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Štefan Schwabik; Ye Guoju

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ON THE STRONG MCSHANE INTEGRAL OF FUNCTIONS WITH VALUES IN A BANACH SPACE

Štefan Schwabik, Praha, and Ye Guoju, Lanzhou

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Dedicated to the seventieth birthday of Ivo Vrkoč

Abstract. The classical Bochner integral is compared with the McShane concept of integration based on Riemann type integral sums. It turns out that the Bochner integrable functions form a proper subclass of the set of functions which are McShane integrable provided the Banach space to which the values of functions belong is infinite-dimensional. The Bochner integrable functions are characterized by using gauge techniques.

The situation is different in the case of finite-dimensional valued vector functions.

Keywords: Bochner integral, strong McShane integral

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1. Partitions, systems and gauges

Let an interval $[a,b]\subset\mathbb{R}$, $-\infty<a<b<+\infty$ be given. A pair (τ,J) of a point $\tau\in\mathbb{R}$ and a compact interval $J\subset\mathbb{R}$ is called a *tagged interval*, τ is the *tag* of J.

A finite collection $\{(\tau_j,J_j),\ j=1,\dots,p\}$ of tagged intervals is called an *M-system* on $[a,b]$ if

$$\operatorname{Int}(J_i)\cap\operatorname{Int}(J_j)=\emptyset \text{ for } i\neq j.$$

($\operatorname{Int}(J)$ denotes the interior of an interval J.)

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An M-system on $[a,b]$ is called an *M-partition* of $[a,b]$ if

$$\bigcup_{j=1}^{k} J_j = [a,b].$$

An M-partition $\{(\tau_j, J_j),\ j=1,\dots,k\}$ for which

$$\tau_j \in J_j,\ \ j=1,\dots,k$$

is called a *P-partition* of $[a,b]$.

Clearly every P-partition of $[a,b]$ is also an M-partition of $[a,b]$.

Given a positive function $\delta\colon\ [a,b] \to (0,+\infty)$ called a *gauge* on $[a,b]$, a tagged interval (τ, J) with $\tau \in [a,b]$ is said to be *δ-fine* if

$$J \subset (\tau - \delta(\tau), \tau + \delta(\tau)).$$

Using this concept we can speak about *δ-fine M-partitions (or systems)* and *δ-fine P-partitions* $\{(\tau_j, J_j),\ j=1,\dots,k\}$ of the interval $[a,b]$ whenever (τ_j, J_j) is δ-fine for every $j=1,\dots,k$.

It is a well-known fact that given a gauge $\delta\colon\ [a,b] \to (0,+\infty)$ there exists a δ-fine P-partition of $[a,b]$. This result is called *Cousin's lemma*, see e.g. [5], [9] and many other monographs on Henstock-Kurzweil integration.

2. McShane integral, the classes $\mathcal{S}$, $\mathcal{S}^\star$ and the strong McShane integral

Assume that μ is a (nonnegative) measure on $[a,b]$ (e.g. the Lebesgue measure) and that X is a Banach space with a norm $\|\cdot\|_X$.

Definition 1. A function $f\colon\ [a,b] \to X$ is said to be McShane integrable on $[a,b]$ if there is an element $I \in X$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\Bigl\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - I\Bigr\|_X < \varepsilon$$

for every δ-fine M-partition $\{(t_i, J_i),\ i=1,\dots,k\}$ of $[a,b]$.

We denote $I = (\mathcal{M})\int_a^b f\,\mathrm{d}\mu$ in this case.

Denote further by $\mathcal{M} = \mathcal{M}([a,b];X)$ the set of functions $f\colon\ [a,b] \to X$ which are McShane integrable on $[a,b]$.

Definition 2 (see [7]). By $\mathcal{S}^\star = \mathcal{S}^\star([a,b];X)$ we denote the set of functions $f\colon [a,b] \to X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i) - f(s_j)\|_X \mu(J_i \cap L_j) < \varepsilon$$

for any δ-fine M-partitions $\{(t_i, J_i),\ i = 1,\dots,k\}$ and $\{(s_j, L_j),\ j = 1,\dots,l\}$ of $[a,b]$.

Definition 3 (see [7]). By $\mathcal{S} = \mathcal{S}([a,b];X)$ we denote the set of functions $f\colon [a,b] \to X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\Big\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j)\Big\|_X < \varepsilon$$

for any δ-fine M-partitions $\{(t_i, J_i),\ i = 1,\dots,k\}$ and $\{(s_j, L_j),\ j = 1,\dots,l\}$ of $[a,b]$.

Lemma 4. *If $f \in \mathcal{S}^\star$ then $f \in \mathcal{S}$, i.e. $\mathcal{S}^\star \subset \mathcal{S}$.*

Proof. If $\{(t_i, J_i),\ i = 1,\dots,k\}$ and $\{(s_j, L_j),\ j = 1,\dots,l\}$ are δ-fine M-partitions of $[a,b]$ we have

$$\mu(J_i) = \sum_{j=1}^{l} \mu(J_i \cap L_j) \qquad \text{and} \qquad \mu(L_j) = \sum_{i=1}^{k} \mu(J_i \cap L_j).$$

Hence

$$\begin{aligned}
&\Big\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j)\Big\|_X \\
&\quad = \Big\|\sum_{j=1}^{l}\sum_{i=1}^{k} f(t_i)\mu(J_i \cap L_j) - \sum_{i=1}^{k}\sum_{j=1}^{l} f(s_j)\mu(J_i \cap L_j)\|_X \\
&\quad = \Big\|\sum_{j=1}^{l}\sum_{i=1}^{k} (f(t_i) - f(s_j))\mu(J_i \cap L_j)\Big\|_X \leqslant \sum_{j=1}^{l}\sum_{i=1}^{k} \|f(t_i) - f(s_j)\|_X \mu(J_i \cap L_j)
\end{aligned}$$

and by Definitions 2 and 3 this yields the statement. □

Proposition 5.

(a) *If $f \in \mathcal{S}$ then there is an element $I \in X$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that*

$$\Big\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - I\Big\|_X < \varepsilon$$

for every δ-fine M-partition $\{(t_i, J_i),\ i = 1,\dots,k\}$ of $[a,b]$, i.e. $f \in \mathcal{M}$.

(b) *If $f \in \mathcal{M}$ then $f \in \mathcal{S}$.*

P r o o f. (a) Let $\varepsilon > 0$ be given and assume that δ is the gauge which corresponds to $\varepsilon/2$ by the definition of the class of functions $\mathcal{S}$.

Denote

$$S(\varepsilon) = \Big\{ S(f,D) = \sum_{i=1}^{k} f(t_i)\mu(J_i);\ D = \{(t_i,J_i),\ i=1,\ldots,k\} \text{ an arbitrary } \delta\text{-fine M-partition of } [a,b] \Big\}.$$

The set $S(\varepsilon) \subset X$ is nonempty because by Cousin's lemma there exists a δ-fine M-partition $\{(t_i,J_i),\ i=1,\ldots,k\}$ of $[a,b]$. Since by definition of $\mathcal{S}$ and δ we have

$$\Big\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j) \Big\|_X < \frac{\varepsilon}{2}$$

for any δ-fine M-partitions $\{(t_i,J_i),\ i=1,\ldots,k\}$ and $\{(s_j,L_j),\ j=1,\ldots,l\}$ of $[a,b]$, we have also

$$\operatorname{diam} S(\varepsilon) < \frac{\varepsilon}{2}$$

(by $\operatorname{diam} S(\varepsilon)$ the diameter of the set $S(\varepsilon)$ is denoted). Further, evidently

$$S(\varepsilon_1) \subset S(\varepsilon_2),$$

provided $\varepsilon_1 < \varepsilon_2$. Hence the set

$$\bigcap_{\varepsilon>0} \overline{S(\varepsilon)} = I \in X$$

consists of a single point because the space X is complete ($\overline{S(\varepsilon)}$ denotes the closure of the set $S(\varepsilon)$ in X).

For the integral sum $S(f,D)$ we get

$$\Big\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - I \Big\|_X < \frac{\varepsilon}{2}$$

whenever $D = \{(t_i,J_i),\ i=1,\ldots,k\}$ is an arbitrary δ-fine M-partition of $[a,b]$.

(b) If $f \in \mathcal{M}$ then there is an $I \in X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\Big\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - I \Big\|_X < \frac{\varepsilon}{2}$$

whenever $D = \{(t_i,J_i),\ i=1,\ldots,k\}$ is an arbitrary δ-fine M-partition of $[a,b]$.

If we have two δ-fine M-partitions $\{(t_i, J_i),\ i = 1, \ldots, k\}$ and $\{(s_j, L_j), j = 1, \ldots, l\}$ of $[a, b]$ then

$$\begin{aligned}\Big\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j)\Big\|_X \\ \leqslant \Big\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - I\Big\|_X + \Big\|\sum_{j=1}^{l} f(s_j)\mu(L_j) - I\Big\|_X < \varepsilon\end{aligned}$$

and $f \in \mathcal{S}$. $\square$

Corollary 6. *If $f \in \mathcal{S}^*$ then $f \in \mathcal{M} = \mathcal{S}$, i.e. $\mathcal{S}^* \subset \mathcal{M} = \mathcal{S}$.*

Proof. This follows from Proposition 5 and Lemma 4. $\square$

It is easy to show that the McShane integral has the usual properties, especially we have:

If $f, g\colon [a, b] \to X$ and the integrals $(\mathcal{M}) \int_a^b f \,\mathrm{d}\mu$ and $(\mathcal{M}) \int_a^b g \,\mathrm{d}\mu$ exist then for $c_1, c_2 \in \mathbb{R}$ the integral $(\mathcal{M}) \int_a^b (c_1 f + c_2 g) \,\mathrm{d}\mu$ exists and

$$(\mathcal{M}) \int_a^b (c_1 f + c_2 g) \,\mathrm{d}\mu = c_1 (\mathcal{M}) \int_a^b f \,\mathrm{d}\mu + c_2 (\mathcal{M}) \int_a^b g \,\mathrm{d}\mu.$$

If the integral $(\mathcal{M}) \int_a^b f \,\mathrm{d}\mu$ exists and $[c, d] \subset [a, b]$ is an interval, then also the integral $(\mathcal{M}) \int_c^d f \,\mathrm{d}\mu$ exists.

This makes it possible to define the indefinite McShane integral $F\colon [a, b] \to X$ by the relation

$$F(t) = (\mathcal{M}) \int_a^t f \,\mathrm{d}\mu, \ \ t \in [a, b]$$

and for an interval $J = [c, d] \subset [a, b]$ the interval function

$$F[J] = F(d) - F(c) = (\mathcal{M}) \int_c^d f \,\mathrm{d}\mu.$$

Definition 7. A function $f\colon [a, b] \to X$ is said to be *strongly McShane integrable* on $[a, b]$ if f is McShane integrable on $[a, b]$ ($f \in \mathcal{M}$) and if for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\sum_{i=1}^{k} \|f(t_i)\mu(J_i) - F[J_i]\|_X < \varepsilon$$

for every δ-fine M-partition $\{(t_i, J_i),\ i = 1, \ldots, k\}$ of $[a, b]$.

Denote by $\mathcal{SM} = \mathcal{SM}([a,b];X)$ the set of functions $f\colon [a,b] \to X$ which are strongly McShane integrable on $[a,b]$.

Lemma 8. *If $f \in \mathcal{SM}$ then $f \in \mathcal{S}^*$.*

P r o o f. If $f \in \mathcal{SM}$ then by definition for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\sum_{i=1}^{k} \|f(t_i)\mu(J_i) - F[J_i]\|_X < \frac{\varepsilon}{2}$$

for every δ-fine M-partition $\{(t_i,J_i),\ i=1,\ldots,k\}$ of $[a,b]$. If we have two δ-fine M-partitions $\{(t_i,J_i),\ i=1,\ldots,k\}$ and $\{(s_j,L_j),\ j=1,\ldots,l\}$ of $[a,b]$ then

$$\begin{aligned}
\sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i) - f(s_j)\|_X \mu(J_i\cap L_j) &= \sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i)\mu(J_i\cap L_j) - f(s_j)\mu(J_i\cap L_j)\|_X \\
= \sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i)\mu(J_i\cap L_j) - F[J_i\cap L_j] &+ F[J_i\cap L_j] - f(s_j)\mu(J_i\cap L_j)\|_X \\
\leqslant \sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i)\mu(J_i\cap L_j) - F[J_i\cap L_j]\|_X & \\
+ \sum_{i=1}^{k}\sum_{j=1}^{l} \|F[J_i\cap L_j] - f(s_j)\mu(J_i\cap L_j)\|_X &< \varepsilon
\end{aligned}$$

because evidently $\{(t_i, J_i\cap L_j),\ i=1,\ldots,k,\ j=1,\ldots,l\}$ and $\{(s_j, J_i\cap L_j),\ j=1,\ldots,l,\ i=1,\ldots,k\}$ are δ-fine M-partitions of $[a,b]$.

Hence $f \in \mathcal{S}^*$. $\square$

Lemma 9. *If $f \in \mathcal{S}^*$ then $f \in \mathcal{SM}$.*

P r o o f. If $f \in \mathcal{S}^*$ then, by Definition 2, for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i) - f(s_j)\|_X \mu(J_i\cap L_j) < \frac{\varepsilon}{2}$$

for any δ-fine M-partitions $\{(t_i,J_i),\ i=1,\ldots,k\}$ and $\{(s_j,L_j),\ j=1,\ldots,l\}$ of $[a,b]$.

Assume that $\{(t_i,J_i),\ i=1,\ldots,k\}$ is an arbitrary δ-fine M-partition of $[a,b]$.

By Corollary 6 we have $f \in \mathcal{M}$ and therefore f is McShane integrable over every interval J_i, $i=1,\ldots,k$. Hence for the given $\varepsilon > 0$ there is a gauge δ' on

$[a,b]$ such that $\delta'(t) \leqslant \delta(t)$ for $t \in [a,b]$ and such that for any δ'-fine M-partition $\{(s_j^{(i)}, L_j^{(i)}),\ j = 1, \ldots, l^{(i)}\}$ of the intervals J_i we have

$$\left\| \sum_{j=1}^{l^{(i)}} [f(s_j^{(i)})\mu(L_j^{(i)}) - F[L_j^{(i)}]] \right\|_X < \frac{\varepsilon}{2k}.$$

Note that $\{(s_j^{(i)}, L_j^{(i)}),\ j = 1, \ldots, l^{(i)},\ i = 1, \ldots, k\}$ is a δ-fine M-partition of the interval $[a,b]$ and that for any $i = 1, \ldots, k$ we have

$$f(t_i)\mu(J_i) = \sum_{j=1}^{l^{(i)}} f(t_i)\mu(J_i \cap L_j^{(i)})$$

and, because of the additivity of the indefinite integral F, also

$$F[J_i] = \sum_{j=1}^{l^{(i)}} F[J_i \cap L_j^{(i)}].$$

Hence

$$\begin{aligned}
\sum_{i=1}^{k} \|f(t_i)\mu(J_i) - F[J_i]\|_X &= \sum_{i=1}^{k} \left\| \sum_{j=1}^{l^{(i)}} f(t_i)\mu(J_i \cap L_j^{(i)}) - \sum_{j=1}^{l^{(i)}} F[J_i \cap L_j^{(i)}] \right\|_X \\
&= \sum_{i=1}^{k} \left\| \sum_{j=1}^{l^{(i)}} (f(t_i) - f(s_j^{(i)}))\mu(J_i \cap L_j^{(i)}) + \sum_{j=1}^{l^{(i)}} [f(s_j^{(i)})\mu(J_i \cap L_j^{(i)}) - F[J_i \cap L_j^{(i)}]] \right\|_X \\
&\leqslant \sum_{i=1}^{k} \left\| \sum_{j=1}^{l^{(i)}} (f(t_i) - f(s_j^{(i)}))\mu(J_i \cap L_j^{(i)}) \right\|_X \\
&\quad + \sum_{i=1}^{k} \left\| \sum_{j=1}^{l^{(i)}} [f(s_j^{(i)})\mu(J_i \cap L_j^{(i)}) - F[J_i \cap L_j^{(i)}]] \right\|_X \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{l^{(i)}} \|(f(t_i) - f(s_j^{(i)}))\mu(J_i \cap L_j^{(i)})\|_X \\
&\quad + \sum_{i=1}^{k} \left\| \sum_{j=1}^{l^{(i)}} [f(s_j^{(i)})\mu(J_i \cap L_j^{(i)}) - F[J_i \cap L_j^{(i)}]] \right\|_X < \frac{\varepsilon}{2} + \sum_{i=1}^{k} \frac{\varepsilon}{2k} = \varepsilon.
\end{aligned}$$

This shows that $f \in \mathcal{SM}$. $\square$

Using Lemma 8 and 9 we immediately obtain

Corollary 10. *$f \in \mathcal{S}^*$ if and only if $f \in \mathcal{SM}$, i.e. $\mathcal{S}^* = \mathcal{SM}$.*

3. Comparison of various integrals

Our aim now is to compare the concepts of Bochner integral, McShane integral described above and the variational McShane integral introduced in [8].

Denote by $\mathcal{B}$ the space of all Bochner integrable functions $f\colon [a,b] \to X$.

We use the concept of the Bochner integral as it is presented in the book [6] of S. Lang and its slight modification from [7]. This concept is based on the elementary definition of the integral of the so called simple functions and on the completion of the linear space of simple functions on the interval $[a,b]$ with respect to the L_1-seminorm given by $\|f\|_1 = \int_a^b \|f\| \,\mathrm{d}\mu$ for a simple function f. (For a more detailed account of these well-known facts see [6] or [7]).

In [7] the following fact was shown.

Proposition 11. *If $f \in \mathcal{B}$ then also $f \in \mathcal{S}^\star = \mathcal{SM} \subset \mathcal{M}$ and*

$$(\mathcal{B}) \int_a^b f \,\mathrm{d}\mu = (\mathcal{M}) \int_a^b f \,\mathrm{d}\mu.$$

So if X is a general Banach space then $\mathcal{B} \subset \mathcal{S}^\star = \mathcal{SM} \subset \mathcal{M}$.

On the other hand, the following statement holds.

Proposition 12. *If $f \in \mathcal{S}^\star$ then also $f \in \mathcal{B}$ and*

$$(\mathcal{B}) \int_a^b f \,\mathrm{d}\mu = (\mathcal{M}) \int_a^b f \,\mathrm{d}\mu.$$

The proof of Proposition 12 can be found in [7]. Here we don't repeat it.

From Propositions 11 and 12 we get

Corollary 13. *$f \in \mathcal{S}^*$ if and only if $f \in \mathcal{B}$, i.e. $\mathcal{S}^* = \mathcal{SM} = \mathcal{B}$.*

Now we recall the variational McShane integral introduced in [8].

Let $\mathcal{I}$ be a collection of all closed intervals that are contained in $[a,b]$ and let $F_1, F_2\colon \mathcal{I} \times [a,b] \to X$ be interval-point functions.

Functions $F_1, F_2\colon \mathcal{I} \times [a,b] \to X$ are said to be *McShane variationally equivalent* if $V_M(F_1 - F_2) = 0$, where

$$V_M(F_1 - F_2) = \inf_{\delta} \sup_{T} \sum_{j=1}^{k} \|F_1(\tau_i, J_i) - F_2(\tau_i, J_i)\|$$

(the sup is taken over all McShane δ-fine partitions $T = \{(\tau_j, J_j),\ j = 1, \dots, k\}$ and the inf is taken over all gauges δ on $[a,b]$).

Definition 14. A function $f\colon [a,b] \to X$ is called *McShane variationally integrable* ($\mathcal{MV}$-integrable) on $[a,b]$ if there exists an additive interval function $F\colon \mathcal{I} \to X$ such that the interval-point function $f(t)\mu(I)$ and $F[I]$ are McShane variationally equivalent, $F[I]$ being the indefinite $\mathcal{MV}$-integral of f.

Theorem 15. *If $f\colon [a,b] \to X$, then the following assertions are equivalent.*
(1) $f \in \mathcal{S}^\star$,
(2) $f \in \mathcal{SM}$,
(3) *the function f is $\mathcal{MV}$-integrable on $[a,b]$,*
(4) $f \in \mathcal{B}$.

P r o o f. By Corollary 10 it follows that (1) and (2) are equivalent. By Corollary 13, it follows that (1) and (4) are equivalent. By Theorem 2 from [8], it follows that (4) and (3) are equivalent. So we obtain the statement of our theorem, i.e. we have $\mathcal{S}^\star = \mathcal{SM} = \mathcal{B} = \mathcal{MV}$. □

In this way different definitions of Bochner integrability are obtained making in fact a link to the McShane variational integral presented in [8] ($\mathcal{MV}$) using the McShane variational equivalence.

Moreover, the following result was proved in [7].

Proposition 16. *If X is a finite dimensional Banach space, then $\mathcal{S}^\star = \mathcal{SM} = \mathcal{S} = \mathcal{M}$.*

In [7] (like Skvortsov and Solodov have done it in [8]) the results of Dvoretzky and Rogers from [1] have been used to prove

Proposition 17. *Given a Banach space X then $\mathcal{S}^\star = \mathcal{SM} \subset \mathcal{S} = \mathcal{M}$ and $\mathcal{S}^\star = \mathcal{SM} = \mathcal{S} = \mathcal{M}$ if and only if the dimension of X is finite.*

So with respect to the above considerations and to Theorem 15 we obtain

Theorem 18. *If X is a Banach space then the classes of functions $\mathcal{B}$, $\mathcal{S}^\star$, $\mathcal{SM}$, $\mathcal{MV}$ and $\mathcal{M}$ coincide if and only if X is finite-dimensional.*

In other words, this says that if X is a finite-dimensional Banach space then the $\mathcal{B}$-integral and the $\mathcal{M}$-integral are equivalent (and this occurs if the integrand belongs to the equivalent classes of functons $\mathcal{S}^\star$, $\mathcal{SM}$, $\mathcal{MV}$). So we get various characterizations of the Bochner integrable functions $f\colon [a,b] \to X$ in this case.

Remark. All the integrals presented in this paper and also our results can be generalized to the case of functions defined on intervals in $\mathbb{R}^n$, i.e. to the case when $f\colon I_0 \to X$ ($I_0 \subset \mathbb{R}^n$ is an interval and X is a Banach space) because we don't use any special property of the one dimensional space in the above statements. So this generalization from $\mathbb{R}^1$ to $\mathbb{R}^n$ is very natural.

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Authors' addresses: Š. Schwabik, Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `schwabik@math.cas.cz`; Ye Guoju, Northwest Normal University, Lanzhou, Peoples Republic of China, e-mail: `yeguoju@math.cas.cz`, `yeguoju@21cn.com`.