Andrzej Walendziak

Weak congruences of an algebra with the CEP and the WCIP

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 1, 117–127

Persistent URL: http://dml.cz/dmlcz/127706

WEAK CONGRUENCES OF AN ALGEBRA WITH THE CEP AND THE WCIP

Andrzej Walendziak, Warszawa

(Received February 4, 1999)

Abstract. Here we consider the weak congruence lattice $C_W(A)$ of an algebra A with the congruence extension property (the CEP for short) and the weak congruence intersection property (briefly the WCIP). In the first section we give necessary and sufficient conditions for the semimodularity of that lattice. In the second part we characterize algebras whose weak congruences form complemented lattices.

Keywords: weak congruence, CEP, WCIP, semimodular lattice, complemented lattice

MSC 2000: 08A30, 06C10, 06C15

0.

For a given algebra A, let $\mathrm{Con}(A)$ and $\mathrm{Sub}(A)$ be the lattices of all congruences and subalgebras of A, respectively. $C_W(A)$ is the lattice of all weak congruences on A, i.e., symmetric and transitive binary relations on A, satisfying the usual substitution property extended to the nullary operations of A. It is obvious that $C_W(A)$ coincides with the set of all congruences on all subalgebras of A, i.e., $\alpha \in C_W(A)$ if and only if there is a (unique) $B \in \mathrm{Sub}(A)$ such that $\alpha \in \mathrm{Con}(B)$. It was proved in [9] that $C_W(A)$ is an algebraic lattice. We know (see [1], p. 14) that every algebraic lattice is weakly atomic (a lattice L is weakly atomic, if for every pair of elements $a, b \in L$ with $a < b$, there exist elements $u, v \in L$ such that $a \leqslant u < v \leqslant b$ and $\{x \in L\colon\ u < x < v\} = \emptyset$).

The diagonal relation $\Delta = \{(a,a)\colon\ a \in A\}$ is a codistributive element in $C_W(A)$, i.e., for all $\alpha, \beta \in C_W(A)$,

$$(\alpha \vee \beta) \wedge \Delta = (\alpha \wedge \Delta) \vee (\beta \wedge \Delta).$$

It is well known that $\mathrm{Con}(A) = [\Delta)$ (a filter in $C_W(A)$ generated by Δ) and $\mathrm{Sub}(A)$ is isomorphic with $(\Delta]$ where $(\Delta]$ is an ideal in $C_W(A)$ generated by Δ.

For $B \in \mathrm{Sub}(A)$, let

$$\Delta_B = B^2 \wedge \Delta.$$

We check at once that if $B, C \in \mathrm{Sub}(A)$, then

$$\Delta_{B\wedge C} = \Delta_B \wedge \Delta_C \ \text{ and } \ \Delta_{B\vee C} = \Delta_B \vee \Delta_C. \tag{1}$$

If $\alpha \in C_W(A)$ and $\alpha \leqslant B^2$, then we put

$$\alpha_B = \alpha \vee \Delta_B.$$

An algebra A has the CEP if every congruence on a subalgebra of A is a restriction of a congruence on A. We recall the following characterizations of the CEP.

Proposition 1 (see [5], Lemma 8)**.** *The following statements are equivalent for an algebra A:*

(i) *A has the CEP.*

(ii) *If $\alpha, \beta \in \mathrm{Con}(B)$, $B \in \mathrm{Sub}(A)$, then $\alpha \vee \Delta = \beta \vee \Delta$ implies $\alpha = \beta$.*

(iii) *For all $\alpha, \beta \in C_W(A)$, $\alpha \vee (\Delta \wedge \beta) = (\alpha \vee \Delta) \wedge (\alpha \vee \beta)$.*

(iv) *If $B, C \in \mathrm{Sub}(A)$ $(B \leqslant C)$, then for $\alpha \in \mathrm{Con}(B)$, $\alpha = \alpha_C \wedge B^2$.*

A is said to have the WCIP ([4], [6] and [7]) if for all $\alpha \in C_W(A)$ and $\beta \in \mathrm{Con}(A)$

$$(\alpha \wedge \beta)_A = \alpha_A \wedge \beta.$$

It is easy to see that every simple algebra has the WCIP. Consequently, each field satisfies the WCIP. Fields also have the CEP (see [2]). We know that unary algebras (see [7]) and Hamiltonian groups (see [3]) belong to the class of all algebras which satisfy the CEP and the WCIP.

Proposition 2 (cf. [8], Theorem 5)**.** *An algebra A satisfies the CEP and the WCIP if and only if the mapping*

$$h_B \colon \ \beta \in \mathrm{Con}(A) \to \beta \vee \Delta$$

is an isomorphism from $\mathrm{Con}(B)$ onto $(B^2 \vee \Delta]$, where $(B^2 \vee \Delta]$ is an ideal in $C_W(A)$.

As a preparation we also need the following statement.

Proposition 3. *If an algebra A has the CEP, then it has the WCIP if and only if for every $\alpha \in \mathrm{Con}(B)$, $\beta \in \mathrm{Con}(C)$ ($B, C \in \mathrm{Sub}(A)$ and $B \leqslant C$),*

$$(\alpha \wedge \beta)_C = \alpha_C \wedge \beta. \tag{2}$$

P r o o f. Let A have the WCIP and let B and C be subalgebras of A such that $B \leqslant C$. Let $\alpha \in \mathrm{Con}(B)$ and $\beta \in \mathrm{Con}(C)$. By Proposition 1, $\beta = \beta_A \wedge C^2$. Hence $\alpha \wedge \beta = \alpha \wedge \beta_A$. Since A has the WCIP, we have

$$(\alpha \wedge \beta)_A = (\alpha \wedge \beta_A)_A = \alpha_A \wedge \beta_A.$$

Therefore,

$$(\alpha \wedge \beta)_A \wedge C^2 = \alpha_A \wedge \beta_A \wedge C^2. \tag{3}$$

Proposition 1 now gives

$$\alpha_C = \alpha_A \wedge C^2 \quad \text{and} \quad (\alpha \wedge \beta)_C = (\alpha \wedge \beta)_A \wedge C^2. \tag{4}$$

Applying (3) and (4) we obtain

$$(\alpha \wedge \beta)_C = (\alpha \wedge \beta)_A \wedge C^2 = \alpha_A \wedge \beta_A \wedge C^2 = (\alpha_A \wedge C^2) \wedge (\beta_A \wedge C^2) = \alpha_C \wedge \beta.$$

Thus (2) holds. The converse is obvious. □

1.

Let L be a lattice and let $a, b \in L$. We say that b covers a if $a < b$ and $\{x \in L\colon\ a < x < b\} = \emptyset$; in this case we write $a \prec b$. Recall that a lattice L is semimodular if whenever a and b are elements of L and $a \wedge b \prec a$ we have $b \prec a \vee b$.

Proposition 4. *Let A be an algebra such that for every $\alpha, \beta \in C_W(A)$, if $\alpha \wedge \Delta = \beta \wedge \Delta$, then*

$$\alpha \prec \beta \ \textit{implies}\ \alpha_A \prec \beta_A. \tag{5}$$

Then A has the CEP.

P r o o f. If A does not satisfy the *CEP*, then by Proposition 1 there are $\gamma, \delta \in \mathrm{Con}(B)$ ($B \in \mathrm{Sub}(A)$) such that $\gamma \neq \delta$ and $\gamma_A = \delta_A$. Without loss of generality we can assume that $\gamma < \gamma \vee \delta$. Since $C_W(A)$ is a weakly atomic lattice, there exist $\alpha, \beta \in C_W(A)$ with $\gamma \leqslant \alpha \prec \beta \leqslant \gamma \vee \delta$. We have

$$\gamma \wedge \Delta \leqslant \alpha \wedge \Delta \leqslant \beta \wedge \Delta \leqslant (\gamma \vee \delta) \wedge \Delta.$$

Since Δ is a codistributive element of $C_W(A)$, we obtain

$$(\gamma \vee \delta) \wedge \Delta = (\gamma \wedge \Delta) \vee (\delta \wedge \Delta) = \gamma \wedge \Delta.$$

Therefore, $\alpha \wedge \Delta = \beta \wedge \Delta$. From (5) it follows that $\alpha_A \prec \beta_A$. On the other hand,

$$\gamma \vee \Delta \leqslant \alpha \vee \Delta \leqslant \beta \vee \Delta \leqslant (\gamma \vee \delta) \vee \Delta = \gamma \vee \Delta,$$

and hence $\alpha_A \prec \beta_A$. This contradiction shows that A has the CEP. $\square$

Remark 1. The assumption of finite length of $C_W(A)$ can be dropped in Theorem 4 [7].

As a preparation we need the following lemma.

Lemma 1. *Let A be an algebra satisfying the CEP and let $B, C \in \mathrm{Sub}(A)$, $B \neq C$. If $\alpha \in \mathrm{Con}(A)$ and $\beta \in \mathrm{Con}(C)$, then $\alpha \prec \beta$ if and only if $B \prec C$ and $\beta = \alpha \vee \Delta_C$.*

Proof. Let $\alpha \prec \beta$. It is easily seen that $B < C$. Assume that there exists a subalgebra D such that

$$B < D < C. \tag{6}$$

From this we deduce that

$$\alpha \leqslant \alpha \vee \Delta_D \leqslant \alpha \vee \Delta_C \leqslant \beta.$$

If $\alpha = \alpha \vee \Delta_D$, then $\Delta_D \leqslant \alpha \leqslant B^2$, and therefore $D \leqslant B$, contrary to (6). Thus

$$\alpha \vee \Delta_D = \alpha \vee \Delta_C = \beta.$$

Hence $\Delta_C = \alpha \vee \Delta_D \leqslant D^2$. Consequently, $C \leqslant D$. This contradiction shows that

$$B \prec C \text{ and } \beta = \alpha \vee \Delta_C. \tag{7}$$

Now, let (7) hold. We prove that $\alpha \prec \beta$. On the contrary, suppose that there is a weak congruence $\gamma \in C_W(A)$ with $\alpha < \gamma < \beta$. Let $\gamma \in \mathrm{Con}(E)$ for some $E \in \mathrm{Sub}(A)$.

It is easy to check that $B \leqslant E \leqslant C$. Since $B \prec C$, we have $B = E$. Therefore, $\alpha, \gamma \in \mathrm{Con}(B)$ and hence

$$\alpha \vee (\Delta \wedge \gamma) = \alpha \vee \Delta_B = \alpha.$$

By Proposition 1, $\alpha \vee (\Delta \wedge \gamma) = (\alpha \vee \Delta) \wedge \gamma$. Then

$$\alpha = (\alpha \vee \Delta) \wedge \gamma.$$

Since $\gamma < \beta = \alpha \vee \Delta_C \leqslant \alpha \vee \Delta$, we see that $(\alpha \vee \Delta) \wedge \gamma = \gamma$. Thus $\alpha = \gamma$, contrary to our assumption that $\alpha < \gamma < \beta$. Consequently, $\alpha \prec \beta$. □

Theorem 1. *Let A be an algebra satisfying:*

(i) $\mathrm{Sub}(A)$ *is a semimodular lattice;*

(ii) $\mathrm{Con}(A)$ *is a semimodular lattice for every* $B \in \mathrm{Sub}(A)$*;*

(iii) *for all* $B, C \in \mathrm{Sub}(A)$, $B \leqslant C$, *if* $\alpha, \beta \in \mathrm{Con}(B)$ *with* $\alpha \prec \beta$, *then* $\alpha_C \prec \beta_C$.

Then the lattice $C_W(A)$ is semimodular.

P r o o f. Take $\alpha \in \mathrm{Con}(B)$, $\beta \in \mathrm{Con}(C)$ $(B, C \in \mathrm{Sub}(A))$ and assume that $\alpha \wedge \beta \prec \alpha$. We shall consider two cases.

Case 1. $B \leqslant C$.

It is clear that $\alpha \wedge \beta \in \mathrm{Con}(B)$. By assumption (iii),

$$(\alpha \wedge \beta)_C \prec \alpha_C. \tag{8}$$

Observe that $\alpha_C \nleqslant \beta$. Indeed, if $\alpha_C \leqslant \beta$, then $\alpha \vee \beta = \alpha_C \vee \beta = \beta$ and hence $\alpha \wedge \beta = \alpha$, a contradiction. We have

$$(\alpha \wedge \beta)_C \leqslant \alpha_C \wedge \beta < \alpha_C,$$

and from (8) it follows that

$$\alpha_C \wedge \beta \prec \alpha_C. \tag{9}$$

By (ii), $\mathrm{Con}(C)$ is a semimodular lattice. Since $\alpha_C, \beta \in \mathrm{Con}(C)$, (9) gives $\beta \prec \alpha_C \vee \beta = \alpha \vee \beta$.

Case 2. $B \nleqslant C$.

We have $\alpha \wedge \beta \in \mathrm{Con}(B \wedge C)$ and $\alpha \wedge \beta \prec \alpha$. By Proposition 4, A has the CEP. From Lemma 1 we conclude that $B \wedge C \prec B$ and

$$\alpha = (\alpha \wedge \beta) \vee \Delta_B. \tag{10}$$

By (i), $\mathrm{Sub}(A)$ is a semimodular lattice. Then

$$C \prec B \vee C.$$

Lemma 1 now shows that

$$\beta \prec \beta \vee \Delta_{B \vee C}.$$

Applying (1) and (10) we obtain

$$\beta \vee \Delta_{B\vee C} = \beta \vee \Delta_B \vee \Delta_C = \beta \vee \Delta_B = \beta \vee (\alpha \wedge \beta) \vee \Delta_B = \alpha \vee \beta.$$

Consequently, $\beta \prec \alpha \vee \beta$.

Thus $C_W(A)$ is a semimodular lattice. □

Lemma 2. *If A has the CEP and the WCIP, then condition* (iii) *of Theorem* 1 *is satisfied.*

Proof. Let $B, C \in \mathrm{Sub}(A)$, $B \leqslant C$ and let $\alpha, \beta \in \mathrm{Con}(B)$ such that $\alpha \prec \beta$. Suppose that there exists a weak congruence γ with $\alpha_C \leqslant \gamma \leqslant \beta_C$. Obviously, $\gamma \in \mathrm{Con}(C)$. From Proposition 1 it follows that

$$\alpha = \alpha_C \wedge B^2 \leqslant \gamma \wedge B^2 \leqslant \beta_C \wedge B^2 = \beta.$$

If $\alpha = \gamma \wedge B^2$, then by Proposition 3,

$$\alpha_C = (\gamma \wedge B^2)_C = \gamma \wedge (B^2)_C.$$

Since $\gamma \leqslant \beta_C \leqslant (B^2)_C$, we have $\gamma = \alpha_C$. Similarly, if $\beta = \gamma \wedge B^2$, then $\gamma = \beta_C$. Hence $\alpha_C \prec \beta_C$, and thus (iii) holds. □

Theorem 2. *If an algebra A satisfies the CEP and the WCIP, then $C_W(A)$ is semimodular if and only if the lattices* $\mathrm{Con}(A)$ *and* $\mathrm{Sub}(A)$ *are semimodular.*

Proof. Let $\mathrm{Con}(A)$ and $\mathrm{Sub}(A)$ be semimodular lattices. By Proposition 2, $\mathrm{Con}(B) \cong (B^2 \vee \Delta]$ for every $B \in \mathrm{Sub}(A)$. Theorefore $\mathrm{Con}(B)$ is a semimodular lattice. Thus conditions (i) and (ii) (given in Theorem 1) are satisfied. From Lemma 2 we obtain (iii). Now, Theorem 1 shows that the lattice $C_W(A)$ is semimodular.

The converse follows from the fact that $\mathrm{Con}(A)$ and $\mathrm{Sub}(A)$ are convex sublattices of $C_W(A)$. □

The following example shows that the assumption of the CEP cannot be dropped in the previous theorem.

Example 1. Let $G_1 = (\{a,b,c,d\}, \cdot)$ be the groupoid given by the Cayley table below, with constant a. G_1 has the lattice of weak congruences $C_W(G_1)$ shown in Figure 1.

$\cdot$	a	b	c	d
a	a	b	c	c
b	a	b	c	d
c	c	c	a	c
d	a	b	b	d

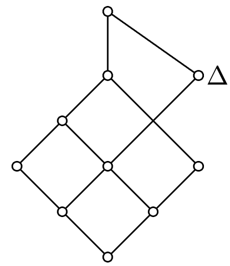

Fig. 1.

It is easy to see that the groupoid G_1 satisfies the WCIP but fails on the CEP. G_1 has distributive lattices of congruences and subgroupoids but $C_W(G_1)$ is not semimodular.

Remark. Symmetric group S_3 has a modular lattice of subgroups and a distributive one of normal subgroups. It satisfies the CEP, but fails on the WCIP (see Example 4 of [9]). The lattice $C_W(S_3)$ is not semimodular. This example shows that the assumption of the WCIP cannot be dropped in Theorem 2.

2.

In this part we give necessary and sufficient conditions under which the weak congruence lattice of an algebra is complemented. As a preparation we need the following lemma.

Lemma 3. *Let A satisfy the CEP and the WCIP and let B be a subalgebra of A such that $B^2 \vee \Delta = A^2$. Then*

$$\mathrm{Con}(B) \cong \mathrm{Con}(A)$$

under $h_B\colon\ \beta \in \mathrm{Con}(B) \to \beta \vee \Delta$.

P r o o f. Since A has the CEP and the WCIP, Proposition 2 shows that h_B is an isomorphism from $\mathrm{Con}(B)$ onto $(B^2 \vee \Delta]$. By assumption, $B^2 \vee \Delta = A^2$. Therefore $\mathrm{Con}(B) \cong \mathrm{Con}(A)$ under h_B. $\square$

Proposition 5. *Let A be an algebra satisfying:*

(i) *A has the CEP and the WCIP,*

(ii) *for every $B \in \mathrm{Sub}(A)$, $B^2 \vee \Delta = A^2$,*

(iii) $\mathrm{Con}(A)$ *is a complemented lattice,*

(iv) $\mathrm{Sub}(A)$ *is a complemented lattice.*

Then the lattice $C_W(A)$ is complemented.

P r o o f. Let α be a weak congruence of A. Then there exists $B \in \mathrm{Sub}(A)$ such that $\alpha \in \mathrm{Con}(B)$. By (iv), there is $C \in \mathrm{Sub}(A)$ with $B \vee C = A$ and $B \wedge C = M$, where M is the least subalgebra of A. Since A satisfies (i) and (ii) we can apply Lemma 3 and we get

$$\mathrm{Con}(B) \cong \mathrm{Con}(A).$$

We conclude from (iii) that $\mathrm{Con}(B)$ is a complemented lattice. Let β be a complement of α in $\mathrm{Con}(B)$. Lemma 3 now shows that

$$\mathrm{Con}(C) \cong \mathrm{Con}(A)$$

under $h_C\colon \gamma \in \mathrm{Con}(C) \to \gamma \vee \Delta$. From this we see that there is $\gamma \in \mathrm{Con}(C)$ such that $\gamma_A = \beta_A$. We shall prove that γ is a complement of α in $C_W(A)$. Indeed, we have

$$\alpha \vee \gamma \geqslant \Delta_B \vee \Delta_C = \text{ (by (1)) } \Delta_{B\vee C} = \Delta_A = \Delta,$$

and hence

$$\alpha \vee \gamma = \alpha \vee \gamma \vee \Delta = \alpha_A \vee \gamma_A = \alpha_A \vee \beta_A = (\alpha \vee \beta)_A = (B^2)_A = A^2.$$

Thus

$$\alpha \vee \gamma = A^2. \tag{11}$$

Moreover,

$$\Delta_M \leqslant \alpha \wedge \gamma \leqslant B^2 \wedge C^2 = (B \wedge C)^2 = M^2$$

and

$$\begin{aligned} \alpha \wedge \gamma &\leqslant \alpha_A \wedge \gamma_A = \alpha_A \wedge \beta_A \text{ (by the WCIP)} \\ &= (\alpha_A \wedge \beta)_A = (\alpha_A \wedge B^2 \wedge \beta)_A \text{ (by the CEP and Proposition 1)} \\ &= (\alpha \vee \beta)_A = (\Delta_M)_A = \Delta. \end{aligned}$$

Consequently, $\Delta_M \leqslant \alpha \wedge \gamma \leqslant M^2 \wedge \Delta = \Delta_M$. Then

$$\alpha \wedge \gamma = \Delta_M.$$

From this and (11) we obtain that γ is a complement of α in $C_W(A)$. Thus $C_W(A)$ is a complemented lattice. □

Proposition 6. *Let A be an algebra having the CEP and the WCIP. If $C_W(A)$ is a complemented lattice, then the conditions* (ii), (iii) *and* (iv) *(given in Proposition* 5*) are satisfied.*

P r o o f. Denote by M the least subalgebra of A. To prove (ii), let $D \in \mathrm{Sub}(A)$ and let α be a complement of Δ in $C_W(A)$. Then

$$\alpha \vee \Delta = A^2 \text{ and } \alpha \wedge \Delta = \Delta_M.$$

Observe that $\alpha \in \mathrm{Con}(M)$. Indeed, if

$$\Delta_C \leqslant \alpha \leqslant C^2$$

for some $C \in \mathrm{Sub}(A)$, then $\Delta_C = \Delta_C \wedge \Delta \leqslant \alpha \wedge \Delta = \Delta_M$. Consequently, $C = M$, i.e., $\alpha \in \mathrm{Con}(M)$. From this we have $\alpha \leqslant M^2$, and hence

$$A^2 = \alpha \vee \Delta \leqslant M^2 \vee \Delta \leqslant B^2 \vee \Delta \leqslant A^2.$$

Therefore,

$$B^2 \vee \Delta = A^2,$$

i.e., (ii) holds.

To prove (iii), let $\alpha \in \mathrm{Con}(A)$. Denote by β a complement of $\alpha \wedge M^2$ in $C_W(A)$. Then

$$(\alpha \wedge M^2) \vee \beta = A^2 \tag{12}$$

and

$$\alpha \wedge \beta \wedge M^2 = \Delta_M. \tag{13}$$

Since Δ is a codistributive element of $C_W(A)$, we have

$$\begin{aligned}\Delta = \Delta \wedge A^2 = \Delta \wedge [(\alpha \wedge M^2) \vee \beta] &= (\alpha \wedge M^2 \wedge \Delta) \vee (\beta \wedge \Delta)\\ &= \Delta_M \vee (\beta \wedge \Delta) = \beta \wedge \Delta.\end{aligned}$$

Therefore, $\Delta \leqslant \beta$, and hence $\beta \in \mathrm{Con}(A)$. From (12) we conclude that

$$\alpha \vee \beta = A^2.$$

Applying (13) and the WCIP we get

$$\Delta = \Delta_M \vee \Delta = (\alpha \wedge \beta \wedge M^2) \vee \Delta = (\alpha \wedge \beta) \wedge (M^2 \vee \Delta).$$

By (ii), $M^2 \vee \Delta = A^2$. Consequently,

$$\alpha \wedge \beta = \Delta.$$

From this and (14) we obtain that β is a complement of α in $C_W(A)$. Thus (iii) is satisfied.

To prove (iv), let B be a subalgebra of A. Denote by α a complement of Δ_B in $C_W(A)$, i.e.,

$$\alpha \vee \Delta_B = A^2 \text{ and } \alpha \wedge \Delta_B = \Delta_M.$$

Let C be an subalgebra of A such that $\alpha \in \mathrm{Con}(C)$. Since $B^2 \vee C^2 \geqslant \Delta_B \vee \alpha = A^2$, we have $B \vee C = A$. Applying (1) yields

$$\Delta_M = \alpha \wedge \Delta_B \geqslant \Delta_C \wedge \Delta_B = \Delta_{B \wedge C}.$$

Hence $B \wedge C = M$. Consequently, C is a complement of B in $\mathrm{Sub}(A)$. $\square$

Summarizing the results of this section we have the following theorem.

Theorem 3. *An algebra A with the CEP and the WCIP has a complemented lattice of weak congruences if and only if it satisfies conditions* (ii), (iii) *and* (iv) *from Proposition* 5.

Example 2. Let $G_2 = (\{a, b, c, d\}, \cdot)$ be the groupoid given by the Cayley table below, with constants a and b. The weak congruence lattice of G_2 is given in Fig. 2. This lattice is complemented. Moreover, it is a Boolean lattice.

$\cdot$	a	b	c	d
a	a	b	a	a
b	a	b	c	b
c	b	c	c	d
d	d	b	c	d

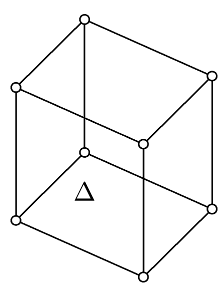

Fig. 2.

References

[1] *P. Crawley and R. P. Dilworth*: Algebraic Theory of Lattices. Prentice Hall, Englewood Cliffs, New Jersey, 1973.

[2] *E. W. Kiss, L. Marki, P. Pröhle and W. Tholen*: Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness and injectivity. Studia Sci. Math Hung. *18* (1983), 79–141.

[3] *B. Šešelja and G. Vojvodič*: A note on some lattice characterizations of Hamiltonian groups. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. *19* (1989), 179–184.

[4] *B. Šešelja and G. Vojvodič*: CEP and homomorphic images of algebras. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. *19* (1989), 75–80.

[5] *B. Šešelja and A. Tepavčevič*: Special elements of the lattice and lattice identities. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. *20* (1990), 21–29.

[6] *B. Šešelja and A. Tepavčevič*: Weak congruences and homomorphisms. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. *20* (1990), 61–69.

[7] *B. Šešelja and A. Tepavčevič*: On CEP and semimodularity in the lattice of weak congruences. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. *22* (1992), 95–106.

[8] *G. Vojvodič and B. Šešelja*: Subalgebras and congruences via diagonal relation. In: Algebra and Logic, Proc. of Sarajevo Conf. 1987, pp. 169–177.

[9] *G. Vojvodič and B. Šešelja*: On the lattice of weak congruence relations. Algebra Universalis *25* (1988), 121–130.

Author's address: Institute of Mathematics and Physics, University of Podlasie, 08110 Siedlce, Poland, e-mail: `imif@ap.siedlce.pl`.