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A NOTE ON NORMAL VARIETIES OF MONOUNARY ALGEBRAS

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Abstract. A variety is called normal if no laws of the form $s = t$ are valid in it where s is a variable and t is not a variable. Let L denote the lattice of all varieties of monounary algebras (A, f) and let V be a non-trivial non-normal element of L. Then V is of the form $\mathrm{Mod}(f^n(x) = x)$ with some $n > 0$. It is shown that the smallest normal variety containing V is contained in $\mathrm{HSC}(\mathrm{Mod}(f^{mn}(x) = x))$ for every $m > 1$ where C denotes the operator of forming choice algebras. Moreover, it is proved that the sublattice of L consisting of all normal elements of L is isomorphic to L.

Keywords: monounary algebra, variety, normal variety, choice algebra

MSC 2000: 08A60, 08B15

1. Introduction and motivation

The concept of normal identity was introduced by I. I. Mel'nik ([7]) and the so called normally presented varieties were studied by E. Graczyńska ([2]) and J. Płonka (cf. the references in [2]). For any variety V let $\mathrm{Id}_N V$ denote the set of all normal identities holding in V and $N(V)$ the model of $\mathrm{Id}_N V$. As pointed out in [2] and [7], $N(V)$ is a variety covering V in the lattice of all varieties of the same type. The first author was interested in the construction of $N(V)$ by using the so called choice algebras (cf. [1]). Unfortunately, this construction (valid for algebras of a larger type) fails for monounary algebras (see [4], [6] and [8]). The aim of this paper is to improve this situation by showing how $N(V)$ can be obtained from V in this case. Moreover, we show that the lattice of all varieties of monounary algebras and that of all normally presented monounary algebras are isomorphic.

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2. Varieties of monounary algebras

A monounary algebra is an algebra of type (1). In what follows let L denote the lattice of all varieties of such algebras. We first summarize some well-known facts about L (cf. [3] and [5]):

L consists exactly of the following varieties:

$$\begin{aligned} V_i &:= \mathrm{Mod}(f^i(x) = f^i(y)) \text{ for } i \geqslant 0, \\ V_{ij} &:= \mathrm{Mod}(f^j(x) = f^i(x)) \text{ for } 0 \leqslant i \leqslant j \\ \text{and } \overline{V} &:= \mathrm{Mod}\emptyset. \end{aligned}$$

(Here and in the sequel $\mathrm{Mod}\Sigma$ denotes the class of all monounary algebras satisfying Σ and f^0 denotes the identity mapping.)

We have

$$\begin{aligned} V_i &\subseteq V_j \text{ iff } i \leqslant j, \\ V_i &\subseteq V_{jk} \text{ iff } i \leqslant j, \\ V_{ij} &\not\subseteq V_k, \\ V_{ij} &\subseteq V_{kl} \text{ iff both } i \leqslant k \text{ and } j - i \mid l - k \end{aligned}$$

and hence

$$\begin{aligned} V_i &\prec V_j \text{ iff } j = i + 1, \\ V_i &\prec V_{jk} \text{ iff } (j,k) = (i, i+1), \\ V_{ij} &\not\prec V_k, \\ V_{ij} &\prec V_{kl} \text{ iff either } (k,l) = (i+1, j+1) \text{ or } (k,l) = (i, i + p(j-i)) \text{ with } p \text{ prime.} \end{aligned}$$

If for a class K of monounary algebras, $\Sigma(K)$ denotes the set of all laws holding in K then we have

$$\begin{aligned} \Sigma(V_i) = \{f^j(x) = f^k(y) \mid j,k \geqslant i\} &\cup \{f^j(x) = f^k(x) \mid j,k \geqslant i\} \\ &\cup \{f^j(x) = f^j(x) \mid j \geqslant 0\} \end{aligned}$$

and

$$\Sigma(V_{ij}) = \{f^k(x) = f^l(x) \mid k,l \geqslant i; j - i \mid k - l\} \cup \{f^k(x) = f^k(x) \mid k \geqslant 0\}.$$

We can now prove

Theorem 1. *L is distributive but not pseudocomplemented, has exactly two atoms and no coatoms and has both infinite chains and infinite antichains.*

P r o o f. The above description of L yields

$$\begin{aligned} V_i \vee V_j &= V_{\max(i,j)}, \\ V_i \wedge V_j &= V_{\min(i,j)}, \\ V_i \vee V_{jk} &= V_{\max(i,j),\max(i,j)+k-j}, \\ V_i \wedge V_{jk} &= V_{\min(i,j)}, \\ V_{ij} \vee V_{kl} &= V_{\max(i,k),\max(i,k)+\operatorname{lcm}(j-i,l-k)} \end{aligned}$$

and

$$V_{ij} \wedge V_{kl} = V_{\min(i,k),\min(i,k)+\gcd(j-i,l-k)}.$$

Distinguishing all (finitely many) possible cases, distributivity of L can now be checked. Since $\{V \in L \mid V \wedge V_1 = V_0\} = \{V_0\} \cup \{V_{0j} \mid j > 0\}$ it follows that V_1 has no pseudocomplement and hence L is not pseudocomplemented. Of course, V_1 and V_{01} are the only atoms of L and no coatoms exist in L. $\{V_i \mid i \geqslant 0\}$ is an infinite chain and $\{L_{0p} \mid p \text{ prime}\}$ an infinite antichain in L. □

Remarks. (i) The fact that L has exactly two atoms was already remarked in [3]. In that paper it was also proved how to calculate the join and the meet of two elements of L.

(ii) Since L is complete and distributive, but not pseudocomplemented, $\wedge$ is not infinitely distributive with respect to $\vee$ in L.

(iii) Since

$$\begin{aligned} \{V \in L \mid V \wedge V_0 = V_0\} &= L, \\ \{V \in L \mid V \wedge V_i = V_0\} &= \{V_0\} \cup \{V_{0j} \mid j > 0\} \text{ for } i > 0, \\ \{V \in L \mid V \wedge V_{0j} = V_0\} &= \{V_i \mid i \geqslant 0\}, \\ \{V \in L \mid V \wedge V_{ij} = V_0\} &= \{V_0\} \text{ for } i > 0 \\ \text{and } \{V \in L \mid V \wedge \overline{V} = V_0\} &= \{V_0\}, \end{aligned}$$

the only elements V of L having a pseudocomplement V^* are V_0, V_{ij} for $i > 0$ and $\overline{V}$: $V_0^* = \overline{V}$ and $V_{ij}^* = \overline{V}^* = V_0$ for $i > 0$.

A variety is called normal if no laws of the form $s = t$ are valid in it where s is a variable and t is not a variable. For every variety V let $N(V)$ denote the smallest normal variety (of the same type as V) containing V.

Remark. From the results in Section 2 it follows that the non-normal elements of L are exactly V_0 and V_{0j} $(j > 0)$ and that $N(V_0) = V_1$ and $N(V_{0j}) = V_{1,j+1}$ $(j > 0)$.

Next, we want to explain the concept of a choice algebra:

Let M be a set and θ an equivalence relation on M. A choice function on M/θ is a mapping φ from M/θ to M such that $\varphi(B) \in B$ for every $B \in M/\theta$. Let (A, F) be an algebra, $\theta \in \mathrm{Con}A$ and let φ be a choice function on A/θ. Then the algebra (A, F^*) where $F^* := \{f^* \mid f \in F\}$ and where for every n-ary $f \in F$, the operation f^* is defined by $f^*(a_1, \ldots, a_n) := \varphi([f(a_1, \ldots, a_n)]\theta)$ for all $a_1, \ldots, a_n \in A$, is called the choice algebra corresponding to A and θ. (It is well-known that choice algebras corresponding to both the same algebra and the same congruence are isomorphic.) For every class K of algebras let $\mathrm{C}(K)$ denote the class of all choice algebras corresponding to members of K.

Let (A, f) be a monounary algebra, k, l positive integers and $m > 0$ a cardinal number. (A, f) is called a k-cycle if $|A| = k$ and if there exists $a \in A$ such that $a, f(a), \ldots, f^{k-1}(a)$ are mutually distinct and $f^k(a) = a$. (A, f) is called a k-cycle with an l-element chain if there exists $b \in A$ such that $b, f(b), \ldots, f^l(b)$ are mutually distinct and $A \setminus \{b, f(b), , f^{l-1}(b)\}$ is a k-cycle. (A, f) is called a k-cycle with m meeting l-element chains if there exists an m-element subset C of A such that for every $c \in C$ $c, f(c), \ldots, f^l(c)$ are mutually distinct, $\{c, f(c), \ldots, f^{l-1}(c)\}$, $c \in C$, are mutually disjoint, $f^l(c) = f^l(d)$ for all $c, d \in C$ and $A \setminus \bigcup c \in C\{c, f(c), \ldots, f^{l-1}(c)\}$ is a k-cycle.

For every variety V and every set X let $F_V(X)$ denote the free algebra over X with respect to V. The following can be easily seen:

Lemma 2. *For any non-empty set X the following conditions* (i)–(iii) *hold:*

(i) *For $i > 0$, $F_{V_i}(X)$ is a 1-cycle with $|X|$ meeting i-element chains.*

(ii) *$F_{V_{0j}}(X)$ is the disjoint union of $|X|$ j-cycles.*

(iii) *For $i > 0$, $F_{V_{ij}}(X)$ is the disjoint union of $|X|$ $(j - i)$-cycles with an i-element chain.*

Now we can prove our main theorem:

Theorem 3. *Let V be a non-trivial non-normal element of L. Then V is of the form $V = \mathrm{Mod}(f^n(x) = x)$ for some $n > 0$ and $N(V) \subseteq \mathrm{HSC}(\mathrm{Mod}(f^{mn}(x) = x))$ holds for every $m > 1$.*

P r o o f. Let V be a non-trivial non-normal element of L. Then $V = V_{0n}$ for some $n > 0$ and $N(V) = V_{1,n+1}$. Let $A \in N(V)$. (Without loss of generality, $A \neq \emptyset$.) Then $A \in \mathrm{H}(F_1)$ where $F_1 := F_{N(V)}(A)$. Let $m > 1$. From Lemma 2 it follows that F_1 is the disjoint union of $|A|$ n-cycles with a one-element chain and that $F_2 := F_{V_{0,mn}}(A)$ is the disjoint union of $|A|$ mn-cycles. Let θ denote the equivalence relation on F_2 corresponding to the partition $\{\{x, f^n(x)\} \mid x \in F_2\}$ of F_2. Then $\theta \in \mathrm{Con}F_2$. Let F_2^* denote the choice algebra corresponding to F_2 and θ. It is easy to see that $F_1 \in \mathrm{IS}(F_2^*)$. So we finally arrive at

$$A \in \mathrm{H}(F_1) \subseteq \mathrm{HIS}(F_2^*) \subseteq \mathrm{HISC}(F_2) \subseteq \mathrm{HISC}(V_{0,mn}) = \mathrm{HSC}(V_{0,mn}).$$

Since A was an arbitrary member of $N(V)$, the proof is complete. □

Finally, we mention an interesting result concerning the lattice of all normal varieties of monounary algebras:

Theorem 4. *The sublattice of L consisting of all normal elements of L is isomorphic to L.*

P r o o f. Let L' denote this sublattice. Then the mapping assigning V_{i+1} to V_i, $V_{i+1,j+1}$ to V_{ij} and $\overline{V}$ to $\overline{V}$ is an order isomorphism and hence also a lattice isomorphism from L to L'. □

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