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NEW PROOF OF A CHARACTERIZATION OF GEODETIC GRAPHS

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Abstract. In [3], the present author used a binary operation as a tool for characterizing geodetic graphs. In this paper a new proof of the main result of the paper cited above is presented. The new proof is shorter and simpler.

Keywords: geodetic graphs, shortest paths, binary operations

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By a graph we will mean a graph in the sense of [1], i.e. a finite undirected graph without loops or multiple edges. Let G be a graph (with vertex set $V(G)$ and edge set $E(G)$). Then G is said to be *geodetic* if it is connected and there exists exactly one shortest $u-v$ path for each ordered pair of $u, v \in V(G)$.

Let G be a geodetic graph. Following [3], we say that $*$ is the *proper operation* of G if $*$ is the binary operation on $V(G)$ defined as follows:

$$u * v = u \text{ if } u = v,$$
$$u * v \text{ is the second vertex on the shortest } u-v \text{ path if } u \neq v.$$

for all $u, v \in V(G)$. Thus, if $*$ is the proper operation of G, then $\{x, x*y\} \in E(G)$ for all ordered pairs of distinct $x, y \in V(G)$.

Let G be a graph. Following [3], we say that $*$ is a binary operation *associated with* G if $*$ is a binary operation on $V(G)$ and

$$E(G) = \{\{u, v\};\ u, v \in V(G), u \neq v, u * v = v \text{ and } v * u = u\}.$$

It is easy to see that if G is a geodetic graph, then the proper operation of G is associated with G.

The following theorem, which was proved in [3], gives a characterization of geodetic graphs and their proper operations:

Theorem. *Let G be a graph, and let $*$ be a binary operation associated with G. Put $U = V(G)$. Then G is geodetic and $*$ is the proper operation of G if and only if G is connected and $*$ satisfies the following Axioms* (A)–(D)*:*

(A) *if $u, v \in U$, then $(u * v) * u = u$;*

(B) *if $u, v \in U$, then $u = v$ or $(u * v) * v \neq u$;*

(C) *if $u, v \in U$, then $v * u = u$ or $u * (v * u) = u * v$;*

(D) *if $u, v, w \in U$ and $w * v = v$, then $u * v = u * w$ or $w * (u * v) = v$.*

As was shown in [3], the condition that G is connected cannot be omitted in this theorem.

The proof of this theorem given in [3] is rather long and complicated. In the present paper we will give a new proof. This proof (including the proofs of the lemmas) is shorter and simpler.

The following lemma was presented in [3] without proof (note that only Axioms (A) and (B) are utilized in the proof):

Lemma 1. *Let $*$ be a binary operation on a nonempty set U, and let $*$ satisfy Axioms* (A)–(D)*. Then*

$$u * v = u \quad \text{if and only if} \quad u = v$$

and

$$u * v = v \quad \text{if and only if} \quad v * u = u$$

for all $u, v \in U$.

The next lemma was proved in [3]:

Lemma 2. *Let $*$ be a binary operation on a nonempty set U, and let $*$ satisfy Axioms* (A)–(D)*. Let $u_1, \dots, u_h, u_{h+1}, v, w \in U$, where $h \geqslant 1$. Assume that*

$$u_1 \neq u_2, \ \dots, \ u_h \neq u_{h+1}$$

and

$$u_2 = u_1 * v, \ \dots, \ u_{h+1} = u_h * v.$$

If $w * u_1 = v$, then

$$w * u_2 = \dots = w * u_{h+1} = v$$

and

$$u_2 = u_1 * w, \ \dots, \ u_{h+1} = u_h * w.$$

P r o o f (outlined). Consider g, $1 \leqslant g \leqslant h$, and assume that $u_g * v = u_{g+1}$ and $w * u_g = v$. Since $u_g \neq u_{g+1}$, Lemma 1 implies that $u_g \neq v$ and therefore, $w * u_g \neq u_g$. By Axiom (C), $u_g * (w * u_g) = u_g * w$. Hence $u_g * w = u_g * v = u_{g+1}$. Since $w * u_g \neq u_g$, Lemma 1 implies that $u_g * w \neq w$. By Axiom (C), $w * (u_g * w) = w * u_g$. Hence $w * u_{g+1} = w * u_g = v$.

Proceding by the induction on g, we will prove the lemma. □

We will need three more lemmas.

Lemma 3. *Let $*$ be a binary operation on a nonempty set U, and let $*$ satisfy Axioms* (A)–(D). *Consider $u_1, \ldots, u_{h+1} \in U$, where $h \geqslant 1$, such that*

$$u_1 \neq u_2, \ \ldots, \ u_h \neq u_{h+1}$$

and

$$u_h = u_{h+1} * u_1, \ \ldots, \ u_1 = u_2 * u_1.$$

Then

$$u_2 = u_1 * u_{h+1}, \ \ldots, \ u_{h+1} = u_h * u_{h+1}. \tag{1}$$

P r o o f. We proceed by induction on h. If $h = 1$, the result follows from Lemma 1. Let $h \geqslant 2$. Since

$$u_{h-1} = u_h * u_1, \ \ldots, \ u_1 = u_2 * u_1,$$

it follows from the induction hypothesis that

$$u_2 = u_1 * u_h, \ \ldots, \ u_h = u_{h-1} * u_h.$$

Since $u_{h+1} * u_1 = u_h$, Lemma 2 implies that

$$u_2 = u_1 * u_{h+1}, \ \ldots, \ u_h = u_{h-1} * u_{h+1}.$$

By Axiom (A), $u_{h+1} = (u_{h+1} * u_1) * u_{h+1}$. We get $u_{h+1} = u_h * u_{h+1}$. Hence (1) holds. □

The next lemma is similar to Lemma 5 of [3], but our proof will be different and shorter.

Lemma 4. *Let G be a connected graph, let $*$ be a binary operation associated with a connected graph G, and let $*$ satisfy Axioms* (A)–(D). *Consider arbitrary* *distinct* $u, v \in V(G)$. *Then there exist pairwise distinct* $u_1, \ldots, u_{m+1} \in V(G)$, $m \geqslant 1$, *such that* $u_1 = u$, $u_{m+1} = v$ *and*

$$u_2 = u_1 * v, \ \ldots, \ u_{m+1} = u_m * v.$$

P r o o f. Suppose, to the contrary, that the lemma is false. Since $V(G)$ is finite, it is easy to see that there exist $v_1, \ldots, v_{k+1} \in V(G)$, where $k \geqslant 1$, such that $v_1 = u$,

$$v_2 = v_1 * v \neq v, \ \ldots, \ v_{k+1} = v_k * v \neq v \tag{2}$$

and there exists j, $1 \leqslant j \leqslant k$, with the property that $v_j = v_{k+1}$ and the vertices $v_j, v_{j+1}, \ldots, v_k$ are pairwise distinct. By virtue of Lemma 1, $j < k$. Let d denote the distance function of G. For each $w \in V(G)$, we denote

$$e(w) = \min\{d(w, v_i); \ j \leqslant i \leqslant k\}.$$

Moreover, we denote by Z the set of all $z \in V(G)$ such that

$$v_{j+1} = v_j * z, \ \ldots, \ v_{k+1} = v_k * z.$$

As follows from (2), $v \in Z$. Consider an arbitrary $x \in Z$ such that

$$e(x) \leqslant e(z) \quad \text{for all } z \in Z. \tag{3}$$

Since $x \in Z$, Lemma 1 implies that $e(x) \geqslant 1$. Therefore, there exists $y \in V(G)$ such that $e(y) = e(x) - 1$ and $\{x, y\} \in E(G)$. By Lemma 1, $y * x = x$. By (3), $y \notin Z$. Recall that $v_{k+1} = v_j$. Without loss of generality, we may assume that $v_{k+1} \neq v_k * y$. Thus $v_k * x \neq v_k * y$. By Axiom (D),

$$x = y * (v_k * x) = y * v_{k+1} = y * v_j.$$

Since $v_j \neq v_{j+1}, \ldots, v_k \neq v_{k+1}$, Lemma 2 implies that

$$v_{j+1} = v_j * y, \ \ldots, \ v_k * y = v_{k+1}.$$

Thus $y \in Z$, which is a contradiction. □

Let G be a graph, let $*$ be a binary operation associated with G, and let $*$ satisfy Axioms (A)–(D). We will say that

$$(u_1, \ldots, u_k, u_{k+1})$$

is a $(*)$-path in G, if $k \geqslant 1$, $u_1, \ldots, u_k, u_{k+1}$ are pairwise distinct vertices of G, and

$$u_2 = u_1 * u_{k+1}, \ \ldots, \ u_{k+1} = u_k * u_{k+1}.$$

Obviously, every $(*)$-path in G is a path in G.

Let G be a connected graph, and let d denote its distance function. For every ordered pair $x, y \in V(G)$, we denote

$$A_G(x,y) = \{x\}, \quad \text{if } x = y$$

and

$$A_G(x,y) = \{z \in V(G);\ d(x,z) = 1 \text{ and } d(z,y) = d(x,y) - 1\}, \text{ if } x \neq y.$$

(Note that if $x \neq y$, then $A_G(x,y)$ is $N_1(x,y)$ in the sense of [2].)

The next lemma is the main one:

Lemma 5. *Let G be a connected graph, let $*$ be a binary operation associated with G, and let $*$ satisfy Axioms* (A)–(D)*. Consider arbitrary $u, v \in V(G)$. Then*

$$(4) \qquad A_G(u,v) = \{u * v\}.$$

Proof. Let P_* denote the set of all $(*)$-paths in G, and let d denote the distance function of G. Put $n = d(u,v)$. We proceed by induction on n. Clearly, if $n \leqslant 1$, then (4) holds. Let $n \geqslant 2$. We assume that

$$(5) \qquad A_G(x,y) = \{x * y\} \text{ for all } x, y \in V(G) \quad \text{such that } d(x,y) < n.$$

Suppose, to the contrary, that $A_G(u,v) \neq \{u * v\}$. Then there exists $w \in A_G(u,v)$ such that $w \neq u * v$. By Lemma 4, there exist $u_1, \ldots, u_m, u_{m+1} \in V(G)$ such that $m \geqslant n$, $u_1 = u$, $u_{m+1} = v$, and $(u_1, \ldots, u_m, u_{m+1}) \in P_*$. Since $w \in A_G(u,v)$, there exist $u_{m+2}, \ldots, u_{m+n}, u_{m+n+1} \in V(G)$ such that $u_{m+n} = w$, $u_{m+n+1} = u$ and $(u_{m+n+1}, u_{m+n}, \ldots, u_{m+1})$ is a shortest $u - v$ path in G.

Obviously, $u_{m+n+1} = u_1$. Put $u_{m+n+2} = u_2$, $u_{m+n+3} = u_3, \ldots, u_{m+2n+1} = u_{n+1}$. Define

$$\alpha_h = (u_h, \ldots, u_{h+m-1}, u_{h+m}) \quad \text{and} \quad \beta_h = (u_{h+m+n}, u_{h+m+n-1}, \ldots, u_{h+m})$$

for each $h = 1, \ldots, n+1$.

Obviously, $\alpha_1 \in P_*$. Since $w \neq u * v$ and $u_2 = u * v$, we have $u_2 \neq u_{m+n}$ and $\beta_1 \notin P_*$.

Since $\alpha_1 \in P_*$, Lemma 3 implies that

$$(u_{m+1}, u_m, \ldots, u_2, u_1) \in P_*.$$

Therefore, $(u_m, \ldots, u_2, u_1) \in P_*$. Applying Lemma 3 again, we have

$$(u_1, u_2, \ldots, u_m) \in P_*.$$

Hence $u_2 = u_1 * u_m$.

Clearly, $w \in A_G(u, u_{m+2})$. Since $d(u, u_{m+2}) = n - 1$, it follows from (5) that $A_G(u, u_{m+2}) = \{u * u_{m+2}\}$ and thus $w = u * u_{m+2}$. Since $w \neq u_2 = u * u_m$, we get $u_m \neq u_{m+2}$.

Recall that α_1 and β_1 are paths in G. Thus

$$u_3 \neq u_1, u_4 \neq u_2, \ \ldots, \ u_{m+n+2} \neq u_{m+n}. \tag{6}$$

Assume that $\alpha_{n+1} \in P_*$. Then also $(u_{m+1}, \ldots, u_{m+n}, u_{m+n+1}) \in P_*$ and, by Lemma 3, $\beta_1 \in P_*$; a contradiction. Thus $\alpha_{n+1} \notin P_*$. Since $\alpha_1 \in P_*$ and $\beta_1 \notin P_*$, there exists i, $1 \leqslant i \leqslant n$ such that

$$\alpha_i \in P_* \text{ and } \beta_i \notin P_*, \tag{7}$$

and

$$\alpha_{i+1} \notin P_* \text{ or } \beta_{i+1} \in P_*. \tag{8}$$

Let $\alpha_{i+1} \in P_*$. By (8), $\beta_{i+1} \in P_*$. Since $u_{i+1} = u_{i+m+n+1}$ and $u_{i+2} = u_{i+m+n+2}$, we have $u_{i+m+n+2} = u_{i+2} = u_{i+1} * u_{i+m+1} = u_{i+m+n+1} * u_{i+m+1} = u_{i+m+n}$, which contradicts (6). Thus $\alpha_{i+1} \notin P_*$.

By (7), $\alpha_i \in P_*$. This implies that $u_{i+1} = u_i * u_{i+m}$ and

$$u_{i+2} = u_{i+1} * u_{i+m}, \ \ldots, \ u_{i+m} = u_{i+m-1} * u_{i+m}.$$

Let $u_{i+m} = u_{i+m+1} * u_{i+1}$. By Lemma 2,

$$u_{i+2} = u_{i+1} * u_{i+m+1}, \ \ldots, \ u_{i+m} = u_{i+m-1} * u_{i+m+1}.$$

Since $u_{i+m+1} = u_{i+m} * u_{i+m+1}$, we get $\alpha_{i+1} \in P_*$, which is a contradiction. Thus $u_{i+m} \neq u_{i+m+1} * u_{i+1}$.

Since $u_{i+1} = u_i * u_{i+m}$, we get $u_{i+m} \neq u_{i+m+1} * (u_i * u_{i+m})$. By Axiom (D),

$$u_i * u_{i+m+1} = u_i * u_{i+m}. \tag{9}$$

Since $u_i = u_{i+m+n}$, we have $d(u_i, u_{i+m+1}) \leqslant n - 1$.

First, let $d(u_i, u_{i+m+1}) = n - 1$. By (5), $A_G(u_{i+m+n}, u_{i+m+1}) = \{u_{i+m+n} * u_{i+m+1}\}$ and thus $u_{i+m+n} * u_{i+m+1} = u_{i+m+n-1}$. It follows from (9) that $u_{i+m+n+1} = u_{i+m+n-1}$, which contradicts (6).

Now, let $d(u_i, u_{i+m+1}) < n - 1$.Thus $d(u_i, u_{i+m}) < n$. Applying (5) step by step, we see that α_i is a shortest $u_i - u_{i+m}$ path in G. We get $n \leqslant m = d(u_i, u_{i+m}) < n$; a contradiction.

Thus (4) holds. □

P r o o f of the theorem. Put $U = V(G)$. Let G be geodetic and let $*$ be its proper operation. Then G is connected. It is easy to see that $*$ satisfies Axioms (A), (B) and (C). Moreover, it is not difficult to show that $*$ satisfies also Axiom (D); this verification can be found in [3].

Conversely, let G be connected and let $*$ satisfy Axioms (A)–(D). By Lemma 5, $|A_G(x, y)| = 1$, for all $x, y \in U$. It is easy to prove, by induction on $d(x, y)$, that G is geodetic. By Lemma 5, $A_G(x, y) = \{x * y\}$ for all $x, y \in U$. This implies that $*$ is the proper operation of G, which completes the proof of the theorem. □

Remark. Obviously, every tree is a geodetic graph. For trees, a stronger result can be found in [4].

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