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# EXTREMAL METRICS AND MODULUS

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*Abstract.* We give a new proof of Beurling's result related to the equality of the extremal length and the Dirichlet integral of solution of a mixed Dirichlet-Neuman problem.

Our approach is influenced by Gehring's work in $\mathbb{R}^3$ space. Also, some generalizations of Gehring's result are presented.

*Keywords*: extremal distance, conformal capacity, Beurling theorem

*MSC 2000*: 30A15, 30C85

## Introduction

Beurling proved the following result (see Ahlfors [1]):

**Theorem 0.1.** (Beurling's theorem) *Let $\Omega$ be a region in the complex plane bounded by a finite number of analytic Jordan curves, let $E_0$ and $E_1$ be disjoint and consist of a finite number of closed arcs or curves in the boundary of $\Omega$. Then the extremal distance $d_\Omega(E_0, E_1)$ is the reciprocal of the Dirichlet integral*

$$D(u) = \iint_\Omega (u_x^2 + u_y^2)\,\mathrm{d}x\,\mathrm{d}y,$$

*where $u$ satisfies*

(i) *$u$ is bounded and harmonic in $\Omega$,*

(ii) *$u$ has a continous extension to $\Omega \cup E_0^\circ \cup E_1^\circ$, and $u = 0$ on $E_0$ and $u = 1$ on $E_1$,*

(iii) *the normal derivative $\frac{\partial u}{\partial n}$ exists and vanishes on $C_\circ$ ($C$ denotes the full boundary of $\Omega$, $C_\circ = C - (E_0 \cup E_1)$, and $E_0^\circ$ and $E_1^\circ$ denote the relative interiors of $E_0$ and $E_1$ as subsets of $C$).*

The proof is based on two important ingredients:

1) existence of a solution of a mixed Dirichlet-Neuman problem (we denote it by $u$),

2) decomposition of the domain to rings and quadrilateral subdomains using, in fact, the orthogonal and vertical trajectories of the quadratic differential defined by $u$.

For the theory of trajectories of holomorphic quadratic differentials see Gardiner [7] and Strebel [5].

Our first purpose was to give a more elementary proof of this result (that is, with no use of these two subjects), using a minimizing sequence (see for example Courant's book [6]), and to derive some equalities not contained in the proof of Beurling's theorem.

During our work on this problem we became aware of Gehring's papers ([2], [3]), which strongly influenced our research.

In [2] and [3] Gehring proved that Väisälä's definition of extremal distance between $E_0$ and $E_1$ in $\Omega$ (see [9]) is essentially equivalent to Dirichlet's integral definition due to Loewner (see [10]) if $\Omega$ is a ring domain in $\mathbb{R}^3$, and $E_0$ and $E_1$ are boundary components of $\Omega$ (cf. also [4]). Gehring used this result to study quasiconformal mappings in space.

We generalize this result to the setting of smooth domains in $\mathbb{R}^n$. An application of this result gives a short proof of Beurling's Theorem.

As we understand, there are additional technical difficulties if we work with general domains instead of ring domains. Because of that, we need Lemma 2.1.

## 1. Notation

**Definition 1.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$ and $\Gamma$ a set whose elements $\gamma$ are rectifiable arcs in $\Omega$. Let $\varrho$ be a nonnegative Borel measurable function in $\Omega$ (such $\varrho$ we will call a metric). We can define the *$\varrho$-length* of $\gamma$ by

$$L(\gamma, \varrho) = \int_\gamma \varrho \, |\mathrm{d}x|,$$

the *$\varrho$-volume* of $\Omega$ as

$$V(\Omega, \varrho) = \int_\Omega \varrho^n \, \mathrm{d}V(x),$$

where $\mathrm{d}V$ is the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$, and the *minimum length* of $\Gamma$ by $L(\Gamma, \varrho) = \inf_{\gamma \in \Gamma} L(\gamma, \varrho)$. The *modulus* of $\Gamma$ in $\Omega$ is defined by $\operatorname{mod}_\Omega(\Gamma) = \inf_\varrho \frac{V(\Omega, \varrho)}{L(\Gamma, \varrho)^n}$ where $\varrho$ is subject to the condition $0 < V(\Omega, \varrho) < \infty$. The *extremal length* of $\Gamma$ in $\Omega$ is defined as $\Lambda_\Omega(\Gamma) = \operatorname{mod}_\Omega(\Gamma)^{\frac{1}{1-n}}$.

**Definition 1.2.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $E_0$, $E_1$ be two sets in the closure of $\Omega$. Take $\Gamma$ to be the set of all connected arcs in $\Omega$ which join $E_0$ and $E_1$, i.e. each $\gamma \in \Gamma$ has one endpoint in $E_0$ and one in $E_1$, and all other points of $\gamma$ are in $\Omega$. The extremal length $\Lambda(\Gamma)$ is called the *extremal distance* of $E_0$ and $E_1$ in $\Omega$, and we denote it by $d_\Omega(E_0, E_1)$.

Now, let $\Omega$ be a bounded region whose boundary consists of a finite number of $C^1$ hypersurfaces, and $E_0, E_1$ are disjoint, and each is a finite union of closed hypersurfaces contained in the boundary of $\Omega$. Then we define the *conformal $n$-capacity* of $\Omega$ as

$$C[\Omega, E_0, E_1] = \inf_u \int_\Omega |\nabla u|^n \,\mathrm{d}V(x),$$

where the infimum is taken over all functions $u\colon \Omega \to \mathbb{R}$ which are differentiable in $\Omega$, continuous in $\overline{\Omega}$ and have boundary values 0 on $E_0$ and 1 on $E_1$.

From now on let $\Gamma$ be the family of arcs in $\Omega$ which join $E_0$ and $E_1$.

**Definition 1.3.** If $u$ is continuous and ACL in $\Omega$, and $u$ has boundary values 0 on $E_0$ and 1 on $E_1$, we say that $u$ is an admissible function for the domain $\Omega$ with respect to $E_0$ and $E_1$ and denote it by $u \in \mathrm{E}(\Omega, E_0, E_1)$.

## 2. Extremal distance and conformal capacity

In this section we want to prove that

$$d_\Omega(E_0, E_1) = C[\Omega, E_0, E_1]^{\frac{1}{1-n}}.$$

**Lemma 2.1.** *Let $f$ be a metric in $\Omega$ and $V(\Omega, f) < \infty$. Then there exists a neighborhood $U$ of $\partial\Omega$, a metric $\tilde{f}$ on $U$, and a diffeomorphism $A$ of $U$ onto itself such that*

i) $\tilde{f} = f$ *on* $U \bigcap \Omega = U'$,

ii) *$A$ is the identity on $\partial\Omega$ and $A(U') = U''$, where $U'' = U \bigcap \Omega^c$,*

iii) *for every rectifiable curve $\gamma$ in $U''$ we have*

$$L(\gamma, \tilde{f}) \geqslant L(A(\gamma), f),$$

iv) $V(U'', \tilde{f}) \leqslant K\, V(U', f)$, *where $K$ is a finite constant,*

v) *$K$ and $U$ do not depend on $f$.*

P r o o f. The Tubular theorem (see [8]) yields that there exists a neighborhood $U$ of $\partial\Omega$ such that there exists a diffeomorphism $H$ from $U$ onto $(-1,1)\times\partial\Omega$ and $H(x)=(0,x)$ for $x\in\partial\Omega$. For $U$ small enough, we have for the Jacobian $J_H$ of $H$ that $0<m<|J_H|<M<\infty$. Let $S$ be the mapping from $(-1,1)\times\partial\Omega$ onto itself defined as $S((t,x))=(-t,x)$. Define $A$ as $A=H^{-1}\circ S\circ H$. We obtain that $A\circ A=\mathrm{id}$ and $A(U')=U''$. For the Jacobian $J_A$ of $A$ we have $\frac{m}{M}<|J_A|<\frac{M}{m}$, and it folows that $|A'(x)|^n\leqslant K\,|J_A(x)|$ for some $K<+\infty$.

Let now $x$ be from $U''$. Define $\tilde f(x)$ as $\tilde f(x)=f(A(x))\,|A'(x)|$. Then for a rectifiable curve $\gamma$ in $U''$ we have

$$\int_\gamma \tilde f(x)\,|\mathrm{d}x| = \int_\gamma f(A(x))\,|A'(x)|\,|\mathrm{d}x| \geqslant \int_{A(\gamma)} f(y)\,|\mathrm{d}y|.$$

We also conclude that

$$\begin{aligned}\int_{U''}\tilde f^n(x)\,\mathrm{d}V(x) &= \int_{U''} f^n(A(x))\,|A'(x)|^n\,\mathrm{d}V(x)\\ &\leqslant K\int_{U''} f^n(A(x))\,|J_A(x)|\,\mathrm{d}V(x) = K\int_{U'} f^n(y)\,\mathrm{d}V(y).\end{aligned}$$

□

From now on, we suppose that any metric $f$ is defined in some neighborhood of the domain $\overline{\Omega}$ (namely, $\Omega^*=\Omega\bigcup U$), and that we have a diffeomorpfism $A$ of each outside boundary strip small enough onto an appropriate inside boundary strip.

**Lemma 2.2.** *Let $S_r$ be a spherical surface of radius $r$, and let $f$ be a metric on $S_r$. Then each pair of points $P$ and $Q$ on $S_r$ can be joined by a circular arc $\alpha\subset S_r$ such that*

$$\Big(\int_\alpha f(x)\,|\mathrm{d}x|\Big)^n \leqslant A\,r\int_{S_r} f^n(x)\,\mathrm{d}\sigma_r(x),$$

*where $\mathrm{d}\sigma_r$ is the Lebesgue measure on $S_r$ and $A$ is a constant depending only on $n$.*

P r o o f. Let $d(P,Q)=\inf_\beta(L(\beta,f))$, where infimum is taken over all circular arcs on $S_r$ which join the points $P$ and $Q$. We suppose that this infimum is positive (the case when it is zero is left to the reader). Then there exists a circular arc $\alpha$ such that $L(\alpha,f)\leqslant 2\,d(P,Q)$.

Without loss of generality, we can assume that $r=1$ and $P=(0,0,\dots,0,1)$, and denote $\mathbb{S}_1$ by $\mathbb{S}$.

Now we map $\mathbb{S}$ stereographically by $p$ onto $Z=\mathbb{R}^{n-1}$. Then $P$ corresponds to $\infty$, $Q$ to some point $a$, and hence we obtain

$$d(P,Q)\leqslant L(\beta,f)=\int_\beta f(x)\,|\mathrm{d}x| = \int_{\beta'} f(y)\,\frac{2|\mathrm{d}y|}{1+|y|^2},$$

where $\beta$ is a circular arc joining $P$ and $Q$, and $\beta' = p(\beta)$. Then $\beta'$ is the straight line joining $a$ and $\infty$, i.e. $\beta'(t) = a + t\,v$, where $v \in \mathbb{S}^{n-2} = \{x \in \mathbb{R}^{n-1} : |x| = 1\}$ and $t$ goes from 0 to $+\infty$. Hence

$$d(P,Q) \leqslant \int_0^{+\infty} f(y)\,\frac{2\,\mathrm{d}t}{1+|y|^2}, \qquad y = a + t\,v.$$

Integrating with respect to $v \in \mathbb{S}^{n-2}$ and applying Fubini's theorem we conclude

$$d(P,Q) \leqslant \frac{2}{\sigma_{n-2}} \int_{\mathbb{S}^{n-2}} \left( \int_0^{+\infty} \frac{f(y)\,\mathrm{d}t}{1+|y|^2} \right) \mathrm{d}\sigma(v) = \frac{2}{\sigma_{n-2}} \int_Z \frac{f(y)\,\mathrm{d}V(y)}{|y-a|^{n-2}(1+|y|^2)},$$

where $\sigma_{n-2}$ is the $n-2$ dimensional Lebesgue volume of $\mathbb{S}^{n-2}$. By Hölder's inequality we see that the last itegral on the right hand side is majorized by

$$\frac{A^{\frac{1}{n}}}{2} \left( \int_Z f^n(y)\,\frac{\mathrm{d}V(y)}{(1+|y|^2)^{n-1}} \right)^{\frac{1}{n}} = \frac{A^{\frac{1}{n}}}{2} \left( \int_S f^n(x)\,\mathrm{d}\sigma(x) \right)^{\frac{1}{n}},$$

where $\mathrm{d}V$ is the Lebesgue measure in $\mathbb{R}^{n-1}$ and $\mathrm{d}\sigma = \mathrm{d}\sigma_1$, and

$$A^{\frac{1}{n}} = \frac{4}{\sigma_{n-2}} \sup_{a \in Z} \left( \int_Z \frac{\mathrm{d}V(y)}{|y-a|^{\frac{n(n-2)}{n-1}} (1+|y|^2)^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}}.$$

We leave it to the reader to verify that $A$ is finite.

Then we conclude that $\left(\int_\alpha f(x)\,|\mathrm{d}x|\right)^n \leqslant A \int_S f^n(x)\,\mathrm{d}\sigma(x)$. □

**Lemma 2.3.** *Let $\beta$ be a rectifiable curve in $\Omega$ whose one endpoint $A_0$ is in $E_0$ and the other $A_1$ in $E_1$. Let $f$ be any metric in $\Omega$. Then for each $a > 0$ there exists $b > 0$ such that, if we translate the curve $\beta$ by a vector $t$, $|t| < b$ (notation $\beta_t$), then*

$$\int_{\beta_t} f\,|\mathrm{d}x| \geqslant L(\Gamma, f) - a,$$

*where $\Gamma$ is the family of all rectifiable Jordan arcs joining $E_0$ and $E_1$ inside $\Omega$.*

**Remark.** If we work with a ring domain, where $E_0$ and $E_1$ are boundary components, then, if there is part of the curve $\beta_t$ outside $\Omega$ then $\beta_t$ must intersect the corresponding boundary component, and we can choose the appropriate part of $\beta_t$ which joins components (see [3] and [4]).

As we understand, in general we need an additional consideration because there is a possibility that $\beta_t$ has a part outside $\Omega$ without intersection with $E_0$ or $E_1$.

P r o o f. Fix $a > 0$ and choose $\varepsilon > 0$ such that $\varepsilon = \frac{a^n \ln 2}{2^n A}$. There exists $b > 0$ such that

(i) the distance between $E_0$ and $E_1$ is greater than $4b$,

(ii) the diameter of each component of $E_0$ and $E_1$ is greater than $4b$,

(iii) $\iint_{|x-y|<2b} f^n \,\mathrm{d}V(x) < \varepsilon$ for each $y \in \overline{\Omega}$ (in fact, $\mu(A) = \int_A f^n \,\mathrm{d}V$ is an absolutely continuous measure with respect to the Lebesgue measure),

(iv) the outside boundary strip $V''$ is more than $4b$ thick.

By the Fubini theorem we have

$$\int_{b<|x-y|<2b} f^n(x) \,\mathrm{d}V(x) = \int_b^{2b} \frac{dr}{r} \int_{S_r} r f^n \,\mathrm{d}\sigma_r,$$

where $S_r$ is the sphere of radius $r$ with center at $y$.

So, then there exists $r_0 \in (b, 2b)$ such that

$$r_0 \int_{S_{r_0}} f^n \,\mathrm{d}\sigma_{r_0} \int_b^{2b} \frac{dr}{r} = r_0 \ln 2 \int_{S_{r_0}} f^n \,\mathrm{d}\sigma_{r_0} < \varepsilon,$$

i.e.

$$A\, r_0 \int_{S_{r_0}} f^n \,\mathrm{d}\sigma_{r_0} < \frac{A\varepsilon}{\ln 2} = \frac{a^n}{2^n}.$$

If we apply the above argument to $y = A_0$ then there exists $r_0 \in (b, 2b)$ such that

$$A r_0 \int_{S_{r_0}} f^n \,\mathrm{d}\sigma_{r_0} < \frac{a^n}{2^n}.$$

Let $B_0 \in S_{r_0} \cap \beta_t$ and $T_0 \in S_{r_0} \cap E_0$ (these intersections exist because the diameters of $\beta_t$ and the components of $E_0$ are greater than $4b$). Then by Lemma 2.2 we can choose an arc $\alpha_0$ on $S_{r_0}$ joining $T_0$ and $B_0$ such that $L(\alpha_0, f) < \frac{a}{2}$.

In a similar way we can find a sphere $S_{r_1}$ with center at $A_1$ and radius $r_1 \in (b, 2b)$, and choose a curve $\alpha_1$ which joins the point $B_1$ of the curve $\beta_t$ and the point $T_1$ on $E_1$, such that $L(\alpha, f) < \frac{a}{2}$.

From the arc $\alpha_0 + \beta_t + \alpha_1$ we choose a subarc $\gamma$ which joins $E_0$ and $E_1$. Of course, $\gamma$ is in $\Omega^*$ (which is a neighborhood of $\overline{\Omega}$). Every subarc of $\gamma$ which is not in $\Omega$ can be mapped by $A$ to be in $\Omega$ (we obtain a new arc $\gamma'$). Because $\gamma' \in \Gamma$ and by Lemma 2.1 we have

$$\int_\gamma f\,|\mathrm{d}x| \geqslant \int_{\gamma'} f\,|\mathrm{d}x| \geqslant L(\Gamma, f) \tag{1}$$

and by (1) we conclude

$$\int_{\beta_t} f\,|\mathrm{d}x| \geqslant \int_{\gamma} f\,|\mathrm{d}x| - \int_{\alpha_0} f\,|\mathrm{d}x| - \int_{\alpha_1} f\,|\mathrm{d}x|$$
$$\geqslant L(\Gamma, f) - \frac{a}{2} - \frac{a}{2} = L(\Gamma, f) - a,$$

which yields the desired conclusion. □

**Proposition 2.1.** *Under the above conditions we have*

$$\operatorname{mod}_\Omega(\Gamma) = d_\Omega(E_0, E_1)^{1-n} = \inf_g \frac{V(\Omega, g)}{L(\Omega, g)^n},$$

*where the infimum is taken over all continuous metrics $g$ in $\Omega$.*

P r o o f. Suppose that $0 < a < 1$ and $f$ is any metric defined in $\Omega$. Choose $b$ as in Lemma 2.3.

Define $g$ by

$$g(x) = \frac{1}{m(U_b)} \int_{U_b} f(x+y)\,\mathrm{d}V(y),$$

where $U_b = \{x\colon\ |x| < b\}$.

Then $g$ is bounded and continuous. By Fubini's theorem for any $\beta \in \Gamma$ we have

$$\int_\beta g\,|\mathrm{d}x| = \int_\beta \Big( \frac{1}{m(U_b)} \int_{U_b} f(x+y)\,\mathrm{d}V(y) \Big) |\mathrm{d}x| \tag{2}$$
$$= \frac{1}{m(U_b)} \int_{U_b} \Big( \int_{\beta_y} f(x) |\mathrm{d}x| \Big)\,\mathrm{d}V(y),$$

where $\beta_y$ denotes the translation of $\beta$ through the vector $y$.

Now Lemma 2.3 implies that $\int_{\beta_y} f|\mathrm{d}x| \geqslant L(\Gamma, f) - a$ for each $|y| < b$, and we have by (2)

$$L(\beta, g) = \int_\beta g\,|\mathrm{d}x| \geqslant L(\Gamma, f) - a, \tag{3}$$

and if we take the infimum in (3) over all such $\beta$, we obtain

$$L(\Gamma, g) \geqslant L(\Gamma, f) - a. \tag{4}$$

Further, by Jensen's inequality we have

$$V(\Omega, g) = \int_\Omega g^n(x)\,\mathrm{d}V(x) \leqslant \frac{1}{m(U_b)} \int_{U_b} \int_\Omega f^n(x+y)\,\mathrm{d}V(x)\,\mathrm{d}V(y) \tag{5}$$
$$\leqslant \int_{\Omega_b} f^n(x)\,\mathrm{d}V(x) = V(\Omega_b, f),$$

where $\Omega_b$ is a $b$-neighborhood of $\overline{\Omega}$, and, by Lemma 2.1, $V(\Omega_b, f) \to V(\Omega, f)$ when $b \to 0$. By (4) and (5) we have

$$\frac{V(\Omega, g)}{L(\Gamma, g)^n} \leqslant \frac{V(\Omega_b, f)}{(L(\Gamma, f) - a)^n} \to \frac{V(\Omega, f)}{L(\Gamma, f)^n} \tag{6}$$

when $a \to 0$. From (6) we easily obtain the desired conclusion. □

**Proposition 2.2.** *Under the above conditions we have*

$$\inf_g \frac{V(\Omega, g)}{L(\Gamma, g)^n} = \inf_h \frac{V(\Omega, h)}{L(\Gamma, h)^n},$$

*where $g$ is any continuous metric and $h$ is a metric from $C^\infty(\Omega)$.*

P r o o f. Since $g$ could be defined in a neighborhood $\Omega^*$ of $\overline{\Omega}$ then $g$ can be aproximated by nonnegative $C^\infty$-functions uniformly in the whole $\overline{\Omega}$. Let $h_k \rightrightarrows g$ in $\overline{\Omega}$ when $k \to \infty, h_k \in C^\infty(\Omega^*)$. Then

$$V(\Omega, h_k) \to V(\Omega, g),$$

and $L(\beta, h_k) \to L(\beta, g)$ for all $\beta \in \Gamma$, and also

$$L(\Gamma, h_k) \to L(\Gamma, g), \ k \to \infty.$$

Hence

$$\frac{V(\Omega, h_k)}{L(\Gamma, h_k)^n} \to \frac{V(\Omega, g)}{L(\Gamma, g)^n}, k \to \infty,$$

and we have the desired conclusion. □

**Proposition 2.3.** *Under the above conditions we have*

$$\inf_h \frac{V(\Omega, h)}{L(\Gamma, h)^n} = \inf_u \int_\Omega |\nabla u|^n \, \mathrm{d}V(x),$$

*where $h$ is any $C^\infty$-metric and $u \in \mathrm{E}(\Omega, E_0, E_1)$.*

P r o o f. We can define a function $m$ by

$$m(x) = \inf_\beta \int_\beta h(y) \, |\mathrm{d}y|$$

and $u$ by

$$u(x) = \min\Big(1, \frac{m(x)}{L(\Gamma, h)}\Big)$$

for each $x \in \overline{\Omega}$, where $\beta$ is any Jordan arc joining $x$ and $E_0$ inside $\Omega$. Now, $u$ satisfies the uniform Lipschitz condition and $u = 0$ on $E_0$ and $u = 1$ on $E_1$. Hence, $u \in \mathrm{E}(\Omega, E_0, E_1)$ and since $|\nabla u| \leqslant \frac{h}{L(\Gamma,h)}$ a.e. in $\Omega$ we have

$$\int_\Omega |\nabla u|^n \, \mathrm{d}V(x) \leqslant \frac{1}{(L(\Gamma, h))^n} \int_\Omega h^n \, \mathrm{d}V(x) = \frac{V(\Omega, h)}{L(\Gamma, h)^n}.$$

We have proved the proposition. □

**Proposition 2.4.** *Under the above conditions we have*

$$C[\Omega, E_0, E_1] = \inf_u \int_\Omega |\nabla u|^n \, \mathrm{d}V(x), \tag{7}$$

*where the infimum is taken over all* $u \in \mathrm{E}(\Omega, E_0, E_1)$.

P r o o f. For $u \in \mathrm{E}(\Omega, E_0, E_1)$ one can conclude that $u$ can be extended to a neighborhood $\Omega^*$ of $\overline{\Omega}$ such that $u$ remains continuous and ACL. We may assume that $|\nabla u|$ is $L^n$-integrable over $\Omega^*$. Next fix $0 < a < \frac{1}{2}$ and let

$$v = \begin{cases} 0, & \text{if } u < a \\ \dfrac{u-a}{1-2a}, & \text{if } a \leqslant u \leqslant 1-a \qquad \text{on } \overline{\Omega}. \\ 1, & \text{if } 1-a < u \end{cases} \tag{8}$$

The set where $a \leqslant u \leqslant 1 - a$ is a bounded subset of $\mathbb{R}^n$ and lies at a distance $b$ from $E_0 \cup E_1$. Let

$$\omega(x) = \frac{1}{m(U_c)} \int_{U_c} v(x+y) \, \mathrm{d}V(y),$$

where $c < b$.

This function is continuously differentiable in $\Omega$ and has boundary values 0 on $E_0$ and 1 on $E_1$. From (8) we see that $v$ is ACL everywhere and by Hölder's inequality we obtain that $|\nabla v|$ is $L^n$-integrable over each compact set. Hence, we can apply Fubini's theorem to conclude that

$$\nabla\omega(x) = \frac{1}{m(U_c)} \int_{U_c} \nabla v(x+y) \, \mathrm{d}V(y)$$

for each $x \in \Omega$. Then applying Jensen's inequality we obtain

$$\int_\Omega |\nabla\omega(x)|^n \, \mathrm{d}V(x) \leqslant \frac{1}{m(U_c)} \int_{U_c} \int_\Omega |\nabla v(x+y)|^n \, \mathrm{d}V(x) \, \mathrm{d}V(y).$$

The inner integral on the right hand side is majorized by

$$\int_{\Omega_c} |\nabla v(x)|^n \, \mathrm{d}V(x) \leqslant \frac{1}{(1-2a)^n} \int_{\Omega_c} |\nabla u(x)|^n \, \mathrm{d}V(x)$$

for each $y$ in $U_c$. Hence

$$\int_{\Omega} |\nabla \omega|^n \, \mathrm{d}V(x) \leqslant \frac{1}{(1-2a)^n} \int_{\Omega_c} |\nabla u|^n \, \mathrm{d}V(x)$$

and

$$C[\Omega, E_0, E_1] \leqslant \frac{1}{(1-2a)^n} \int_{\Omega_c} |\nabla u|^n \, \mathrm{d}V(x).$$

Letting $a \to 0$ we have

$$C[\Omega, E_0, E_1] \leqslant \int_{\Omega} |\nabla u|^n \, \mathrm{d}V(x). \tag{9}$$

Because the infimum on the right hand side of (7) is over a wider class of functions than on the left hand side we have the inequality

$$C[\Omega, E_0, E_1] \geqslant \inf_u \int_{\Omega} |\nabla u|^n \, \mathrm{d}V(x). \tag{10}$$

By (9) and (10) we have the desired conclusion. □

**Theorem 2.1.** *If $\Omega$ is a bounded domain whose boundary consists of a finite number of $C^1$ hypersurfaces, and if $E_0$ and $E_1$ are disjoint subsets of the boundary of $\Omega$ consisting of a finite number of closed hypersurfaces, then we have*

$$\operatorname{mod}_{\Omega}(\Gamma) = \inf_f \frac{V(\Omega, f)}{L(\Gamma, f)^n} = C[\Omega, E_0, E_1], \tag{11}$$

*where $f$ is any metric in $\Omega$ and $\Gamma$ is the family of Jordan arcs joining $E_0$ and $E_1$ inside $\Omega$.*

Proof. It follows by Propositions 2.1, 2.2, 2.3 and 2.4. □

The case $n = 2$ of the above Theorem enables us to give a short proof of Theorem 1.1. In fact, the proof immediately follows from Theorem 1.3 [6], which gives a solution of a mixed Dirichlet-Neuman problem.

The proof of Theorem 1.3 in Courant's book [6] is based on using minimizing sequences. We believe that we can use minimizing sequences as Gehring in [2] to show the existence of the extremal admissible function $u \in \mathrm{E}(\Omega, E_0, E_1)$ such that

$$C[\Omega, E_0, E_1] = \int_{\Omega} |\nabla u|^n \, \mathrm{d}V.$$

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