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SYMMETRIES IN FINITE ORDER VARIATIONAL SEQUENCES

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Abstract. We refer to Krupka's variational sequence, i.e. the quotient of the de Rham sequence on a finite order jet space with respect to a 'variationally trivial' subsequence. Among the morphisms of the variational sequence there are the Euler-Lagrange operator and the Helmholtz operator.

In this note we show that the Lie derivative operator passes to the quotient in the variational sequence. Then we define the *variational Lie derivative* as an operator on the sheaves of the variational sequence. Explicit representations of this operator give us some abstract versions of Noether's theorems, which can be interpreted in terms of conserved currents for Lagrangians and Euler-Lagrange morphisms.

Keywords: fibered manifold, jet space, variational sequence, symmetries, conservation laws, Euler-Lagrange morphism, Helmholtz morphism

MSC 2000: 58A12, 58A20, 58E30, 58J10

1. Introduction

The geometrical formulations of the Calculus of Variations on fibered manifolds include a large class of theories for which the Euler-Lagrange operator is a morphism of an exact sequence [1, 11, 17, 20, 21, 22]. This viewpoint allows one to overcome several problems of the Lagrangian formulations in Mechanics and Field Theories [19].

We consider the recent formulation by Krupka [11]. This has two main features. First, it is stated on finite order jets of the fibering, rather than on infinite order jets like most of the others. Moreover, it is conceptually very simple, because the exact

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sequence, or *variational sequence*, is defined as a quotient of the de Rham sequence on a finite order jet space with respect to an intrinsically defined subsequence, the choice of which is inspired by the Calculus of Variations.

We show that the Lie derivative operator with respect to fiber-preserving vector fields passes to the quotient, yielding a new operator on the sheaves of the variational sequence, namely the *variational Lie derivative*. This idea has already been exploited by Vinogradov although in a different formalism [21, 22].

We make use of the representation given in [24] of the quotient sheaves of the variational sequence as concrete sheaves of forms. In this way, we provide explicit formulae for the quotient Lie derivative operators, as well as some abstract versions of Noether's theorems. Finally, we interpret Noether's theorems in terms of conserved currents for Lagrangians and Euler-Lagrange morphisms. In particular we recast, in a coherent and comprehensive scheme, some results previously found by various authors in different frameworks [1, 9, 10, 12, 13, 15, 18, 19].

We stress that the algebraic methods used here allow a synthetic and clear understanding of concepts whose full meaning could hardly be reached by means of coordinate expressions alone.

Throughout the paper the structure forms on jet spaces [14] as well as the horizontal and the vertical differential [16] will be used as fundamental tools.

Manifolds and maps between manifolds are C^∞. All morphisms of fibered manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. As for sheaves, we will use the definitions and the main results given in [26]. In particular, we will be concerned only with sheaves of $\mathbb{R}$-vector spaces. Thus, by 'sheaf morphism' we will shortly mean a morphism of sheaves of $\mathbb{R}$-vector spaces. Section 2 reviews earlier results, while Sections 3 and 4 contain the new results. An example is provided in Section 5.

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2. Jet spaces and variational sequences

In this section we recall some basic facts about jet spaces [3, 14, 16] and Krupka's formulation of the finite order variational sequence [11, 24].

2.1. Jet spaces.

Here we introduce jet spaces of a fibered manifold and the sheaves of forms on the r-th order jet space. Moreover, we recall the notion of the horizontal and the vertical differential [16].

Our framework is a fibered manifold $\pi\colon \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

For $r \geqslant 0$ we are concerned with the r-jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^r_s\colon J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geqslant s$, $\pi^r\colon J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^r_{r-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \ldots$ run from 1 to n and they label base coordinates, while Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i) the local bases of vector fields and 1-forms on $\boldsymbol{Y}$ induced by an adapted chart, respectively.

We denote multi-indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $0 \leqslant \alpha_\mu$, $\mu = 1, \ldots, n$; by an abuse of notation, we denote by λ the multi-index such that $\alpha_\mu = 0$ if $\mu \neq \lambda$, $\alpha_\mu = 1$ if $\mu = \lambda$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \ldots + \alpha_n$ and $\boldsymbol{\alpha}! := \alpha_1! \ldots \alpha_n!$.

The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\alpha}})$, with $0 \leqslant |\boldsymbol{\alpha}| \leqslant r$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

In the theory of variational sequences a fundamental role is played by the *contact maps* on jet spaces (see [3, 11, 12, 14]). Namely, for $r \geqslant 1$, we consider the natural complementary fibered morphisms over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$Д\colon J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}, \qquad \vartheta\colon J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y},$$

with coordinate expressions, for $0 \leqslant |\boldsymbol{\alpha}| \leqslant r-1$, given by

$$Д = d^\lambda \otimes Д_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda}\partial^{\boldsymbol{\alpha}}_j), \quad \vartheta = \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda}d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j.$$

We have

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \Big(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X}\Big) \oplus \operatorname{im}\vartheta^*_r, \tag{1}$$

where $\vartheta^*_r\colon J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y} \to J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y}$.

If $f\colon J_r\boldsymbol{Y} \to \mathbb{R}$ is a function, then we set $D_\lambda f := (Д)_\lambda f$, $D_{\boldsymbol{\alpha}+\lambda} f := D_\lambda D_{\boldsymbol{\alpha}} f$, where D_λ is the standard *formal derivative*. Given a vector field $Z\colon J_r\boldsymbol{Y} \to TJ_r\boldsymbol{Y}$, the splitting 1 yields $Z \circ \pi^{r+1}_r = Z_H + Z_V$ where, if $Z = Z^\gamma\partial_\gamma + Z^i_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}}_i$, then we have $Z_H = Z^\gamma D_\gamma$ and $Z_V = (Z^i_{\boldsymbol{\alpha}} - y^i_{\boldsymbol{\alpha}+\gamma}Z^\gamma)\partial^{\boldsymbol{\alpha}}_i$.

The splitting 1 induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$, where d_H and d_V are defined to be the *horizontal* and the *vertical differential.* The action of d_H and d_V on functions and 1-forms on $J_r\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, e.g., [24] for more details).

A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair $(\Xi, \bar{\Xi})$, where $\Xi\colon \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\bar{\Xi}\colon \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields and Ξ is a fibered morphism over $\bar{\Xi}$. The coordinate expression of a projectable vector field is then $\Xi = \Xi^\lambda \partial_\lambda + \Xi^i \partial_i$, $\bar{\Xi} = \Xi^\lambda \partial_\lambda$, where Ξ^λ will depend only on coordinates on $\boldsymbol{X}$. If there is no danger of confusion, we will denote a projectable vector field $(\Xi, \bar{\Xi})$ simply by Ξ.

A projectable vector field $(\Xi, \bar{\Xi})$ can be conveniently prolonged to a projectable vector field $(j_r\Xi, \bar{\Xi})$, whose coordinate expression turns out to be

$$j_r\Xi = \Xi^\lambda \partial_\lambda + \Big(D_{\boldsymbol{\alpha}}\Xi^i - \sum_{\boldsymbol{\beta}+\boldsymbol{\gamma}=\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\boldsymbol{\gamma}!} D_{\boldsymbol{\beta}}\Xi^\mu y^i_{\boldsymbol{\gamma}+\mu}\Big)\partial^{\boldsymbol{\alpha}}_i,$$

where $\boldsymbol{\beta} \neq 0$ and $0 \leqslant |\boldsymbol{\alpha}| \leqslant r$ (see [10, 14, 16]); in particular, we have the expressions $(j_r\Xi)_H = \Xi^\lambda D_\lambda$, $(j_r\Xi)_V = D_{\boldsymbol{\alpha}}(\Xi_V)^i \partial^{\boldsymbol{\alpha}}_i$ with $(\Xi_V)^i = \Xi^i - y^i_\lambda \Xi^\lambda$. From now on, by an abuse of notation, we will drop the parentheses in $(j_r\Xi)_H$ and $(j_r\Xi)_V$ and write simply $j_r\Xi_H$ and $j_r\Xi_V$.

2.2. Variational sequences.

We will be now concerned with some distinguished sheaves of forms on jet spaces [3, 11, 12, 16, 24]. Notice that we will consider sheaves on $J_r\boldsymbol{Y}$ with respect to the topology generated by open sets of the kind $(\pi^r_0)^{-1}(\boldsymbol{U})$, with $\boldsymbol{U} \subset \boldsymbol{Y}$ open in $\boldsymbol{Y}$. This is due to the topological triviality of the fibre of $J_r\boldsymbol{Y} \to \boldsymbol{Y}$.

i. For $r \geqslant 0$ we consider the standard sheaves $\overset{p}{\Lambda}_r$ of p-forms on $J_r\boldsymbol{Y}$.

ii. For $0 \leqslant s \leqslant r$ we consider the sheaves $\overset{p}{\mathcal{H}}_{(r,s)}$ and $\overset{p}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibered morphisms over π^r_s and π^r of the type $\alpha\colon J_r\boldsymbol{Y} \to \overset{p}{\bigwedge} T^*J_s\boldsymbol{Y}$ and $\beta\colon J_r\boldsymbol{Y} \to \overset{p}{\bigwedge} T^*\boldsymbol{X}$, respectively.

iii. For $0 \leqslant s < r$ we consider the subsheaf $\overset{p}{\mathcal{C}}_{(r,s)} \subset \overset{p}{\mathcal{H}}_{(r,s)}$ of *contact forms*, i.e. of sections $\alpha \in \overset{p}{\mathcal{H}}_{(r,s)}$ with values into $\overset{p}{\bigwedge}(\operatorname{im}\vartheta^*_{s+1})$. We have the distinguished subsheaf $\overset{p}{\mathcal{C}}_r \subset \overset{p}{\mathcal{C}}_{(r+1,r)}$ of local fibered morphisms $\alpha \in \overset{p}{\mathcal{C}}_{(r+1,r)}$ such that $\alpha = \overset{p}{\bigwedge}\vartheta^*_{r+1} \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} \overset{p}{\bigwedge} V^*J_r\boldsymbol{Y} \to J_{r+1}\boldsymbol{Y}$ which projects down onto $J_r\boldsymbol{Y}$.

According to [24], the fibered splitting 1 yields the sheaf splitting $\overset{p}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_{(r+1,r)} \wedge \overset{t}{\mathcal{H}}_{r+1}$, which restricts to the inclusion $\overset{p}{\Lambda}_r \subset \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_r \wedge \overset{t}{\mathcal{H}}{}^h_{r+1}$, where $\overset{p}{\mathcal{H}}{}^h_{r+1} := h(\overset{p}{\Lambda}_r)$ for $0 < p \leqslant n$ and h is defined to be the restriction to $\overset{p}{\Lambda}_r$ of the projection of the above splitting onto the non-trivial summand with the highest value of t. We also define the map $v := \operatorname{id} - h$.

We recall now the theory of variational sequences on finite order jet spaces, as it was developed by Krupka in [11].

By an abuse of notation, let us denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$ (see [26]). We set $\overset{*}{\Theta}_r := \ker h + d\ker h$.

In [11] it is proved that the following diagram is commutative and that its rows and columns are exact:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_r \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \overset{I}{\Theta}_r \arrow[r, "d"] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \overset{I}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r \arrow[r, "\mathcal{E}_2"] \arrow[d] & \cdots \arrow[r, "\mathcal{E}_{I-1}"] & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r \arrow[r, "\mathcal{E}_I"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & &
\end{tikzcd}
$$

Definition 2.1. The top row of the above diagram is said to be the *contact sequence*, and the bottom row is said to be the r-th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$.

Notice that, in general, the highest integer I depends on the dimension of the fibers of $J_r\boldsymbol{Y} \to \boldsymbol{X}$.

Remark 2.2. If $0 \leqslant p \leqslant n$, then $d\ker h \subset \ker h$, and $\alpha \in \ker h$ if and only if $(j_r\sigma)^*\alpha = 0$ for every section $\sigma\colon \boldsymbol{X} \to \boldsymbol{Y}$; this partly shows the relation of $\overset{p}{\Theta}_r$ to the Calculus of Variations [3, 6, 10, 19].

2.3. Representation of the variational sequence.

Isomorphisms of the quotient sheaves with suitable sheaves of forms can be given by means of decomposition formulae.

The sheaf morphism h yields the isomorphisms [23, 24]:

$$I_p\colon\ \overset{p}{\Lambda}_r/\overset{p}{\Theta}_r \to \overset{p}{\mathcal{H}}{}^h_{r+1}\colon\ [\alpha] \mapsto h(\alpha), \qquad 0 \leqslant p \leqslant n. \tag{2}$$

Let $\beta \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. Then there is a unique pair of sheaf morphisms [3, 6, 9, 10, 21, 22]

$$E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1}, \qquad F_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1}, \tag{3}$$

such that $(\pi^{2r+1}_{r+1})^*\alpha = E_\alpha - F_\alpha$ and F_α is locally of the form $F_\alpha = d_H p_\alpha$ with $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r}$.

The above formula yields the isomorphism

$$(4) \qquad I_{n+1}\colon \overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r \to \overset{n+1}{\mathcal{V}}_r\colon [\alpha] \mapsto E_{h(\alpha)}.$$

See [24] for the expression of $\overset{n+1}{\mathcal{V}}_r$.

Let $\eta \in \overset{n+1}{\mathcal{V}}_r$. It has been proved [24] that there exists $\tilde{H}_{d\eta} \in \overset{1}{\mathcal{C}}_{(4r+1,2r+1)} \otimes \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1}$ such that, for all vertical vector fields $\Sigma\colon \boldsymbol{Y} \to V\boldsymbol{Y}$, we have $E_{\widehat{d\eta}} = \tilde{H}_{d\eta}(j_{2r+1}\Sigma)$, where $\widehat{d\eta} := j_{2r+1}\Sigma \rfloor d\eta$. Hence there is a unique pair of sheaf morphisms

$$(5) \qquad H_{d\eta} \in \overset{1}{\mathcal{C}}_{(4r+1,2r+1)} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1}, \quad G_{d\eta} \in \overset{2}{\mathcal{C}}_{(4r+1,2r+1)} \wedge \overset{n}{\mathcal{H}}_{4r+1},$$

such that $(\pi^{4r+1}_{2r+1})^*(d\eta) = H_{d\eta} - G_{d\eta}$ and $H_{d\eta} = \frac{1}{2}\, A(\tilde{H}_{d\eta})$, where A is the skew-symmetrisation map. Moreover, $G_{d\eta}$ is locally of the type $G_{d\eta} = d_H q_{d\eta}$, where $q_{d\eta} \in \overset{2}{\mathcal{C}}_{4r} \wedge \overset{n-1}{\mathcal{H}}_{4r}$. We recall that $\tilde{H}_{d\eta} = 0$ if and only if $H_{d\eta} = 0$. Remark that, in general, the sections p_α and $q_{d\eta}$ are not uniquely characterized (see, e.g. [3, 9, 24]).

Moreover, we have the isomorphism $I_{n+2}\colon \mathcal{E}_{n+1}(\overset{n+1}{\mathcal{V}}_r) \to \overset{n+2}{\mathcal{V}}_r$, where $\overset{n+2}{\mathcal{V}}_r$ is the image of the injective morphism $\mathcal{E}_{n+1}(\overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r) \to \overset{n+2}{\mathcal{V}}_r\colon [d\alpha] \mapsto H_{dE_{h(\alpha)}}$.

By means of the above isomorphisms I_p, $0 \leqslant p \leqslant n+2$, we obtain a representation of the 'short' variational sequence [24], namely

$$0 \longrightarrow \mathbb{R} \longrightarrow \overset{0}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_0} \overset{1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_1} \cdots \xrightarrow{\mathcal{E}_n} \overset{n+1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_{n+1}} \overset{n+2}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_{n+2}} 0.$$

Notice that the morphisms $\mathcal{E}_p$ can be read through the above isomorphisms. In particular, $\mathcal{E}_p(h(\alpha)) = h(d\alpha)$ if $0 \leqslant p \leqslant n-1$.

We can interpret the above sequence in terms of the Calculus of Variations (see [3, 6, 9, 16, 24]).

A section $\lambda \in \overset{n}{\mathcal{V}}_r$ is just a Lagrangian of order $(r+1)$ of the standard literature; $\mathcal{E}_n(\lambda) \in \overset{n+1}{\mathcal{V}}_r$ is the standard higher order Euler-Lagrange morphism associated with λ. Due to the exactness of the above sequence, if λ is variationally trivial (i.e., $\mathcal{E}_n(\lambda) = 0$ holds), then (locally) $\lambda = d_H\varepsilon$, with $\varepsilon \in \overset{n-1}{\mathcal{V}}_r$.

Let $\alpha \in \overset{n+1}{\Lambda}_r$, i.e. $h(\alpha) \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}^h_{r+1}$. We call $E_{h(\alpha)} \in \overset{n+1}{\mathcal{V}}_r$ an *Euler-Lagrange type morphism*; it is defined on $J_{2r+1}\boldsymbol{Y}$. We say $p_{h(\alpha)}$ is a *momentum* associated with $E_{h(\alpha)}$.

Let $\eta \in \overset{n+1}{\mathcal{V}}_r$. We call $\mathcal{E}_{n+1}(\eta) = H_{d\eta}$ the *Helmholtz morphism* associated with η; it is defined on $J_{4r+1}\boldsymbol{Y}$. Again, the exactness of the sequence implies that, if η is locally variational (i.e. , $\mathcal{E}_{n+1}(\eta) = H_{d\eta} = 0 = \tilde{H}_{d\eta}$ holds), then (locally) $\eta = \mathcal{E}_n(\lambda)$ with $\lambda \in \overset{n}{\mathcal{V}}_r$.

Remark 2.3. Let $s \leqslant r$. Then the inclusions $\overset{p}{\Lambda}_s \subset \overset{p}{\Lambda}_r$ and $\overset{p}{\Theta}_s \subset \overset{p}{\Theta}_r$ yield the injective sheaf morphisms (see [11]) $\chi^r_s\colon \big(\overset{p}{\Lambda}_s/\overset{p}{\Theta}_s\big) \to \big(\overset{p}{\Lambda}_r/\overset{p}{\Theta}_r\big)\colon [\alpha] \mapsto [\pi^{r*}_s\alpha]$, hence the inclusions

$$\kappa^r_s\colon \overset{p}{\mathcal{V}}_s \to \overset{p}{\mathcal{V}}_r \tag{6}$$

for $s \leqslant r$.

3. Variational Lie derivative

In this section we give a representation of the Lie derivative operator in the variational sequence. We consider a projectable vector field $(\Xi, \bar{\Xi})$ on $\boldsymbol{Y}$ and take into account the Lie derivative with respect to the prolongation $j_r\Xi$ of Ξ. In fact, such a prolonged vector field preserves the fiberings π^s_r, π^r; hence it preserves the splitting 1. So, we shall prove that the operator $L_{j_r\Xi}$ passes to the quotient, yielding an operator on the quotient sheaves of the sequence. Moreover, we give the expression of this operator on the representations of the finite order variational sequence.

3.1. Representation of Lie derivative.

Let $L_{j_r\Xi}$ be the Lie derivative operator with respect to the r-th prolongation $j_r\Xi$ of a projectable vector field $(\Xi, \bar{\Xi})$ on $\boldsymbol{Y}$.

As a consequence of earlier results due to Krupka [10], we show that the operator $L_{j_r\Xi}$ defines an operator $\overline{L}_{j_r\Xi}$ on the quotient spaces $\overset{p}{\Lambda}_r/\overset{p}{\Theta}_r$. First we prove a lemma.

Lemma 3.1. *We have*

$$\overline{L}_{j_r\Xi}\colon \overset{p}{\Lambda}_r/\overset{p}{\Theta}_r \to \overset{p}{\Lambda}_r/\overset{p}{\Theta}_r\colon [\alpha] \mapsto \overline{L}_{j_r\Xi}([\alpha]) = [L_{j_r\Xi}\alpha].$$

P r o o f. We have to prove that $L_{j_r\Xi}(\overset{p}{\Theta}_r) \subset \overset{p}{\Theta}_r$. We recall that $\overset{*}{\Theta}_r = \ker h + d\ker h$ and that $\alpha \in d\ker h$ if and only if, locally, $\alpha = d\beta$, where $\beta \in \ker h$. Moreover, it is evident that $L_{j_r\Xi}(d\ker h) = d(L_{j_r\Xi}\ker h)$. So, we have only to prove that $L_{j_r\Xi}(\ker h) \subset \ker h$. To see this, let us consider the inclusion

$$\ker h \subset \bigoplus_{t=0}^{q} \overset{p-t}{\mathcal{C}}_r \wedge \overset{t}{\mathcal{H}}{}^h_{r+1},$$

where $q = p - 1$ if $0 < p \leqslant n$, and $q = p - n - 1$ if $p > n$.

The above inclusion, together with the standard result [10, 16] $L_{j_r\Xi}(\overset{1}{\mathcal{C}}_r) \subset \overset{1}{\mathcal{C}}_r$, shows that $L_{j_r\Xi}(\ker h)$ has no components in $\overset{p}{\mathcal{H}}{}^h_{r+1}$ if $0 \leqslant p \leqslant n$, or in $\overset{p-n}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$ if $p > n$. $\square$

This allows us to prove

Proposition 3.2. *Let $[\alpha] \in \overset{p}{\Lambda}_r / \overset{p}{\Theta}_r$. The isomorphisms I_p enable us to represent the map $\overline{L}_{j_r\Xi}$ on $\overset{p}{\mathcal{V}}_r$ as follows:*

$$(7) \qquad \tilde{\mathcal{L}}_{j_r\Xi} \colon \overset{p}{\mathcal{V}}_r \to \overset{p}{\mathcal{V}}_r \colon I_p([\alpha]) \mapsto \tilde{\mathcal{L}}_{j_r\Xi}(I_p([\alpha])) := I_p(\overline{L}_{j_r\Xi}[\alpha]) = I_p([L_{j_r\Xi}\alpha]).$$

If we set $h(L_{j_r\Xi}\alpha) = \hat{\alpha}$, then we have

$$\begin{aligned} I_p([\alpha]) &= h(\alpha) \mapsto \hat{\alpha} \quad \textit{if} \quad 0 \leqslant p \leqslant n, \\ I_p([\alpha]) &= E_{h(\alpha)} \mapsto E_{\hat{\alpha}} \quad \textit{if} \quad p = n+1, \\ I_p([d\alpha]) &= H_{dE_{h(\alpha)}} \mapsto H_{dE_{\hat{\alpha}}} \quad \textit{if} \quad p = n+2. \end{aligned}$$

Definition 3.3. Let $(\Xi, \underline{\Xi})$ be a projectable vector field. We say that the map $\tilde{\mathcal{L}}_{j_r\Xi}$ defined in the above theorem is the *variational Lie derivative.*

3.2. Noether's theorems.

Variational Lie derivatives allow us to calculate infinitesimal symmetries of forms in the variational sequence. In particular, we are interested in symmetries of Lagrangians and Euler-Lagrange morphisms (up to horizontal forms or divergencies). However, the representation in the above theorem is not very useful in concrete cases; here, we provide formulae for calculating symmetries of such morphisms which generalize Noether's theorems.

The inclusions κ^s_r give rise to new representations of $\overline{L}_{j_r\Xi}$ on $\overset{p}{\mathcal{V}}_s$. We will distinguish between the following cases:

i. $0 \leqslant p \leqslant n - 1$.

ii. $p = n$, which is of great importance for the Calculus of Variations. In fact, the representation yields the classical Noether's theorem.

iii. $p = n + 1$, where the representation yields a formula for the symmetries of an Euler-Lagrange type morphism.

Let again $L_{j_r\Xi}$ be the Lie derivative operator with respect to the r-th prolongation $j_r\Xi$ of a projectable vector field $(\Xi, \underline{\Xi})$ on $\boldsymbol{Y}$. We start with a technical lemma.

Lemma 3.4. *Let $r \leqslant s$. We have*

$$\kappa_r^s \circ \tilde{\mathcal{L}}_{j_r\Xi} = \tilde{\mathcal{L}}_{j_s\Xi} \circ \kappa_r^s = I_p \circ \overline{L}_{j_r\Xi} \circ \chi_r^s \circ I_p^{-1}.$$

Proof. In fact we have $\tilde{\mathcal{L}}_{j_r\Xi} = I_p \circ \overline{L}_{j_r\Xi} \circ I_p^{-1}$, $\kappa_r^s = I_p \circ \chi_r^s \circ I_p^{-1}$. □

Then the following two results hold true (for $p \leqslant n$).

Proposition 3.5. *Let $0 \leqslant p \leqslant n-1$ and $I_p([\alpha]) = h(\alpha) \in \overset{p}{\mathcal{V}}_r$. Then we have*

$$\kappa_r^{r+3} \circ \tilde{\mathcal{L}}_{j_r\Xi}(h(\alpha)) = d_H(\bar{\Xi} \rfloor h(\alpha)) + \bar{\Xi} \rfloor d_H h(\alpha) + j_{r+2}\Xi_V \rfloor d_V h(\alpha).$$

Proof. The proof can be easily performed by means of the splitting 1, which yields $j_r\Xi \circ \pi_r^{r+1} = j_r\Xi_H + j_r\Xi_V$, $(\pi_r^{r+1})^*\alpha = h(\alpha) + v(\alpha)$ and $(\pi_r^{r+1})^* d = d_H + d_V$. We have

$$\begin{aligned}
\kappa_r^{r+3} \circ \tilde{\mathcal{L}}_{j_r\Xi}(h(\alpha)) &= h(L_{j_r\Xi}\alpha) \\
&= h((d_H + d_V)(j_{r+1}\Xi_H + j_{r+1}\Xi_V) \rfloor h(\alpha)) \\
&\quad + h((j_{r+2}\Xi_H + j_{r+2}\Xi_V) \rfloor (d_H + d_V) h(\alpha)) \\
&= d_H(j_{r+1}\Xi_H \rfloor h(\alpha)) + j_{r+2}\Xi_H \rfloor d_H h(\alpha) + j_{r+2}\Xi_V \rfloor d_V h(\alpha).
\end{aligned}$$

□

Theorem 3.6. (First Noether's theorem) *Let $p = n$ and $I_n([\alpha]) = h(\alpha) \in \overset{n}{\mathcal{V}}_r$. Then we have (locally)*

$$\kappa_r^{2r+1} \circ \tilde{\mathcal{L}}_{j_r\Xi}(h(\alpha)) = \Xi_V \rfloor \mathcal{E}(h(\alpha)) + d_H(j_r\Xi_V \rfloor p_{d_V h(\alpha)} + \bar{\Xi} \rfloor h(\alpha)).$$

Proof. We make use of the same techniques used to prove the above proposition, together with the decomposition 3. We have

$$\begin{aligned}
\kappa_r^{2r+1} \circ \tilde{\mathcal{L}}_{j_r\Xi}(h(\alpha)) &= h(L_{j_{r+1}\Xi} h(\alpha))) \\
&= d_H(j_{r+1}\Xi_H \rfloor h(\alpha)) + h(j_{r+2}\Xi_V \rfloor d_V h(\alpha)) \\
&= d_H(\bar{\Xi} \rfloor h(\alpha)) + h(j_{2r+1}\Xi_V \rfloor (E_{d_V h(\alpha)} + F_{d_V h(\alpha)})).
\end{aligned}$$

Since $F_{d_V h(\alpha)} = d_H p_{d_V h(\alpha)}$ at least locally, because of 3 we obtain (locally)

$$\kappa_r^{2r+1} \circ \tilde{\mathcal{L}}_{j_r\Xi}(h(\alpha)) = \Xi_V \rfloor \mathcal{E}(h(\alpha)) + d_H(j_r\Xi_V \rfloor p_{d_V h(\alpha)} + \bar{\Xi} \rfloor h(\alpha)).$$

□

Since we are also interested in the case $p = n+1$, we first prove another technical lemma.

Lemma 3.7. *Let $I_p([\alpha]) \in \overset{p}{\mathcal{V}}_r$ with $p > n$. Then we have (globally)*

$$\kappa_r^{r+3} \circ \tilde{\mathcal{L}}_{j_r\Xi} I_p([\alpha]) = [d_H(\bar{\Xi} \rfloor h(\alpha)) + d_V(j_{r+1}\Xi_V \rfloor h(\alpha)) + j_{r+2}\Xi_V \rfloor d_V h(\alpha)].$$

P r o o f. We can easily prove the lemma by making use of the decomposition 5 and of the isomorphisms I_p. We find

$$\begin{aligned}\kappa_r^{r+3} \circ \tilde{\mathcal{L}}_{j_r\Xi} I_p([\alpha]) &= [h(L_{j_r\Xi}\alpha)] \\ &= [h((d_H + d_V)(j_{r+1}\Xi_H + j_{r+1}\Xi_V) \rfloor h(\alpha)) \\ &\quad + h((j_{r+2}\Xi_H + j_{r+2}\Xi_V) \rfloor (d_H + d_V) h(\alpha))] \\ &= [d_H(\bar{\Xi} \rfloor h(\alpha)) + d_V(j_{r+1}\Xi_V \rfloor h(\alpha)) + j_{r+2}\Xi_V \rfloor d_V h(\alpha)],\end{aligned}$$

where we have used $d_H(j_{r+1}\Xi_H \rfloor h(\alpha)) = d_H(\bar{\Xi} \rfloor h(\alpha))$. □

Remark 3.8. As for the case $p = n+1$, we remark that an Euler-Lagrange type morphism can be given in two ways:

1. as $[\alpha] \in \overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r$, which yields the morphism $E_{h(\alpha)}$;
2. as $\eta \in \overset{n+1}{\mathcal{V}}_r$.

In principle, the two ways are equivalent. But in practical calculations it is very hard to find an $\alpha \in \overset{n+1}{\Lambda}_r$ such that $\eta = E_{h(\alpha)}$ holds.

Theorem 3.9. (Second Noether's theorem) *Let $p = n+1$ and $\alpha \in \overset{n+1}{\Lambda}_r$. Then we have*

$$\kappa_r^{4r+1} \circ \tilde{\mathcal{L}}_{j_r\Xi} I_{n+1}([\alpha]) = \mathcal{E}(j_r\Xi_V \rfloor h(\alpha)) + \tilde{H}_{dE_{h(\alpha)}}(j_{2r+1}\Xi_V).$$

P r o o f. By the above lemma we have

$$\begin{aligned}\kappa_r^{4r+1} \circ \tilde{\mathcal{L}}_{j_r\Xi} I_{n+1}([\alpha]) &= I_{n+1}([\mathcal{E}(j_r\Xi_V \rfloor h(\alpha)) + j_{r+2}\Xi_V \rfloor d_V h(\alpha)]) \\ &= [\mathcal{E}(j_r\Xi_V \rfloor h(\alpha)) + j_{2r+1}\Xi_V \rfloor d_V E_{h(\alpha)}] \\ &= \mathcal{E}(j_r\Xi_V \rfloor h(\alpha)) + \tilde{H}_{dE_{h(\alpha)}}(j_{2r+1}\Xi_V).\end{aligned}$$

□

Remark 3.10. Let $p = n+1$ and $I_{n+1}([\eta]) = \eta \in \overset{n+1}{\mathcal{V}}_r \subset \overset{n+1}{\mathcal{V}}_{2r+1}$. Then we have

$$\kappa_{2r+1}^{4r+1} \circ \tilde{\mathcal{L}}_{j_r\Xi} \eta = \mathcal{E}(\Xi_V \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi_V).$$

3.3. Noether's theorems and variational sequences.

Here we summarize the above results on symmetries in the finite order variational sequence.

Theorem 3.11. *Let* $(\Xi, \bar{\Xi})$ *be a projectable vector field on* $\boldsymbol{Y}$ *and* $r \leqslant s$. *We have the commutative diagram*

$$\begin{tikzcd}
\overset{p}{\mathcal{V}}_r \arrow[rr, "\mathcal{L}_{j_r\Xi}"] \arrow[dr, "\tilde{\mathcal{L}}_{j_r\Xi}"'] & & \overset{p}{\mathcal{V}}_s \\
& \overset{p}{\mathcal{V}}_r \arrow[ur, "\kappa^s_r"'] &
\end{tikzcd}$$

where $\mathcal{L}_{j_r\Xi}$ *is defined as follows:*

(1) *if* $0 \leqslant p \leqslant n-1$ *and* $\mu \in \overset{p}{\mathcal{V}}_r$, *then* $s = r+3$ *and*

$$\mathcal{L}_{j_r\Xi}(\mu) = d_H(\bar{\Xi} \rfloor \mu) + \bar{\Xi} \rfloor d_H\mu + j_{r+2}\Xi_V \rfloor d_V\mu;$$

(2) *if* $p = n$ *and* $\lambda \in \overset{n}{\mathcal{V}}_r$, *then* $s = 2r+1$ *and*

$$\mathcal{L}_{j_r\Xi}(\lambda) = \Xi_V \rfloor \mathcal{E}(\lambda) + d_H(j_r\Xi_V \rfloor p_{d_V\lambda} + \bar{\Xi} \rfloor \lambda);$$

(3) *if* $p = n+1$ *and* $\alpha \in \overset{n+1}{\Lambda}_r$, *then* $s = 4r+1$ *and*

$$\mathcal{L}_{j_r\Xi}(E_{h(\alpha)}) = \mathcal{E}(j_r\Xi_V \rfloor h(\alpha)) + \tilde{H}_{dE_{h(\alpha)}}(j_{2r+1}\Xi_V).$$

4. Relations with the Calculus of Variations

In this section we analyse some consequences of Noether's theorems. In particular, we find the relationship between abstract Noether's theorems and some standard concepts of the Calculus of Variations. The above theorems play an important role in mathematical and physical applications concerning "variationally relevant" symmetries of Lagrangians and Euler-Lagrange morphisms.

Definition 4.1. Let $(\Xi, \bar{\Xi})$ be a projectable vector field on $\boldsymbol{Y}$. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian and $\eta \in \overset{n+1}{\mathcal{V}}_r$ an Euler-Lagrange morphism. Then Ξ is called a *symmetry* of λ (of η) if $\mathcal{L}_{j_{r+1}\Xi}\, \lambda = 0$ (respectively, $\mathcal{L}_{j_{2r+1}\Xi}\, \eta = 0$).

Remark 4.2. Due to $\mathcal{E}\mathcal{L}_{j_r\Xi} = \mathcal{L}_{j_r\Xi}\mathcal{E}$, a symmetry of a Lagrangian λ is also a symmetry of its Euler-Lagrange morphism E_λ (but the converse is not true, see e.g. [10, 13, 18]).

Let now $\eta \in \overset{n+1}{\mathcal{V}}_r$ be an Euler-Lagrange morphism and let $\sigma\colon\ \boldsymbol{X} \to \boldsymbol{Y}$ be a section. We recall that σ is said to be *critical* if $\eta \circ j_{2r+1}\sigma = 0$.

Remark 4.3. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian and $(\Xi, \bar{\Xi})$ a symmetry of λ. Then, by theorem 3.6, i.e. the first Noether's theorem, we have

$$0 = \Xi_V \rfloor \mathcal{E}(\lambda) + d_H(j_r\Xi_V \rfloor p_{d_V\lambda} + \bar{\Xi} \rfloor \lambda).$$

Suppose that the section $\sigma\colon\ \boldsymbol{X} \to \boldsymbol{Y}$ fulfils $(j_{2r+1}\sigma)^*(\Xi_V \rfloor \mathcal{E}(\lambda)) = 0$. This condition holds for all critical sections, but it is not equivalent to the evolution equations since, in principle, it can hold also on some non-critical sections. Then the $(n-1)$-form $\varepsilon = j_r\Xi_V \rfloor p_{d_V\lambda} + \bar{\Xi} \rfloor \lambda$ of $\overset{n-1}{\mathcal{V}}_{2r}$ fulfils the equation

$$d((j_{2r}\sigma)^*(j_r\Xi_V \rfloor p_{d_V\lambda} + \bar{\Xi} \rfloor \lambda)) = 0. \tag{8}$$

If σ is a critical section the above equation (8) admits a physical interpretation as a conservation law along critical sections for the density associated with ε.

Definition 4.4. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian and Ξ a symmetry of λ. Then a sheaf morphism of the type

$$\varepsilon = (j_r\Xi_V \rfloor p_{d_V\lambda} + \bar{\Xi} \rfloor \lambda) \in \overset{n-1}{\mathcal{V}}_{2r}$$

is said to be a *conserved current.*

Remark 4.5. In general, a conserved current is not uniquely defined. In fact, it depends on the choice of $p_{d_V\lambda}$. Moreover, we could add to the conserved current any form $\beta \in \overset{n-1}{\mathcal{V}}_{2r}$ which is variationally closed, i.e. such that $\mathcal{E}_{n-1}(\beta) = 0$ holds. Furthermore, β is locally of the type $\beta = d_H\gamma$, where $\gamma \in \overset{n-2}{\mathcal{V}}_{2r+1}$.

In the sequel we give some preliminary examples for the first and second order Lagrangians making use of the coordinate expressions given in [24].

Remark 4.6. Let $r = 1$ and $\lambda \in \overset{n}{\mathcal{H}}_1 \subset \overset{n}{\mathcal{V}}_1$. Then we have the coordinate expression $\lambda = L\omega$, where $L \in \overset{0}{\Lambda}_1$. Hence we have the following expression for the conserved current:

$$\varepsilon = ((\Xi^i - y^i_\gamma \Xi^\gamma)\partial^\mu_i L + \Xi^\mu L)\omega_\mu.$$

Remark 4.7. Let $r = 2$ and $\lambda \in \overset{n}{\mathcal{H}}_2 \subset \overset{n}{\mathcal{V}}_2$. Then we have the coordinate expression $\lambda = L\omega$, where $L \in \overset{0}{\Lambda}_2$. Hence we have the following expression for the conserved current:

$$\varepsilon = ((\Xi^i - y^i_\gamma \Xi^\gamma)(\partial^\mu_i L - D_\nu \partial^{\mu\nu}_i L) + D_\nu(\Xi^i - y^i_\gamma \Xi^\gamma)\partial^{\mu\nu}_i L + \Xi^\mu L)\omega_\mu.$$

Similar considerations can be made for the Euler-Lagrange morphisms.

Remark 4.8. Let $\eta \in \overset{n+1}{\mathcal{V}}_r$ and let Ξ be a symmetry of η. Then, by Remark 3.10, we have

$$0 = \mathcal{E}(\Xi_V \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi_V).$$

Suppose that η is locally variational, i.e. $H_{d\eta} = \tilde{H}_{d\eta} = 0$; then we have

$$\mathcal{E}(\Xi_V \rfloor \eta) = 0. \tag{9}$$

This implies that $\Xi_V \rfloor \eta$ is variationally trivial. $\Xi_V \rfloor \eta$ is locally of the type $\Xi_V \rfloor \eta = d_H\beta$, where $\beta \in \overset{n-1}{\mathcal{V}}_{r+1}$.

Suppose that the section $\sigma\colon \boldsymbol{X} \to \boldsymbol{Y}$ fulfils $(j_{2r+1}\sigma)^*(\Xi_V \rfloor \eta) = 0$. Then we have $d((j_{2r}\sigma)^*\beta) = 0$ so that, as in the case of Lagrangians, if σ is a critical section, then β is conserved along σ.

Definition 4.9. Let $\eta \in \overset{n+1}{\mathcal{V}}_r$ be an Euler-Lagrange morphism and Ξ a symmetry of η. Then a sheaf morphism of the type $\beta \in \overset{n-1}{\mathcal{V}}_{r+1}$ fulfilling the conditions of the above remark is called a *conserved current.*

Remark 4.10. As in the case of Lagrangians, a conserved current for an Euler-Lagrange morphism is not uniquely defined. In fact, we could add to $\Xi_V \rfloor \eta$ any variationally trivial Lagrangian, obtaining different conserved currents. Moreover, such conserved currents are defined up to variationally trivial $(n-1)$-forms.

Remark 4.11. In general, it is difficult to find a conserved current β of the above type. A possible way to find β is the following one. If η is locally variational, it is possible to find a (local) Lagrangian λ such that $\mathcal{E}(\lambda) = \eta$. Then the conserved current is found with the same procedure as for Lagrangians. We notice that the problem of finding the Lagrangian (inverse problem) is rather difficult (see [11, 12, 17, 24]).

5. An example of physical relevance

In this section we will consider as an example of application the case of the gravitational field interacting with an external "matter" field [4]. We shall show how the formalism developed here enables us to obtain in a very straightforward way well known results about conserved quantities in General Relativity in presence of "matter".

Let $\boldsymbol{Y}$ be a *natural* bundle (see e.g. [7] for a review of naturality). The geometrical framework concerning *natural symmetries* will be developed in detail in a forthcoming paper [5]. Here it is enough to recall that the Lie derivative of a section of $\boldsymbol{Y}$ (see e.g. [4, 7]) can be locally written as

$$\mathcal{L}_\xi y^i = y^i_\gamma \xi^\gamma - \mathcal{Z}^{i\boldsymbol{\alpha}}_\gamma \xi^\gamma_{\boldsymbol{\alpha}},$$

where $\boldsymbol{\alpha}$ is a multi-index of length $|\boldsymbol{\alpha}| = k$ (the order of the natural bundle) and $\mathcal{Z}^{i\boldsymbol{\alpha}}_\lambda \in C^\infty(\boldsymbol{Y})$. Moreover, in this case $-\Xi_V$ is just the Lie derivative of sections of the bundle $\boldsymbol{Y}$.

The following coordinate expressions hold:

$$d_V\lambda = (d_V\lambda)^{\boldsymbol{\alpha}}_i \vartheta^i_{\boldsymbol{\alpha}} \wedge \omega, \qquad E_{d_V\lambda} = \mathcal{E}(\lambda)_i \vartheta^i \wedge \omega, \qquad p_{d_V\lambda} = p(\lambda)^{\boldsymbol{\alpha}\mu}_i \vartheta^i_{\boldsymbol{\alpha}} \wedge \omega_\mu.$$

It is known (see e.g. [9]) that the decomposition formula 3 when applied to $d_V\lambda$ gives

$$(d_V\lambda)^{\boldsymbol{\alpha}}_i = p(\lambda)^{\boldsymbol{\beta}\mu}_i \qquad \boldsymbol{\beta} + \mu = \boldsymbol{\alpha}, |\boldsymbol{\alpha}| = r, \tag{10}$$

$$(d_V\lambda)^{\boldsymbol{\alpha}}_i = p(\lambda)^{\boldsymbol{\beta}\mu}_i + D_\nu p(\lambda)^{\boldsymbol{\alpha}\nu}_i \qquad \boldsymbol{\beta} + \mu = \boldsymbol{\alpha}, |\boldsymbol{\alpha}| = r-1, \tag{11}$$

$$(d_V\lambda)^{\boldsymbol{\alpha}}_i = \mathcal{E}(\lambda)^{\boldsymbol{\alpha}}_i + D_\nu p(\lambda)^{\boldsymbol{\alpha}\nu}_i \qquad |\boldsymbol{\alpha}| = 0. \tag{12}$$

Furthermore, $\mathcal{E}(\lambda)_i = \sum_{|\boldsymbol{\alpha}| \leqslant r} (-1)^{|\boldsymbol{\alpha}|} D_{\boldsymbol{\alpha}} (d_V\lambda)^{\boldsymbol{\alpha}}_i$.

Let us now take $\boldsymbol{Y} = \mathrm{Lor}(\boldsymbol{X}) \underset{\boldsymbol{X}}{\times} \boldsymbol{F}$, where $\mathrm{Lor}(\boldsymbol{X})$ is the bundle of Lorentzian metrics over the space-time $\boldsymbol{X}$ and $\boldsymbol{F}$ is a natural bundle of "matter" fields [4]. Let us consider the *natural* Lagrangian λ defined on the bundle $J_2\boldsymbol{Y}$:

$$\lambda = \lambda(g^{\mu\nu}, \gamma^\alpha_{\mu\nu}, R_{\mu\nu}, \varphi^\Lambda, \varphi^\Lambda_\mu) \tag{13}$$

$$= \lambda_H(g^{\mu\nu}, R_{\mu\nu}) + \lambda_M(g^{\mu\nu}, \gamma^\alpha_{\mu\nu}, \varphi^\Lambda, \varphi^\Lambda_\mu) \tag{14}$$

$$= \lambda_H(j_2 g) + \lambda_M(j_1 g; j_1\varphi), \tag{15}$$

where $\lambda_H = -\frac{1}{2\kappa}\sqrt{g} g^{\alpha\beta} R_{\alpha\beta}$ is the Hilbert Lagrangian, $R_{\alpha\beta}$ is the (formal) Ricci tensor of the metric g given by $R_{\alpha\beta} := R^\mu_{\alpha\mu\beta} = D_\mu \gamma^\mu_{\alpha\beta} - D_\beta \gamma^\mu_{\alpha\mu} + \gamma^\mu_{\nu\mu}\gamma^\nu_{\alpha\beta} - \gamma^\mu_{\nu\beta}\gamma^\nu_{\alpha\mu}$, with $\gamma^{\mu}_{\nu\beta} = \frac{1}{2}g^{\mu\alpha}(D_{\nu}g_{\beta\alpha} - D_{\alpha}g_{\nu\beta} + D_{\beta}g_{\alpha\nu})$ the (formal) Levi-Civita connection of g, $\sqrt{g} = \sqrt{|\det(g^{\mu\nu})|}$, κ is a constant and $\lambda_M(j_1 g; j_1\varphi)$ is the "matter" Lagrangian describing the dynamics of "matter" fields φ interacting with the gravitational field. Then the index i stands for the set of indices $\{\mu\nu, \Lambda\}$ with $\mu, \nu = 1, \dots, \dim \boldsymbol{X}$ and $\Lambda = 1, \dots, \dim \boldsymbol{F}$.

By a direct computational method, we can compare Equations 10–12 with results already obtained e.g. in [4]. In fact, let us make the substitutions

$$\begin{aligned}
(d_V\lambda)_{\Lambda} &= f_{\Lambda} + p^{\mu}_{\Theta}\gamma^{\alpha}_{\beta\mu}\partial_{\Lambda}\mathcal{Z}^{\Theta\beta}_{\alpha},\\
(d_V\lambda)^{\mu}_{\Lambda} &= p(\lambda)^{\mu}_{\Lambda} = p^{\mu}_{\Lambda},\\
(d_V\lambda)^{\alpha}_{\mu\beta} &= \frac{1}{2}(p_{\Lambda\mu}\mathcal{Z}^{\Lambda\alpha}_{\beta} + p_{\Lambda\beta}\mathcal{Z}^{\Lambda\alpha}_{\mu}) \equiv A^{\alpha}_{\beta\mu},\\
(d_V\lambda)_{\alpha\beta} &= -\frac{1}{2\kappa}\sqrt{g}g_{\alpha\beta} = p(\lambda)_{\alpha\beta},\\
(d_V\lambda)_{\mu\nu} &= \frac{1}{2}t_{\mu\nu} - \frac{1}{2\kappa}\sqrt{g}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big),
\end{aligned}$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar curvature and we set $p^{\mathbf{0}\nu}_i := p^{\nu}_i$ for the convenience of notation.

It is easy to verify that equation (8) with the above mentioned substitutions gives

$$\begin{aligned}
\varepsilon^{\sigma} = {} & -\frac{1}{2\kappa}\sqrt{g}g^{\alpha\beta}\pounds_{\xi}u^{\sigma}_{\alpha\beta} + T^{\sigma}_{\mu\nu}\pounds_{\xi}g^{\mu\nu} + p^{\sigma}_{\Lambda}\pounds_{\xi}\varphi^{\Lambda} - \xi^{\sigma}\lambda\\
& + \nabla_{\mu}[\frac{1}{2k}\sqrt{g}(\nabla^{\sigma}\xi^{\mu} - \nabla^{\mu}\xi^{\sigma})]\\
& + E^{\sigma}_{\nu}\xi^{\nu} + E^{\sigma\mu}_{\nu}\nabla_{\mu}\xi^{\nu} - T^{\sigma}_{\nu}\xi^{\nu} + T^{\sigma\mu}_{\nu}\nabla_{\mu}\xi^{\nu} + (d_V\lambda)^{\sigma}_{\nu}\xi^{\nu}.
\end{aligned}$$

Here $u^{\sigma}_{\alpha\beta} = \gamma^{\sigma}_{\alpha\beta} - \frac{1}{2}[\delta^{\sigma}_{\alpha}\gamma^{\nu}_{\nu\beta} + \delta^{\sigma}_{\beta}\gamma^{\nu}_{\nu\alpha}]$, $\nabla_{\nu}\varphi^{\Lambda} = D_{\nu}\varphi^{\Lambda} + \mathcal{Z}^{\Lambda\beta}_{\alpha}(\varphi)\gamma^{\alpha}_{\beta\nu}$ and $T^{\sigma}_{\mu\nu}$ is defined in terms of p^{σ}_{Λ} by means of 11 and 12 in the following way:

$$\begin{aligned}
2T^{\sigma}_{\mu\nu} &= -2[p^{\sigma}_{\Lambda}\mathcal{Z}^{\Lambda}_{(\mu\nu)} + p_{\Lambda(\mu}\mathcal{Z}^{\Lambda\sigma}_{\nu)}] - [p_{\Lambda\nu}\mathcal{Z}^{\Lambda\sigma}_{\mu} + p_{\Lambda\mu}\mathcal{Z}^{\Lambda\sigma}_{\nu}]\\
&= -2A^{\sigma}_{(\mu\nu)} - A^{\sigma}_{\nu\mu} = p(\lambda)^{\sigma}_{\mu\nu}.
\end{aligned}$$

The tensor density $T^{\sigma}_{\mu\nu}$ is related by 12 to the so-called *Hilbert stress tensor* density $T_{\mu\nu} = t_{\mu\nu} - \nabla_{\sigma}T^{\sigma}_{\mu\nu}$ [4]. Furthermore, the tensorial coefficients E^{σ}_{ν} and $E^{\sigma\mu}_{\nu}$ are defined by

$$E^{\sigma}_{\nu} = (d_V\lambda_M)^{\sigma}_{\Lambda}\nabla_{\nu}\varphi^{\Lambda} - \delta^{\sigma}_{\nu}\lambda_M, \quad E^{\sigma\mu}_{\nu} = -(d_V\lambda_M)^{\sigma}_{\Lambda}\mathcal{Z}^{\Lambda\mu}_{\nu}(\varphi),$$

and are called *energy-momentum tensors* of the theory.

Remark 5.1. This approach shows clearly that the density $T^{\sigma}_{\mu\nu}$ expresses the coupling of the matter field with the derivatives of the metric, because it defines the relationship between the momentum $p(\lambda)^{\sigma}_{\mu\nu}$ "associated" to the gravitational Lagrangian and the momentum $p(\lambda)^{\sigma}_{\Lambda}$ "associated" to the matter Lagrangian.

6. Conclusions

The introduction of the variational Lie derivative enables us to collect a wide range of concepts and coordinate formulae. Namely, we obtain a natural framework from the idea that Lie derivatives with respect to prolonged vector fields preserve the contact subsequence.

The representation of the variational sequence yields explicit formulae for the variational Lie derivative. Moreover, we obtain expressions for the conserved currents in a straightforward way. Furthermore, our approach enables us to know which is the degree of arbitrariness [19] when we deal with conserved currents (see Remarks 4.5 and 4.10).

Making use of this natural framework, in a forthcoming paper [5] we shall give a geometrical interpretation of the superpotentials in natural field theories (see [4] and references quoted therein). An extension to gauge-natural field theories will be considered in [2].

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