a) there exists f'_k such that $c(t'_k) = f'_k$. Then $j(t'_k) = j(t_k)$ by Lemma 2, which implies that $f'_k \in A'_{i(0,k)}$, $h_{i(0,k)}(f_k) = f'_k$.

(B) $a(k) \geqslant 1$.

Let $x_1 \in A_{i(1,k)}, \ldots, x_{a(k)} \in A_{i(a(k),k)}$, $x = f_k(x_1, \ldots, x_{a(k)})$. Suppose that $t_l \in \mathbf{S}_{i \in I} A_i$ is such that $c(t_l) = x_l$ for any integer l with the property $1 \leqslant l \leqslant a(k)$, let $t \in \mathbf{S}_{i \in I} A_i$ be such that $c(t) = x$. By the definition of r_k we obtain $(t_1, \ldots, t_{a(k)}, t) \in r_k$. It follows that $(h(t_1), \ldots, h(t_{a(k)}), h(t)) \in r'_k$, which implies the existence of elements $x'_1, \ldots, x'_{a(k)}, x'$ such that $f'_k(x'_1, \ldots, x'_{a(k)}) = x'$, $c(h(t_l)) = x'_l$ for any l with $1 \leqslant l \leqslant$

$a(k)$, and $c(h(t)) = x'$ according to (b). By Lemma 2 we obtain $h_{j(t_l)}(c(t_l)) = x'_l$ for any l such that $1 \leqslant l \leqslant a(k)$, $h_{j(t)}(c(t)) = x'$, i.e. $h_{j(t_l)}(x_l) = x'_l$ for any l with $1 \leqslant l \leqslant a(k)$, $h_{j(t)}(x) = x'$. Clearly, $j(t_l) = i(l,k)$ for any l such that $1 \leqslant l \leqslant a(k)$, $j(t) = i(0,k)$. Hence $f'_k(h_{i(1,k)}(x_1), \ldots, h_{i(a(k),k)}(x_{a(k)})) = f'_k(x'_1, \ldots, x'_{a(k)}) = x' = h_{i(0,k)}(x) = h_{i(0,k)}(f_k(x_1, \ldots, x_{a(k)}))$.

It follows that the mappings h_i of the system $(h_i)_{i\in I}$ satisfy the conditions formulated in the definition of a homomorphism of the heterogeneous algebra $((A_i)_{i\in I}, (f_k)_{\in K})$ into $((A'_i)_{i\in I}, (f'_k)_{k\in K})$. Therefore, (ii) implies (i). □

6. Constructions

We now describe a construction of all homomorphisms of a heterogeneous algebra into a similar one.

Construction 1.

Let $((A_i)_{i\in I}, (f_k)_{k\in K})$, $((A'_i)_{i\in I}, (f'_k)_{k\in K})$ be similar heterogeneous algebras. Construct the relational structures

$$Fo((A_i)_{i\in I}, (f_k)_{k\in K}) = (\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K}),$$
$$Fo((A'_i)_{i\in I}, (f'_k)_{k\in K}) = (\mathbf{S}_{i\in I} A'_i, (r'_k)_{k\in K}).$$

Construct all decomposable strong homomorphisms of the relational structure $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ into $(\mathbf{S}_{i\in I} A'_i, (r'_k)_{k\in K})$.

For any constructed decomposable strong homomorphism of the relational structure $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ into $(\mathbf{S}_{i\in I} A'_i, (r'_k)_{k\in K})$ construct its system of components $(h_i)_{i\in I}$.

The system of mappings $(h_i)_{i\in I}$ is a homomorphism of the heterogeneous algebra $((A_i)_{i\in I}, (f_k)_{k\in K})$ into $((A'_i)_{i\in I}, (f'_k)_{k\in K})$ and any homomorphism of the heterogeneous algebra $((A_i)_{i\in I}, (f_k)_{k\in K})$ into $((A'_i)_{i\in I}, (f'_k)_{k\in K})$ may be constructed in the just described way.

This follows from Theorem 1 and Lemma 4.

Our main problem is how to construct all decomposable strong homomorphisms of the relational structure $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ into $(\mathbf{S}_{i\in I} A'_i, (r'_k)_{k\in K})$. This is solved in [9]. We repeat here the definitions and results that are needed for this purpose.

Let $n \geqslant 0$ be an integer and (A, t) a relational structure where t is an $(n+1)$-ary relation on the set A. Put $\mathbf{P}(A) = \{X;\ X \subseteq A\}$. Furthermore, define

$\mathbf{R}[t] = \{x \in A;\ (x) \in t\}$ if $n = 0$;

$\mathbf{R}[t](X_1, \ldots, X_n) = \{x_{n+1} \in A;$ there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $(x_1, \ldots, x_n, x_{n+1}) \in t\}$ if $n \geqslant 1$ and $X_1, \ldots, X_n$ are in $\mathbf{P}(A)$.

Then $\mathbf{R}[t]$ is an n-ary operation on the set $\mathbf{P}(A)$, i.e., $(\mathbf{P}(A), \mathbf{R}[t])$ is an algebra.

Example 4. Let $(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$, $(\mathbf{S}_{i\in I}A'_i,(r'_k)_{k\in K})$ be the same relational structures as in Example 3. We replace these structures by algebras with one unary and one nullary operation; their carriers are $\mathbf{P}(\mathbf{S}_{i\in I}A_i)$, $\mathbf{P}(\mathbf{S}_{i\in I}A'_i)$, respectively. The unary operation $o_1=\mathbf{R}[r_1]$ defined on the set $\mathbf{P}(\mathbf{S}_{i\in I}A_i)$ has the following table:

$\emptyset$	$\{ao\}$	$\{bo\}$	$\{oa\}$	$\{oc\}$	$\{ao,bo\}$	$\{ao,oa\}$	$\{ao,oc\}$	$\{bo,oa\}$	$\{bo,oc\}$
$\emptyset$	$\{oa\}$	$\{oc\}$	$\emptyset$	$\emptyset$	$\{oa,oc\}$	$\{oa\}$	$\{oa\}$	$\{oc\}$	$\{oc\}$

$\{oa,oc\}$	$\{ao,bo,oa\}$	$\{ao,bo,oc\}$	$\{ao,oa,oc\}$	$\{bo,oa,oc\}$	$\{ao,bo,oa,oc\}$
$\emptyset$	$\{oa,oc\}$	$\{oa,oc\}$	$\{oa\}$	$\{oc\}$	$\{oa,oc\}$

The corresponding mono-unary algebra is represented by Fig. 1.

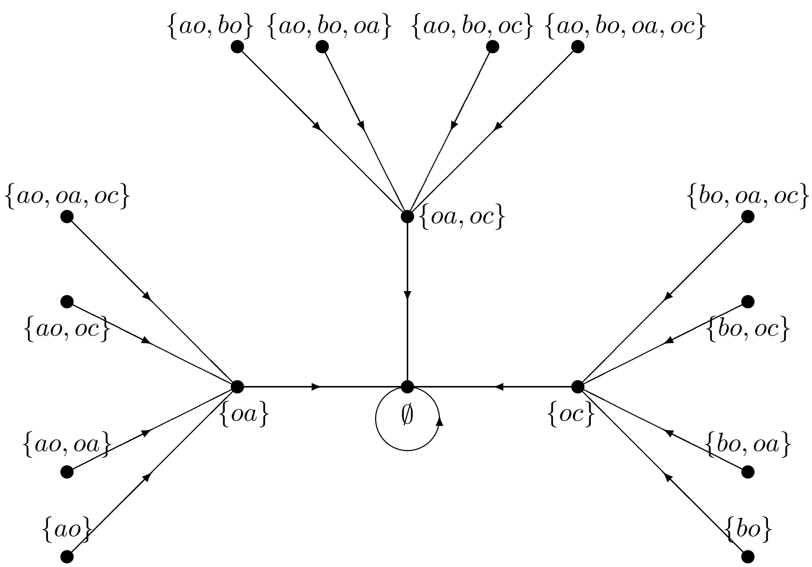

Fig. 1.

The nullary operation $o_2=\mathbf{R}[r_2]$ has the value $\{ao\}$.

Similarly, the unary operation $o'_1=\mathbf{R}[r'_1]$ on the set $\mathbf{P}(\mathbf{S}_{i\in I}A'_i)$ has the following table:

$\emptyset$	$\{a'o'\}$	$\{o'a'\}$	$\{o'c'\}$	$\{a'o',o'a'\}$	$\{a'o',o'c'\}$	$\{o'a',o'c\}$	$\{a'o',o'a',o'c'\}$
$\emptyset$	$\{o'a'\}$	$\emptyset$	$\emptyset$	$\{o'a'\}$	$\{o'a'\}$	$\emptyset$	$\{o'a'\}$

The corresponding mono-unary algebra is represented by Fig. 2.

The nullary operation $o'_2=\mathbf{R}[r'_2]$ has the value $\{a'o'\}$.

The main step in our Construction 1 consists in constructing all strong homomorphisms of one relational structure into another one. The solution of this problem requires some further concepts.

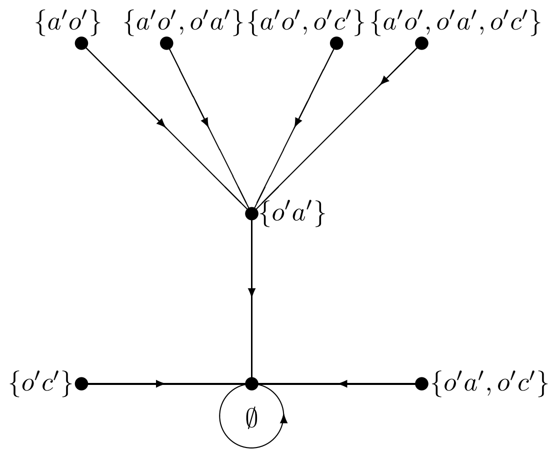

Fig. 2.

Let A, A' be sets. A mapping H of the set $\mathbf{P}(A)$ into $\mathbf{P}(A')$ is said to be *totally additive* if for any $X \in \mathbf{P}(A)$ the condition $H(X) = \bigcup_{x \in X} H(\{x\})$ is satisfied. The mapping H is called *atom preserving* if for any $x \in A$ there exists $x' \in A'$ such that $H(\{x\}) = \{x'\}$.

Let H be a mapping of the set $\mathbf{P}(A)$ into $\mathbf{P}(A')$. Put $\mathbf{Q}[H] = \{(x, x') \in A \times A'; x' \in H(\{x\})\}$. It is easy to see that $\mathbf{Q}[H]$ is a relation from the set A to A'; if H is atom-preserving, $\mathbf{Q}[H]$ is a mapping of the set A into A'.

The construction of all strong homomorphisms of a relational structure into another one can be reduced to a construction of homomorphisms of suitable algebras as follows from

Construction 2.

Let $(A, (r_k)_{k \in K})$, $(A', (r'_k)_{k \in K})$ be similar relational structures.

Construct the algebras $(\mathbf{P}(A), (\mathbf{R}[r_k])_{k \in K})$, $(\mathbf{P}(A'), (\mathbf{R}[r'_k])_{k \in K})$.

Construct all totally additive atom preserving homomorphisms of the algebra $(\mathbf{P}(A), (\mathbf{R}[r_k])_{k \in K})$ into $(\mathbf{P}(A'), (\mathbf{R}[r'_k])_{k \in K})$.

For any constructed totally additive atom preserving homomorphism H of the algebra $(\mathbf{P}(A), (\mathbf{R}[r_k])_{k \in K})$ into $(\mathbf{P}(A'), (\mathbf{R}[r'_k])_{k \in K})$ construct the mapping $\mathbf{Q}[H]$.

Then $\mathbf{Q}[H]$ is a strong homomorphism of the relational structure $(A, (r_k)_{k \in K})$ into $(A', (r'_k)_{k \in K})$ and any strong homomorphism of the relational structure $(A, (r_k)_{k \in K})$ into $(A', (r'_k)_{k \in K})$ may be constructed in the just described way.

This is a consequence of Corollary 2 of [9].

The construction of all homomorphisms of an algebra with a finite number of operations into another algebra of the same type is described in the paper [11]. It is too complicated and, for this reason, it will not be presented here. In our example, we will work with algebras having one nullary and one unary operation. Their homomorphisms can be constructed using [7], [8]. The homomorphisms constructed must be tested in order to choose only totally additive and atom preserving ones.

Example 5. Construct a total additive and atom preserving homomorphism H of the algebra $(\mathbf{P}(\mathbf{S}_{i\in I}A_i),(o_k)_{k\in K})$ into $(\mathbf{P}(\mathbf{S}_{i\in I}A'_i),(o'_k)_{k\in K})$ where these algebras are the same as in Example 4.

Clearly, $H(o_2)=o'_2$, i.e., $H(\{ao\})=\{a'o'\}$. Furthermore, the atoms $\{ao\}$, $\{bo\}$ in the set $\mathbf{P}(\mathbf{S}_{i\in I}A_i)$ (which is ordered by inclusion) satisfy the conditions $o_1(\{ao\})=\{oa\}$, $o_1(\{bo\})=\{oc\}$, i.e., their values are atoms. Since H is a homomorphism preserving atoms, we obtain $o'_1(H(\{ao\}))=H(\{oa\})$, $o'_1(H(\{bo\}))=H(\{oc\})$ where $H(\{ao\})$, $H(\{oa\})$, $H(\{bo\})$, $H(\{oc\})$ are atoms in the set $\mathbf{P}(\mathbf{S}_{i\in I}A'_i)$. Clearly, $\{a'o'\}$ is the only atom in the set $\mathbf{P}(\mathbf{S}_{i\in I}A'_i)$ which is sent to an atom by the mapping o'_1; the other atom is $\{o'a'\}$. It follows that $H(\{ao\})=\{a'o'\}$, $H(\{oa\})=\{o'a'\}$, $H(\{bo\})=\{a'o'\}$, $H(\{oc\})=\{o'a'\}$. Using the total additivity of H we may complete the values of H as follows:

$$\begin{array}{cccccccc}
\emptyset & \{ao\} & \{bo\} & \{oa\} & \{oc\} & \{ao,bo\} & \{ao,oa\} & \{ao,oc\} & \{bo,oa\} \\ \hline
\emptyset & \{a'o'\} & \{a'o'\} & \{o'a'\} & \{o'a'\} & \{a'o'\} & \{a'o',o'a'\} & \{a'o',o'a'\} & \{a'o',o'a'\}
\end{array}$$

$$\begin{array}{cccccc}
\{bo,oc\} & \{oa,oc\} & \{ao,bo,oa\} & \{ao,bo,oc\} & \{ao,oa,oc\} & \{bo,oa,oc\} \\ \hline
\{a'o',o'a'\} & \{o'a'\} & \{a'o',o'a'\} & \{a'o',o'a'\} & \{a'o',o'a'\} & \{a'o',o'a'\}
\end{array}$$

$$\begin{array}{c}
\{ao,bo,oa,oc\} \\ \hline
\{a'o',o'a'\}
\end{array}$$

It is easy to see that H is a total additive and atom preserving homomorphism of the algebra $(\mathbf{P}(\mathbf{S}_{i\in I}A_i),(o_k)_{k\in K})$ into $(\mathbf{P}(\mathbf{S}_{i\in I}A'_i),(o'_k)_{k\in K})$.

Construct $\mathbf{Q}[H]$; it is the mapping h of the set $\mathbf{S}_{i\in I}A_i$ into $\mathbf{S}_{i\in I}A'_i$ such that $h(ao)=a'o'$, $h(bo)=a'o'$, $h(oa)=o'a'$, $h(oc)=o'a'$.

By Construction 2, h is a strong homomorphism of the relational structure $(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$ into $(\mathbf{S}_{i\in I}A'_i,(r'_k)_{k\in K})$.

In this way we are able to construct all strong homomorphisms of the relational structure $(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$ into $(\mathbf{S}_{i\in I}A'_i,(r'_k)_{k\in K})$. Naturally, any constructed strong homomorphism must be tested in order to state whether it is decomposable or not. Any accepted mapping h of the set $\mathbf{S}_{i\in I}A_i$ into $\mathbf{S}_{i\in I}A'_i$ must satisfy the condition $j(t)=j(h(t))$ for any $t\in\mathbf{S}_{i\in I}A_i$ according to Lemma 3.

Example 6. Let the relational structures $(\mathbf{S}_{i\in I}A_i,(r_k)_{k\in K})$, $(\mathbf{S}_{i\in I}A'_i,(r'_k)_{k\in K})$ and the mapping h be the same as in Example 5. In Example 1, we have stated that h is decomposable with components h_1, h_2 where $h_1(a)=a'$, $h_1(b)=a'$,

$h_2(a) = a'$, $h_2(c) = a'$. According to Construction 1, $(h_i)_{i\in I}$ is a homomorphism of the heterogeneous algebra $((A_i)_{i\in I}, (f_k)_{k\in K})$ into $((A'_i)_{i\in I}, (f'_k)_{k\in K})$.

7. Categorical formulations

The fundamental notions of the theory of categories can be found, e.g., in [2] and [6].

Let $I \neq \emptyset$, $K \neq \emptyset$ be sets. Suppose that for any $k \in K$ a nonnegative integer $a(k)$ and a finite sequence $i(k)$ of length $a(k)+1$ formed of elements in I are given: $i(k) = i(0,k)$ if $a(k) = 0$ and $i(k) = i(0,k)i(1,k)\dots i(a(k),k)$ if $a(k) \geqslant 1$ where $i(0,k)$, $i(1,k), \dots, i(a(k),k)$ are elements in I. The sets I, K and the system of finite sequences $(i(k))_{k\in K}$ define the *type* τ of heterogeneous algebras: $\tau = (I, K, (i(k))_{k\in K})$. By a heterogeneous algebra of type $\tau = (I, K, (i(k))_{k\in K})$ we mean a heterogeneous algebra of the form $((A_i)_{i\in I}, (f_k)_{k\in K})$ where the operation f_k has arity $a(k)$ and schema $i(k)$ for any $k \in K$.

Denote by $C(\tau)$ the class of all heterogeneous algebras of the given type τ. Let $((A_i)_{i\in I}, (f_k)_{k\in K})$, $((A'_i)_{i\in I}, (f'_k)_{k\in K})$, $((A''_i)_{i\in I}, (f''_k)_{k\in K})$ be elements in $C(\tau)$, let $(h_i)_{i\in I}$ be a homomorphism of the first algebra into the second, $(h'_i)_{i\in I}$ a homomorphism of the second algebra into the third. Then, clearly, $(h'_i \circ h_i)_{i\in I}$ is a homomorphism of the first algebra into the third. Furthermore, $(\mathrm{id}_{A_i})_{i\in I}$ is a homomorphism of the algebra $((A_i)_{i\in I}, (f_k)_{k\in K})$ into itself. It follows that the class $C(\tau)$ equipped with homomorphisms as morphisms is a category. It will be denoted by $\mathbf{HET}(\tau)$.

Let $\tau = (I, K, (i(k))_{k\in K})$ be a type in the above defined sense. Let $D(\tau)$ denote the class of all relational structures where any of them is defined as follows: We take arbitrary sets A_i where $i \in I$, construct the set $\mathbf{S}_{i\in I} A_i$ and, for any $k \in K$, define a relation r_k of arity $e(k) = a(k) + 1$ in the following way:

(i) If $a(k) = 0$, we put $r_k = \{(s(f_k))\}$ where f_k is an arbitrary element in the set $A_{i(0,k)}$.

(ii) If $a(k) \geqslant 1$, we put $r_k = \{(s(x_1), \dots, s(x_{a(k)}), s(f_k(x_1, \dots, x_{a(k)}))) ;\ x_1 \in A_{i(1,k)}, \dots, x_{a(k)} \in A_{i(a(k),k)}\}$ where f_k is an arbitrary mapping of the set $A_{i(1,k)} \times \dots \times A_{i(a(k),k)}$ into $A_{i(0,k)}$.

Then $D(\tau)$ is the class of all relational structures of the form $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ constructed in the way just described.

Lemma 5. *Let $\tau = (I, K, (i(k))_{k\in K})$ be a type, $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ an object of the class $D(\tau)$. Then there exists a heterogeneous algebra $((A_i)_{i\in I}, (f_k)_{k\in K})$ in the class $C(\tau)$ such that $Fo((A_i)_{i\in I}, (f_k)_{k\in K}) = (\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$.*

P r o o f. The sets A_i can be reconstructed from $\mathbf{S}_{i\in I} A_i$.

Let $k \in K$ be arbitrary.

If $a(k) = 0$, then the arity of r_k is $e(k) = a(k) + 1 = 1$, which implies the existence of an element $f_k \in A_{i(0,k)}$ such that $r_k = \{(s(f_k))\}$ by (i).

Let $a(k) \geqslant 1$; then there exists a mapping f_k of the set $A_{i(1,k)} \times \ldots \times A_{i(a(k),k)}$ into $A_{i(0,k)}$ such that $r_k = \{(s(x_1), \ldots, s(x_{a(k)}), s(f_k(x_1, \ldots, x_{a(k)}));\ x_1 \in A_{i(1,k)}, \ldots, x_{a(k)} \in A_{i(a(k),k)}\}$ by (ii).

In this way, a heterogeneous algebra $((A_i)_{i\in I}, (f_k)_{k\in K})$ is defined that is an element in the class $C(\tau)$. Put $(\mathbf{S}_{i\in I} A_i, (r'_k)_{k\in K}) = Fo((A_i)_{i\in I}, (f_k)_{k\in K})$.

If $a(k) = 0$, then $r'_k = \{(t_k)\}$ where $j(t_k) = i(0,k)$, $c(t_k) = f_k$ according to (a). By Lemma 1 we obtain $t_k = s(c(t_k)) = s(f_k)$ and, hence, $r'_k = \{(s(f_k))\} = r_k$.

Suppose $a(k) \geqslant 1$. Then $r'_k = \{(x_1, \ldots, x_{a(k)}, x_{a(k)+1});\ c(x_1) \in A_{i(1,k)}, \ldots, c(x_{a(k)}) \in A_{i(a(k),k)}, c(x_{a(k)+1}) = f_k(c(x_1), \ldots, c(x_{a(k)})) \in A_{i(0,k)}\} = \{(s(y_1), \ldots, s(y_{a(k)}), s(y_{a(k)+1}));\ c(s(y_1)) \in A_{i(1,k)}, \ldots, c(s(y_{a(k)})) \in A_{i(a(k),k)}, c(s(y_{a(k)+1})) = f_k(c(s(y_1)), \ldots, c(s(y_{a(k)}))) \in A_{i(0,k)}\} = \{(s(y_1), \ldots, s(y_{a(k)}), s(y_{a(k)+1}));\ y_1 \in A_{i(1,k)}, \ldots, y_{a(k)} \in A_{i(a(k),k)}, y_{a(k)+1} = f_k(y_1, \ldots, y_{a(k)}) \in A_{i(0,k)}\} = r_k$ by (b), Lemma 1, and (ii).

Thus, $Fo((A_i)_{i\in I}, (f_k)_{k\in K}) = (\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ holds. $\square$

Let $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$, $(\mathbf{S}_{i\in I} A'_i, (r'_k)_{k\in K})$, $(\mathbf{S}_{i\in I} A''_i, (f''_k)_{k\in K})$ be elements of the class $D(\tau)$. If h is a decomposable homomorphism of the first structure into the second with the system of components $(h_i)_{i\in I}$ and h' is a decomposable homomorphism of the second structure into the third whose system of components equals $(h'_i)_{i\in I}$, then $(h'_i \circ h_i)_{i\in I}$ is a decomposable mapping of the set $\mathbf{S}_{i\in I} A_i$ into $\mathbf{S}_{i\in I} A''_i$. It is easy to see that it is a homomorphism of the relational structure $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ into $(\mathbf{S}_{i\in I} A''_i, (r''_k)_{k\in K})$. Furthermore, $(\mathrm{id}_{A_i})_{i\in I}$ is a decomposable homomorphism of the relational structure $(\mathbf{S}_{i\in I} A_i, (r_k)_{k\in K})$ into itself. It follows that the class $D(\tau)$ equipped with decomposable homomorphisms as morphisms is a category. It will be denoted by $\mathbf{REL}(\tau)$.

Theorem 2. *Let $\tau = (I, K, (i(k))_{k\in K})$ be a type. Then the categories $\mathbf{HET}(\tau)$ and $\mathbf{REL}(\tau)$ are isomorphic.*

P r o o f. The mapping Fo defined in Section 5 maps the class of objects of the category $\mathbf{HET}(\tau)$ into the class of objects of $\mathbf{REL}(\tau)$. By Lemma 5, the mapping Fo is surjective. It is easy to see that it is injective, too: If $((A_i)_{i\in I}, (f_k)_{k\in K})$, $((A'_i)_{i\in I}, (f'_k)_{k\in K})$ are different objects of the category $\mathbf{HET}(\tau)$, then either $(A_i)_{i\in I} \neq (A'_i)_{i\in I}$ which implies that $\mathbf{S}_{i\in I} A_i \neq \mathbf{S}_{i\in I} A'_i$ or $(f_k)_{k\in K} \neq (f'_k)_{k\in K}$ which entails $(r_k)_{k\in K} \neq (r'_k)_{k\in K}$. Thus the mapping Fo is a bijection of the class of all objects in the category $\mathbf{HET}(\tau)$ onto the class of all objects in $\mathbf{REL}(\tau)$.

Let a morphism in the category $\mathbf{HET}(\tau)$ be given. It is a homomorphism $(h_i)_{i\in I}$ of an object $((A_i)_{i\in I},(f_k)_{k\in K})$ into an object $((A'_i)_{i\in I},(f'_k)_{k\in K})$. By Theorem 1 it is a decomposable homomorphism of the relational structure $Fo((A_i)_{i\in I},(f_k)_{k\in K})$ into $Fo((A'_i)_{i\in I},(f'_k)_{k\in K})$. By the same theorem, a decomposable homomorphism $(h_i)_{i\in I}$ of the relational structure $Fo((A_i)_{i\in I},(f_k)_{k\in K})$ into $Fo((A'_i)_{i\in I},(f'_k)_{k\in K})$ is a homomorphism of the algebra $((A_i)_{i\in I},(f_k)_{k\in K})$ into $((A'_i)_{i\in I},(f'_k)_{k\in K})$. Thus, we define the mapping Fm of the class of all morphisms in the category $\mathbf{HET}(\tau)$ into the class of all morphisms in $\mathbf{REL}(\tau)$ to be the identity mapping.

It follows that the functor whose object mapping is Fo and whose morphism mapping is Fm is an isomorphism of the category $\mathbf{HET}(\tau)$ onto the category $\mathbf{REL}(\tau)$. $\square$

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