Franco Flandoli; Ida Minelli

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PROBABILISTIC MODELS OF VORTEX FILAMENTS

Franco Flandoli, Pisa, and Ida Minelli, Padova

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Dedicated to Ivo Vrkoč on the occasion of his 70th birthday

Abstract. A model of vortex filaments based on stochastic processes is presented. In contrast to previous models based on semimartingales, here processes with fractal properties between 1/2 and 1 are used, which include fractional Brownian motion and similar non-Gaussian examples. Stochastic integration for these processes is employed to give a meaning to the kinetic energy.

Keywords: stochastic integration, fractional Brownian motion, p-variation, vortex filaments, statistical fluid mechanics

MSC 2000: 60H05, 60H30, 76M35, 76F55

1. Introduction

Fluid dynamics is an active area of research, still presenting outstanding open problems. A main problem is the *lack of a mathematical description of the three-dimensional structures* that are observed in real fluids or numerical computations. This problem is of interest both in mathematics, physics, engineering and numerics of fluids. For instance, the possible blow-up of solutions (a basic open problem) seems to be associated with the evolution of vortical structures, their fast stretching and folding. As another example in a different direction, there is some hope to understand the statistical properties of turbulent flows by means of statistical mechanics of vortex structures. See [7], [11].

In many simulations and experiments on turbulent flows the vorticity field appears strongly concentrated along thin structures, like filaments; see for instance [22], [20] and the reviews in [7], [11]. These structures appear to be very irregular (see also

[3]), and thus it is natural to attempt a description based on irregular curves, which typically could be trajectories of stochastic processes. A model based on stochastic processes is also natural in view of statistical mechanics of these structures.

Following the ideas, originated by Onsager [19], of the two-dimensional theory of point vortices, A. Chorin introduced vortex filaments described by paths of the self avoiding walk, see [7]. Similarly to the 2-D theory of Onsager, Chorin introduces Gibbs measures of the form

$$\mu_\beta(\mathrm{d}\omega) = \frac{1}{Z_\beta} \mathrm{e}^{-\beta H(\omega)} P(\mathrm{d}\omega) \tag{1}$$

on a measurable space $(\Omega, \mathcal{F})$ of vortex filaments to describe the long-time statistics of turbulent flows. Here H is an approximate expression for the energy of a configuration ω, $(\Omega, \mathcal{F}, P)$ corresponds to the self-avoiding walk, and $Z_\beta = \int_\Omega \mathrm{e}^{-\beta H(\omega)} P(\mathrm{d}\omega)$. The parameter β has usually the meaning of inverse temperature, but in the statistical theories of [19] and [7] it is not the temperature of the fluid, and it may take also negative values. From these phenomenological ensambles it was possible to deduce a number of interesting statistical properties, and even a heuristic confirmation of Kolmogorov (K41) theory [13].

It is natural to try to extend some ideas of Chorin to models based on continuous-time stochastic processes, like Brownian motion, instead of discrete structures on a lattice. Attempts in this direction can be found in the book of Gallavotti [12], Ch. I, Sect. 11, in the paper of P. L. Lions and A. Majda [15], and in the works [9] and [10]. The approach of [15] is limited by a strong idealization (nearly parallel filaments, which partially reduce the problem to an elaborate version of the 2-D case), but the final results of the mean field and the many characterizations in terms of variational problems are outstanding. The approach of Gallavotti, in principle, corresponds to the full problem without idealizations, but it is mainly a heuristic suggestion of a direction of research. The papers [9] and [10] treat 3-D Brownian and more general semimartingale vortex filaments in a rigorous way, with two different approaches.

Some numerical investigations on 3-D flows indicate that vortex filaments have a sort of fractal structure (not yet well understood), with fractal dimensions that are not necessarily those of Brownian curves. For these reasons we try to introduce in the present work a generalization of the vortex structures of [9], [10], based now on processes which are not semimartingales and with paths having fractal characteristics between 1/2 and 1. The case between 0 and 1/2 is interesting as well, but the approach we are going to present is not able to cover it.

The standard example that fulfils the assumptions of this work is the fractional Brownian motion (B_t^H) with Hurst parameter $H \in (\frac{1}{2}, 1)$. However, we only assume a condition on the p-variation of the process, so non-Gaussian examples are admitted. A relevant non-Gaussian example is for instance a process similar to the 3-D fractional Brownian motion but constrained to live in a half space, modelling filaments in a fluid near a solid boundary (vortex structures are often produced near solid boundaries).

The main results of this paper are simply a rigorous definition of such vortex filaments and the proof that the corresponding kinetic energy H is finite. The corresponding Gibbs measures μ_β are then well defined for positive inverse temperatures β, since H is non-negative and finite, with probability one. In contrast to [10], we are still not able to analyse the case of negative β, requiring an exponential integrability that we do not know how to handle without semimartingale properties.

The definition and analysis of the energy H requires stochastic integration with respect to the processes introduced here. We work in the framework of processes with a certain condition on the p-variation and adapt an approach introduced by Bertoin [4], a stochastic analog of a theory of Young [24]. It seems that similar results on the energy H can be obtained by means of the approach of [25]. In a Gaussian framework, in particular in the case of the fractional Brownian motion (B_t^H), stochastic integration can be performed also by means of Malliavin calculus, see [1], [2], and it has been recently applied with success to vortex filaments [18]. We emphasize that here we treat very general processes, in particular non-Gaussian ones, see Section 5.

Remark 1. A good statistical description of vorticity filaments and their Gibbs measures should be the starting point of a *statistical approach to 3-D fluids* along the lines of [7], [15]. A mean field theory as in [15] has been developed in [5]. It would be interesting to study the relations with the theory of processes in a random environment, representing vortex filaments in a surrounding mean vorticity field of lower intensity. Open interesting problems for the models introduced here or in [9] and [10] are the computation of important moments as the structure function or the energy spectrum, the existence of a Hamiltonian dynamic and its relations with Euler equation (see for instance [16] in two dimensions), the existence of Glauber type dynamics and their use for simulations, some form of the invariance principle in connection with the theory of [7], questions about the super or sub diffusive behaviour of the processes defined by these Gibbs measures, and last but not least the attempt to go back to a single filament without cross-section by renormalization. It is also very important to look for more realistic models, not based on a fixed fractal cross-section. All these problems contain nontrivial points and are open at present.

2. Vortex filaments and their energy

In this section we present only non-rigorous arguments, with the aim to explain at a physical level the meaning of the model of vortex filaments introduced here and the associated formula for the energy. Rigorous definitions and results are postponed to the subsequent sections. For a more extensive discussion of the initial part of this section see [7].

An incompressible homogeneous (constant density) fluid is described by a velocity field $u\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and a pressure field $p\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$. The vorticity field $\xi(x) = \operatorname{curl} u(x)$ captures basic features not easily visible at the level of (u,p), and can be used as an alternative variable. If we introduce the vector potential $A(x)$ satisfying $\Delta A(x) = -\xi(x)$ and use the property $\operatorname{curl}\operatorname{curl} A = -\Delta A$ that holds true when A is divergence free, then under suitable assumptions that imply uniqueness we have that $u(x) = \operatorname{curl} A(x)$ is a velocity field, since its curl is $\xi(x)$. Therefore $u(x)$ can be recovered from $\xi(x)$ by means of the Biot-Savard law $u = \operatorname{curl}\Delta^{-1}\xi$.

The kinetic energy H of the fluid is $\frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2\,\mathrm{d}x$. It can be rewritten as

$$H = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\xi(x)\cdot\xi(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y,$$

using the formula $A(x) = 1/(4\pi)\int_{\mathbb{R}^3} \xi(y)/|x-y|\,\mathrm{d}y$. When the vorticity is ideally concentrated on a curve $\gamma(\sigma)$, $\sigma \in [0,1]$, we formally use for the vorticity the model

$$\xi(x) = \Gamma \int_0^1 \delta(x-\gamma(t))\dot\gamma(t)\,\mathrm{d}t$$

for some constant Γ with the meaning of circulation. Then for the energy we formally obtain the expression

$$H = \frac{\Gamma^2}{8\pi}\int_0^1\int_0^1 \frac{\mathrm{d}\gamma(\sigma)\cdot\mathrm{d}\gamma(\sigma')}{|\gamma(\sigma)-\gamma(\sigma')|} \tag{2}$$

where we have written $\mathrm{d}\gamma(t)$ in place of $\dot\gamma(t)\,\mathrm{d}t$ to anticipate the expression for the case of irregular curves. (Notice that the symbol t here and below is not the time of evolution of the fluid but the parameter of the curve.) As a side remark, we notice that there is a second plausible expression for the energy of a curve, just slightly different, that takes care of the correction due to the fact that a vorticity field concentrated on a *open* curve $\gamma(\sigma)$, $\sigma \in [0,1]$, cannot be divergence free, so the incompressibility condition required in the previous rewritings does not hold true. For this second expression see [10]. Since this difference does not play a significant role, we mantain the expression (2) which looks easier.

This expression for the kinetic energy diverges, since the singularity in the denominator is not integrable along the diagonal of $[0,1] \times [0,1]$, and the numerator is not helpful in the case of smooth curves. Non regular curves can be better, in principle. The term $|\gamma(t) - \gamma(s)|$ may be infinitesimal of order less than one, so $1/|\gamma(t) - \gamma(s)|$ is less divergent than in the smooth case, and very fast changes in direction may produce further cancellation in the term $\dot{\gamma}(t) \cdot \dot{\gamma}(s)$. Unfortunately this naïve hope has not been confirmed by rigorous computations until now. The problem partially comes from the very frequent self-intersections and for the major part it seems to come from the integrability requirements on $1/|\gamma(t) - \gamma(s)|$ imposed by the stochastic or generalized integrals appearing in H. So, at present, we have to leave the simple idea of a vortex filament supported by a curve, both if we have in mind smooth or irregular curves.

Following a personal suggestion offered by A. Chorin, a way to introduce a rigorous non-divergent model which reflects properties observed in fluids, is to re-introduce a finite cross-section of the filaments. Physical vortex structures have a cross-section. Since it is small compared to the length of the filament, one is tempted to eliminate the cross-section, but we have seen that this produces divergences. In the next subsection we will see how one can introduce a cross-section in the model.

Remark 2. A related problem is to give a rigorous meaning to the velocity field $u_\gamma(x)$ induced by the vorticity concentrated on the curve γ:

$$u_\gamma(x) := \frac{\Gamma}{4\pi} \int_0^1 \frac{(\gamma(\sigma) - x) \times \mathrm{d}\gamma(\sigma)}{|\gamma(\sigma) - x|^3}.$$

This is the expression used in [12]. It has a meaning when x is outside the curve, but the behaviour as x approaches the curve is not understood and is responible for the motion of the curve itself (the filament moves under the action of the velocity field induced by itself).

Remark 3. It is clear that renormalization could be a way to avoid a cross section and give a meaning to some objects, like the Gibbs measure. In spite of a lot of effort, renormalization seems to be very difficult. In the literature one can find the renormalization of 3-D polymers, see for instance [6], [23], that looks similar to some extent, but here there are new essential difficulties.

2.1. Cross-section.

In order to construct vorticity fields with finite energy but still with an appealing fractal structure and suitable for a probabilistic treatment, we consider a distributional vorticity field formally expressed

$$\xi(x) = \Gamma \int_{\mathbb{R}^3} \left(\int_0^1 \delta(x - y - X_t) \circ \mathrm{d}X_t \right) \varrho(\mathrm{d}y) \tag{3}$$

where ϱ is a probability measure and $(X_t)_{t\in[0,1]}$ is a stochastic process in $\mathbb{R}^3$. The nature of the process will be described in the next rigorous sections. We use Stratonovich type of integrals since they are a more natural generalization of the case of smooth curves, and for a reason related to incompressibility that is described in [9].

We shall try to impose the minimum of regularity on the mollifying measure ϱ. Indeed, consider the case when it is supported by a set $\mathcal{A} \subset \mathbb{R}^3$, so that geometrically the vorticity field is concentrated over the set

$$C_{\mathcal{A}} = \{x + X_t;\ \ x \in \mathcal{A},\ \ t \in [0,1]\}.$$

In this case the set $\mathcal{A}$ has, roughly speaking, the meaning of a cross-section. Complex numerical simulations (see for instance [3]) show that the cross section should be fractal, and not just like a disc. Therefore, to have a model close to the best available numerical understanding of vortex structures, the measure ϱ should be supported by a fractal set. In the case when (X_t) is the Brownian motion, this can be done, see [9], [10]. Here, since we have to use a less powerful stochastic integration theory, we have a less satisfactory result, but still we can consider fractal cross-sections.

The kinetic energy takes now the form

$$H = \frac{\Gamma^2}{8\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(\int_0^1 \int_0^1 \frac{1}{|x + X_t - (y + X_s)|} \circ \mathrm{d}X_s \circ \mathrm{d}X_t \right) \varrho(\mathrm{d}x)\varrho(\mathrm{d}y). \tag{4}$$

If we introduce the *interaction energy* between the curves $(x + X_t)_{t\in[0,1]}$ and $(y + X_t)_{t\in[0,1]}$, with $x \neq y$, formally defined by

$$H_{xy} = \frac{\Gamma^2}{8\pi} \int_0^1 \int_0^1 \frac{1}{|x + X_t - (y + X_s)|} \circ \mathrm{d}X_s \circ \mathrm{d}X_t \tag{5}$$

then we have

$$H = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} H_{xy} \varrho(\mathrm{d}x)\varrho(\mathrm{d}y). \tag{6}$$

It is proved in [9] that, in the case when (X_t) is the Brownian motion, in order to obtain a well defined theory it is necessary and sufficient to assume for the measure ϱ

the condition (called the finite energy condition in potential theory, see [14])

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\,\varrho(\mathrm{d}x)\varrho(\mathrm{d}y) < \infty. \tag{7}$$

There exists a probability measure ϱ supported by $\mathcal{A}$ with finite energy if and only if the capacity of $\mathcal{A}$ is strictly positive. In particular, by Theorem 3.13 of [14], every compact set with Hausdorff dimension $d > 1$ has positive capacity, and therefore supports a probability measure ϱ satisfying (7). In the present paper we have to impose the stronger condition (11), which however still admits the case of a fractal support.

2.2. Spectral representation.

By spectral analysis we can rewrite (4) in a different form that will be easier to handle. Let us set

$$\hat{\varrho}(k) = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}k\cdot x}\,\varrho(\mathrm{d}x).$$

Then the Fourier transform of the vorticity field $\xi(x)$ given by (3) is

$$\hat{\xi}(k) = \Gamma\hat{\varrho}(k)\int_0^1 \mathrm{e}^{\mathrm{i}k\cdot X_t} \circ \mathrm{d}X_t$$

and the energy can be written as

$$H = \frac{\Gamma^2}{8\pi}\int_{\mathbb{R}^3} \mathrm{d}k\,\frac{|\hat{\varrho}(k)|^2}{|k|^2}\left\|\int_0^1 \mathrm{e}^{\mathrm{i}k\cdot X_t} \circ \mathrm{d}X_t\right\|^2_{\mathbb{C}^3}. \tag{8}$$

In the next subsections we will use this expression.

Let us remark that the spectral analysis offers a formula for future investigations of the energy spectrum $E(k)$ (for the definition see [13], [7], [11], for instance)

$$E(k) = \frac{\Gamma^2}{8\pi}\int_{|\mathbf{k}|=k} \mathrm{d}\mathbf{k}\,\frac{|\hat{\varrho}(\mathbf{k})|^2}{|\mathbf{k}|^2}E_\beta\left\|\int_0^1 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot X_t} \circ \mathrm{d}X_t\right\|^2_{\mathbb{C}^3}$$

where we have denoted by $\mathbf{k}$ the 3-D vector previously written as k, and where E_β denotes the expectation with respect to the Gibbs measure μ_β.

3. Some results on integration

In this section we present a variant of a theory developed by J. Bertoin on stochastic integration with respect to processes of bounded α-variation. We will define a stochastic integral with respect to a particular class of processes. For our aims, this variant has the advantage of being quite simple and immediately applicable to the case of a fractional Brownian motion or other processes (see the examples below). In a work in preparation [17], the theory described here is developed in major details.

We consider a closed interval of the real line $\mathbb{R}$, $I=[0,T]$, and a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in I},P)$, where $(\mathcal{F}_t)_{t\in I}$ is a standard filtration. Let $X=(X_t)_{t\in I}$ be an adapted stochastic process defined on such space, with values in $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ (or, more generally, in $(\mathbb{C},\mathcal{B}(\mathbb{C}))$).

For a positive real number α and a subdivision τ of the interval I,

$$\tau=\{t_0,\dots,t_n\colon\ 0=t_0<\dots<t_n=T\},$$

we put

$$Q_\tau^\alpha(X)=|X_0|^\alpha+\sum_{i=0}^{n-1}|X_{t_{i+1}}-X_{t_i}|^\alpha.$$

Definition 4. An adapted process X is *weakly of bounded α-variation on I* if

$$\sup_\tau \mathbb{E}[Q_\tau^\alpha(X)]^{1/\alpha}<+\infty.$$

Remark 5. We recall that, according to Bertoin's definition, an adapted process X, is *of bounded α-variation on I* if it is *continuous*, and $\sup_{\widetilde{\tau}}\mathbb{E}[Q_{\widetilde{\tau}}^\alpha(X)]^{1/\alpha}<+\infty$, where $\widetilde{\tau}$ denotes a *stopping subdivision of* I, that is a collection $T_0,\dots,T_n$ of $(\mathcal{F}_t)$-stopping times such that, for every ω, $\{T_0(\omega),\dots,T_n(\omega)\}$ is a subdivision of I.

We denote by $\mathbb{Q}^{\alpha,d}$ the set of processes which are weakly of bounded α-variation on I. $\mathbb{Q}^{\alpha,d}$ is a vector space and $\sup_\tau\mathbb{E}[Q_\tau^\alpha(X)]^{1/\alpha}$ defines a norm on this space which we denote by $\|\ \|_{\mathbb{Q}^{\alpha,d}}$. It can be proved that $(\mathbb{Q}^{\alpha,d},\|\ \|_{\mathbb{Q}^{\alpha,d}})$ is a Banach space for every $\alpha\geqslant 1$, see [17].

Example 6. If X is a square-integrable $(\mathcal{F}_t)$-martingale with $X_0\equiv 0$, then $X\in\mathbb{Q}^{2,d}$. Indeed, if $\tau=\{t_0,\dots,t_n\}$ is a subdivision of I, we have $\mathbb{E}[X_{t_{i+1}}X_{t_i}]=\mathbb{E}[\mathbb{E}[X_{t_{i+1}}X_{t_i}|\mathcal{F}_{t_i}]=\mathbb{E}[X_{t_i}]^2$ for $i=0,\dots,n$ and so

$$\mathbb{E}\bigg[\sum_{i=0}^{n-1}(X_{t_{i+1}}-X_{t_i})^2\bigg]=\mathbb{E}\bigg[\sum_{i=0}^{n-1}(X_{t_{i+1}}^2-X_{t_i}^2)\bigg]=\mathbb{E}[X_T^2].$$

We fix $1 < \alpha < 2$ and show that, with respect to some processes of $\mathbb{Q}^{\alpha,d}$, there exists a simple way of defining a stochastic integral as a limit of Riemann sums. Suppose that Y is a continuous adapted process satisfying the following condition:

(H) For every $p \geqslant 1$ there exists a constant c_p such that

$$\mathbb{E}[|Y_t - Y_s|^p] \leqslant c_p |t-s|^{p/\alpha} \quad \forall s,t \in I.$$

A simple computation shows that $Y \in \mathbb{Q}^{\gamma,d}$ for every $\gamma \geqslant \alpha$.

Remark 7. Actually, we need the above condition only for some values of p, depending on the result we want to obtain. This values are possibly high, therefore we take the hypothesis (H) in order to simplify the exposition. It will be immediate for the reader, if necessary for some application, to set the weakest hypothesis on p.

Remark 8. The hypothesis (H) on the integrator process is not so far from the one made by Bertoin: indeed, it can be proved [17] that for every process of bounded α-variation there exists a "change of time scale" yielding property (H).

Example 9. Let $Y = (Y_t)_{t\in\mathbb{R}^+}$ be a self-similar process with parameter H (that is, for every real positive number a, the processes $(1/a^H\, Y_{at})_{t\in\mathbb{R}^+}$ and Y have the same law). Suppose that Y has stationary increments and $Y_T \in L^p(P)$ for every $p \geqslant 1$. Then for every $s,t \in I$ and $p \geqslant 1$ we have $\mathbb{E}[|Y_t - Y_s|^p] = \mathbb{E}[|Y_{t-s} - Y_0|^p] = \mathbb{E}[|Y_T|^p](1/T)|t-s|^{Hp}$, and so Y satisfies the condition (H) with $\alpha = 1/H$.

For the sake of simplicity we will suppose that $I = [0,1]$. For two processes $X = (X_t)_{t\in I}$ and $Y = (Y_t)_{t\in I}$ we consider, for $n \in \mathbb{N}$, the random variable

$$J_n = \sum_{k=0}^{2^n-1} X_{k/2^n}\big(Y_{(k+1)/2^n} - Y_{k/2^n}\big).$$

Theorem 10. *If $X \in \mathbb{Q}^{\beta,d}$ and Y satisfies the condition* (H) *with $\beta \geqslant 2$ and $1/\alpha + 1/\beta > 1$, then the sequence $(J_n)_{n\in\mathbb{N}}$ converges in $L^1(P)$ to a random variable J_0^1 which we denote by*

$$\int_0^1 X_t \,\mathrm{d}Y_t.$$

Moreover, we have the inequality

$$\mathbb{E}\left[\left|\int_0^1 X_s \,\mathrm{d}Y_s\right|\right] \leqslant \|X\|_{\mathbb{Q}^{\beta,d}}\, c_{\beta'}^{1/\beta'}(1+C)$$

where β' denotes the conjugate exponent of β, $c_{\beta'}$ is the constant of the hypothesis (H) *and C is a constant depending only on α and β.*

P r o o f. For every n we have

$$
\begin{aligned}
&J_n - J_{n-1}\\
&= \sum_{k=0}^{2^n-1} X_{k/2^n}\bigl(Y_{(k+1)/2^n} - Y_{k/2^n}\bigr) - \sum_{h=0}^{2^{n-1}-1} X_{h/2^{n-1}}\bigl(Y_{(h+1)/2^{n-1}} - Y_{h/2^{n-1}}\bigr)\\
&= \sum_{h=0}^{2^{n-1}-1} X_{2h/2^n}\bigl(Y_{(2h+1)/2^n} - Y_{2h/2^n}\bigr) + \sum_{h=0}^{2^{n-1}-1} X_{(2h+1)/2^n}\bigl(Y_{(2h+2)/2^n} - Y_{(2h+1)/2^n}\bigr)\\
&\quad - \sum_{h=0}^{2^{n-1}-1} X_{2h/2^n}\bigl(Y_{(2h+2)/2^n} - Y_{2h/2^n}\bigr)\\
&= \sum_{h=0}^{2^{n-1}-1} X_{2h/2^n} Y_{(2h+1)/2^n} + \sum_{h=0}^{2^{n-1}-1} X_{(2h+1)/2^n}\bigl(Y_{(2h+2)/2^n} - Y_{(2h+1)/2^n}\bigr)\\
&\quad - \sum_{h=0}^{2^{n-1}-1} X_{2h/2^n} Y_{(2h+2)/2^n}\\
&= \sum_{h=0}^{2^{n-1}-1} \bigl(X_{(2h+1)/2^n} - X_{2h/2^n}\bigr)\bigl(Y_{(2h+2)/2^n} - Y_{(2h+1)/2^n}\bigr).
\end{aligned}
$$

Let us use the following notation: $\Delta_{2h}X = X_{(2h+1)/2^n} - X_{2h/2^n}$ and $\Delta_{2h+1}Y = Y_{(2h+2)/2^n} - Y_{(2h+1)/2^n}$. So we can write

$$
\mathbb{E}\bigl[|J_n - J_{n-1}|\bigr] = \mathbb{E}\Biggl[\Biggl|\sum_{h=0}^{2^{n-1}-1} \Delta_{2h}X\Delta_{2h+1}Y\Biggr|\Biggr] \leqslant \sum_{h=0}^{2^{n-1}-1} \mathbb{E}\bigl[|\Delta_{2h}X||\Delta_{2h+1}Y|\bigr]
$$

and from Hölder's inequality

$$
\begin{aligned}
&\leqslant \sum_{h=0}^{2^{n-1}-1} \mathbb{E}\bigl[|\Delta_{2h}X|^{\beta}\bigr]^{1/\beta}\mathbb{E}\bigl[|\Delta_{2h+1}Y|^{\beta'}\bigr]^{1/\beta'}\\
&\leqslant \sum_{h=0}^{2^{n-1}-1} \mathbb{E}\bigl[|\Delta_{2h}X|^{\beta}\bigr]^{1/\beta} c_{\beta'}^{1/\beta'} 2^{-n/\alpha}.
\end{aligned}
$$

Then, from Young's inequality we obtain

$$
\begin{aligned}
\mathbb{E}\bigl[|J_n - J_{n-1}|\bigr] &\leqslant \Biggl(\sum_{h=0}^{2^{n-1}-1} \mathbb{E}[|\Delta_{2h}X|^{\beta}]\Biggr)^{1/\beta} c_{\beta'}^{1/\beta'} 2^{-n(1/\alpha-1/\beta)}\\
&\leqslant \|X\|_{\mathbb{Q}^{\beta,d}}\, c_{\beta'}^{1/\beta'} 2^{-n(1/\alpha-1/\beta)}.
\end{aligned}
$$

From the assumptions on α and β it follows that $1/\alpha - 1/\beta' > 0$, so the sum $\sum_{1}^{+\infty}(J_n - J_{n-1})$ is absolutely convergent in $L^1(P)$. Hence there exists $J_0^1 \in L^1(P)$ such that

$$\left\| J_0 + \sum_{k=1}^{n}(J_k - J_{k-1}) - J_0^1 \right\|_{L^1} = \|J_n - J_0^1\|_{L^1} \xrightarrow[n\to+\infty]{} 0.$$

Finally, if we put $C = \sum_{n=1}^{+\infty} 2^{-n(1/\alpha-1/\beta')}$, since $\mathbb{E}\big[|J_0^1|\big] = \mathbb{E}\big[|J_0 + \sum_{k=1}^{+\infty}(J_k - J_{k-1})|\big]$, we obtain

$$\mathbb{E}\big[|J_0^1|\big] \leqslant \mathbb{E}\big[|J_0|\big] + \sum_{k=1}^{+\infty}\mathbb{E}\big[|J_k - J_{k-1}|\big] \leqslant \|X\|_{\mathbb{Q}^{\beta,d}}\, c_{\beta'}^{1/\beta'}(1+C).$$

□

Remark 11. The same arguments can be applied to any interval $I = [0,T]$. In particular, from the above theorem it follows that, for every $t \in [0,T]$,

$$\mathbb{E}\big[|J_0^t|\big] \leqslant \|X\|_{\mathbb{Q}^{\beta,d}}\, c_{\beta'}^{1/\beta'}(t + Ct^{1/\beta'}).$$

Indeed, for $t \in [0,T]$, the Riemann sums $(J_n)_n$ are of the type

$$J_n = \sum_{k=0}^{\lfloor 2^n t\rfloor -1} X_{k/2^n}\big(Y_{(k+1)/2^n} - Y_{k/2^n}\big) + X_{\lfloor 2^n t\rfloor}(Y_t - Y_{\lfloor 2^n t\rfloor})$$

and the random variable J_0 satisfies the inequality

$$\begin{aligned}
\mathbb{E}\big[|J_0|\big] &\leqslant \sum_{k=0}^{\lfloor t\rfloor -1}\mathbb{E}\big[|X_k|\,|Y_{k+1} - Y_k|\big] + \mathbb{E}\big[|X_{\lfloor t\rfloor}|, |Y_t - Y_{\lfloor t\rfloor}|\big] \\
&\leqslant \sum_{k=0}^{\lfloor t\rfloor -1}\Big(\mathbb{E}\big[|X_k|^\beta\big]^{1/\beta}\mathbb{E}\big[|Y_{k+1} - Y_k|^{\beta'}\big]^{1/\beta'}\Big) \\
&\quad + \mathbb{E}\big[|X_{\lfloor t\rfloor}|^\beta\big]^{1/\beta}\mathbb{E}\big[|Y_t - Y_{\lfloor t\rfloor}|^{\beta'}\big]^{1/\beta'} \\
&\leqslant t\|X\|_{\mathbb{Q}^{\beta,d}}\, c_{\beta'}^{1/\beta'}.
\end{aligned}$$

We can find similar results for the moments of order greater than one of the integral:

Proposition 12. *Suppose that $\beta > 2$ and $X \in \mathbb{Q}^{\beta,d}$, and let Y be a process which verifies the hypothesis* (H). *If we put $\gamma = 2\beta/(\beta-2)$, then the sequence $(J_n)_{n\in\mathbb{N}}$ converges in $L^2(P)$ to J_0^1 and we have*

$$\left\|\int_0^1 X_t\,\mathrm{d}Y_t\right\|_{L^2} \leqslant \|X\|_{\mathbb{Q}^{\beta,d}}\, c_\gamma^{1/\gamma}(1+C).$$

P r o o f. The proof is similar to the previous one. For every n we have

$$\|J_n - J_{n-1}\|_{L^2} = \left\| \sum_{k=0}^{2^{n-1}-1} \Delta_{2k} X \Delta_{2k+1} Y \right\|_{L^2} \leqslant \sum_{k=0}^{2^{n-1}-1} \|\Delta_{2k} X \Delta_{2k+1} Y\|_{L^2}$$
$$= \sum_{k=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2k} X|^2 |\Delta_{2k+1} Y|^2\big]^{1/2}.$$

The condition $\beta > 2$ implies $|\Delta_{2k} X|^2 \in L^{\beta/2}(P)$ and thus, since $\beta/(\beta-2)$ is the conjugate exponent of $\beta/2$, for the above expression we have

$$\leqslant \sum_{k=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2k} X|^{\beta}\big]^{1/\beta} \mathbb{E}\Big[|\Delta_{2k+1} Y|^{2\beta/(\beta-2)}\Big]^{(\beta-2)/2\beta}$$
$$\times \sum_{k=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2k} X|^{\beta}\big]^{1/\beta} c_{\gamma}^{1/\gamma} \frac{1}{2^{n/\alpha}}$$
$$\leqslant \left(\sum_{k=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2k} X|^{\beta}\big] \right)^{1/\beta} c_{\gamma}^{1/\gamma} \frac{1}{2^{n/\alpha}} 2^{n/\beta'}$$
$$\leqslant \|X\|_{\mathbb{Q}^{\beta,d}} c_{\gamma}^{1/\gamma} 2^{-n(1/\alpha - 1/\beta')}.$$

Thus the sum $\sum\limits_{n=0}^{+\infty} (J_n - J_{n-1})$ is absolutely convergent in $L^2(P)$, therefore $(J_n)_{n\in\mathbb{N}}$ converges in $L^2(P)$ to a random variable which clearly is equal to $\int_0^1 X_t \, \mathrm{d}Y_t$ P-a.s. Moreover,

$$\mathbb{E}\left[\left| \int_0^1 X_t \, \mathrm{d}Y_t \right|^2 \right]^{1/2} = \|J_0^1\|_{L^2} = \left\| J_0 + \sum_{n=0}^{+\infty} J_n - J_{n-1} \right\|_{L^2}$$
$$\leqslant \|J_0\|_{L^2} + \sum_{n=0}^{+\infty} \|J_n - J_{n-1}\|_{L^2}$$
$$\leqslant \|X\|_{\mathbb{Q}^{\beta,d}} \, c_{\gamma}^{1/\gamma} (1 + C).$$

□

Remark 13. We can easily see that Stratonovich integration is equivalent to the one considered here. Indeed, if we denote by $\widetilde{J}_n$ the centered finite sums that should converge to the Stratonovich integral,

$$\widetilde{J}_n = \frac{1}{2} \sum_{k=0}^{2^n-1} \big(X_{(k+1)/2^n} + X_{k/2^n}\big)\big(Y_{(k+1)/2^n} - Y_{k/2^n}\big),$$

we have

$$\widetilde{J}_n = J_n + \frac{1}{2} \sum_{k=0}^{2^n-1} \big(X_{(k+1)/2^n} - X_{k/2^n}\big)\big(Y_{(k+1)/2^n} - Y_{k/2^n}\big)$$

and the sum on the right-hand side converges to 0 in $L^1(P)$.

One can prove that the integral just defined, as a function of the upper integration point, is a process in $\mathbb{Q}^{\alpha,d}$. See [17].

4. Rigorous definition of the energy

Given a real number $1 < \alpha < 2$, we take a stochastic process $Y = (Y_t)_{t\in I}$ with values in $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$ which satisfies a condition of the type (H), that is, for every $p \geqslant 1$ there exists a constant c_p such that

$$\mathbb{E}\big[\|Y_t - Y_s\|^p\big] \leqslant c_p|t-s|^{p/\alpha} \;\; \forall s,t \in I. \tag{9}$$

Remark 14. Here $\|.\|$ is the Euclidean norm in $\mathbb{R}^3$. It is equivalent to assume that each component of the process Y satisfies condition (H) of the previous section.

It is clear that the results of the previous section can be extended to the case of processes with values in $\mathbb{R}^3$ or in $\mathbb{C}$. In particular, if we have a process X in $\mathbb{Q}^{\beta,d}$ with values in $\mathbb{C}$ and with $\beta \geqslant 2$ and $1/\alpha + 1/\beta > 1$, we can define the integral $\int_0^T X_t \,\mathrm{d}Y_t$ as a limit of Riemann sums in $\mathbb{C}^3$.

Let us consider the formal definition of the kinetic energy of a vortex filament having as its core a trajectory of the process Y:

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^3} \bigg\| \int_0^T \mathrm{e}^{\mathrm{i}k\cdot Y_t} \,\mathrm{d}Y_t \bigg\|_{\mathbb{C}^3}^2 \frac{|\hat{\varrho}(k)|^2}{\|k\|^2} \,\mathrm{d}k.$$

We want to establish if the integral

$$\int_0^T \mathrm{e}^{\mathrm{i}k\cdot Y_t} \,\mathrm{d}Y_t \tag{10}$$

is well defined and, in such a case, we want to find a condition on the measure ϱ so that we have

$$\mathbb{E}[\mathcal{H}] = \frac{1}{2}\int_{\mathbb{R}^3} \mathbb{E}\Big[\Big\|\int_0^T \mathrm{e}^{\mathrm{i}k\cdot Y_t}\,\mathrm{d}Y_t\Big\|^2_{\mathbb{C}^3}\Big] \frac{|\hat{\varrho}(k)|^2}{\|k\|^2}\,\mathrm{d}k < +\infty.$$

Theorem 15. *If the process Y satisfies condition* (9) *and for the measure ϱ we have, in addition to* (7),

$$\int_{\mathbb{R}^3} \|k\|^{2\alpha-4}|\hat{\varrho}(k)|^2\,\mathrm{d}k < +\infty, \tag{11}$$

then the random variable $\mathcal{H}$ is well defined and $\mathbb{E}[\mathcal{H}] < +\infty$.

Remark 16. The condition

$$\int_{\mathbb{R}^3} |\hat{\varrho}(k)|^2\,\mathrm{d}k < +\infty \tag{12}$$

is equivalent to assuming that ϱ is absolutely continuous with respect to the Lebesgue measure, with a square integrable density. Therefore condition (11) is intermediate between the finite energy condition (7) and the absolutely continuous case (12). Condition (11) still allows us to consider measures ϱ supported by fractal sets, as we would like to have due to numerical investigations [3].

The proof of the theorem is an immediate consequence of the following

Lemma 17. *Under condition* (9), *the integral* (10) *is well defined and we have the inequality*

$$\mathbb{E}\Big[\Big\|\int_0^T \mathrm{e}^{\mathrm{i}k\cdot Y_t}\,\mathrm{d}Y_t\Big\|^2_{\mathbb{C}^3}\Big] \leqslant 3(c_2 + C\|k\|^{2\alpha-2})T^2$$

where c_2 is the constant from (9) *and C is a constant depending only on α and c_4.*

Proof. From the mean-value theorem it follows that, for every $p \geqslant 1$ and $s, t \in I$,

$$\mathbb{E}\big[|\mathrm{e}^{\mathrm{i}k\cdot Y_t} - \mathrm{e}^{\mathrm{i}k\cdot Y_s}|^p\big] \leqslant \mathbb{E}[\|k\|^p\,\|Y_t - Y_s\|^p] \leqslant \|k\|^p c_p |t-s|^{p/\alpha},$$

that is, if we put $Z_t = \mathrm{e}^{\mathrm{i}k\cdot Y_t}$ for $t \in I$, the process $Z = (Z_t)_{t\in I}$ satisfies the condition (H) introduced in the previous section and so it is a process in $\mathbb{Q}^{\beta,d}$ for every $\beta \geqslant \alpha$. Then the integral (10) is well defined. We can suppose $I = [0,1]$. If we take, for $n \in \mathbb{N}$, the Riemann sum in $\mathbb{C}^3$

$$J_n = \sum_{k=0}^{2^n-1} Z_{k/2^n}\big(Y_{(k+1)/2^n} - Y_{k/2^n}\big),$$

from Proposition 12 we know that the sequence $(J_n)_{n\in\mathbb{N}}$ converges in $L^2(P)$ to the random variable $J_0^1 = \int_0^1 Z_t \,\mathrm{d}Y_t$ and we have the inequality

$$\mathbb{E}\big[\|J_0^1\|_{\mathbb{C}^3}^2\big]^{1/2} \leqslant \mathbb{E}\big[\|J_0\|_{\mathbb{C}^3}^2\big]^{1/2} + \sum_{n=1}^{+\infty} \mathbb{E}\big[\|J_n - J_{n-1}\|_{\mathbb{C}^3}^2\big]^{1/2}.$$

Using the same notation of Proposition 12, for $n \geqslant 1$ we have

$$\begin{aligned}\mathbb{E}\big[\|J_n - J_{n-1}\|_{\mathbb{C}^3}^2\big]^{1/2} &\leqslant \sum_{h=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2h} Z|^2 \|\Delta_{2h+1} Y\|_{\mathbb{R}^3}^2\big]^{1/2} \\ &\leqslant \sum_{h=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2h} Z|^4\big]^{1/4} \mathbb{E}\big[\|\Delta_{2h+1} Y\|_{\mathbb{R}^3}^4\big]^{1/4} \\ &\leqslant \sum_{h=0}^{2^{n-1}-1} \mathbb{E}\big[|\Delta_{2h} Z|^4\big]^{1/4} c_4^{1/4} \frac{1}{2^{n/\alpha}}.\end{aligned}$$

Now let N be an integer such that $2^{N-1} < \|k\| \leqslant 2^N$; we know that

$$\mathbb{E}\big[|\Delta_{2h} Z|^4\big]^{1/4} \leqslant 2^N c_4^{1/4} \frac{1}{2^{n/\alpha}},$$

but for small values of n, more precisely for n such that $2^{N-n/\alpha} > 2$ (that is, for $n < N\alpha$), it would be more convenient to use the bound

$$\mathbb{E}\big[|\Delta_{2h} Z|^4\big]^{1/4} \leqslant 2.$$

Then we can write

$$\begin{aligned}(13)\qquad &\sum_{n=1}^{+\infty} \mathbb{E}\big[\|J_n - J_{n-1}\|_{\mathbb{C}^3}^2\big]^{1/2} \\ &= \sum_{n=1}^{\lfloor N\alpha \rfloor} \mathbb{E}\big[\|J_n - J_{n-1}\|_{\mathbb{C}^3}^2\big]^{1/2} + \sum_{n=\lfloor N\alpha \rfloor+1}^{+\infty} \mathbb{E}\big[\|J_n - J_{n-1}\|_{\mathbb{C}^3}^2\big]^{1/2} \\ &\leqslant \sum_{n=1}^{\lfloor N\alpha \rfloor} 2^{n-1} 2 c_4^{1/4} \frac{1}{2^{n/\alpha}} + \sum_{n=\lfloor N\alpha \rfloor+1}^{+\infty} 2^{n-1} 2^N c_4^{1/2} \frac{1}{2^{2n/\alpha}} \\ &= c_4^{1/4} \sum_{n=1}^{\lfloor N\alpha \rfloor} 2^{n(1-1/\alpha)} + \frac{1}{2} c_4^{1/2} 2^N \sum_{n=\lfloor N\alpha \rfloor+1}^{+\infty} 2^{n(1-2/\alpha)}.\end{aligned}$$

The hypothesis $1<\alpha<2$ implies $2^{1-1/\alpha}>1$ and $2^{1-2/\alpha}<1$, so for the first term of the sum (13) we have

$$c_4^{1/4}\sum_{n=1}^{\lfloor N\alpha\rfloor} 2^{n(1-1/\alpha)} = c_4^{1/4}\frac{(2^{1-1/\alpha})^{\lfloor N\alpha\rfloor+1}-2^{1-1/\alpha}}{2^{1-1/\alpha}-1} \leqslant C_1 2^{N(\alpha-1)}$$

and for the second

$$\frac{1}{2}c_4^{1/2}2^N \sum_{n=\lfloor N\alpha\rfloor+1}^{+\infty} 2^{n(1-2/\alpha)} = \frac{1}{2}c_4^{1/2}2^N\frac{(2^{1-2/\alpha})^{\lfloor N\alpha\rfloor+1}}{1-2^{1-2/\alpha}} \leqslant C_2 2^{N(\alpha-1)}$$

where C_1 and C_2 are two constants depending only on α and c_4. Moreover, we notice that

$$\mathbb{E}\big[\|J_0\|_{\mathbb{C}^3}^2\big]^{1/2} = \mathbb{E}\big[\|\mathrm{e}^{\mathrm{i}k\cdot Y_0}(Y_1-Y_0)\|_{\mathbb{C}^3}^2\big]^{1/2} \leqslant c_2^{1/2}.$$

Since $2^N<2\|k\|$, we obtain the inequality

$$\mathbb{E}\big[\|J_0^1\|_{\mathbb{C}^3}^2\big]^{1/2} \leqslant c_2^{1/2} + (C_1+C_2)2^{\alpha-1}\|k\|^{\alpha-1}.$$

Then, if we denote by C the constant $(C_1+C_2)^2 2^{2\alpha-2}$, we conclude

$$\mathbb{E}\left[\left\|\int_0^T \mathrm{e}^{\mathrm{i}k\cdot Y_t}\,\mathrm{d}Y_t\right\|_{\mathbb{C}^3}^2\right] \leqslant 3(c_2 + C\|k\|^{2\alpha-2})T^2.$$

□

5. Some examples

5.1. Fractional Brownian motion.

The canonical example that fits the framework of this paper is the fractional Brownian motion with Hurst parameter in $(\frac{1}{2},1)$. A normalized *fractional Brownian motion* $B^H=(B_t^H)_{t\in\mathbb{R}}$ with Hurst parameter $H\in(0,1)$ is a stochastic process defined on a probability space $(\Omega,\mathcal{F},P)$ with values in $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and such that

1. B^H is Gaussian,
2. B^H has stationary increments, that is, for every $s,t\in\mathbb{R}$, $s\leqslant t$, the random variables $B_t^H-B_s^H$ and $B_{t-s}^H-B_0^H$ have the same law,
3. $\mathbb{E}[B_t^H]=0$ for every $t\in\mathbb{R}$,
4. $\mathbb{E}[(B_t^H)^2]=|t|^{2H}$ for every $t\in\mathbb{R}$.

We notice that, in the case $H = 1/2$, B^H is a Brownian motion. From these properties we obtain the following expression for the covariance:

$$\operatorname{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

for every $s, t \in \mathbb{R}$. As a consequence, we have that B^H is a self-similar process with parameter H and so, having stationary increments and being Gaussian, when $H \in (\frac{1}{2}, 1)$, it satisfies the condition (H) introduced in the previous sections with $\alpha = 1/H$ (see Example 9). So, for a 3-dimensional process whose components are fractional Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$, the energy is well defined, with this value of α in assumption (11). We do not need any independence of the components, and also the Hurst parameters could be different (in $(\frac{1}{2}, 1)$), with the proper choice of α. In the case of a 3-dimensional fractional Brownian motion, a weaker condition on ϱ has been announced in [18].

5.2. Vortices at a solid boundary.

Since $\big||x| - |y|\big| \leqslant |x-y|$, if a process Y satisfies condition (H) then the same is true for the positive process $|Y|$. So, take a 3-dimensional process $Y^0 = (Y^1, Y^2, Y^3)$ whose components are fractional Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$. Then take the process

$$Y = (Y^1, Y^2, |Y^3|).$$

It satisfies (9), hence the results on the energy hold true with $\alpha = 1/H$. The process Y may model a vortex constrained to live in a half space due to the presence of a solid boundary. Such vortices are commonly observed in experiments and simulations.

A variant of this example consists in the process

$$Y' = (Y^1, Y^2, |B|)$$

where (Y^1, Y^2) is as above and B is a 3-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. This process is more convenient if we think that the vortex should not touch the boundary many times. Variants with fractional Brownian bridges can be also defined, to model hairpin vortices at solid boundaries with end-point on the boundary.

These examples are non-Gaussian. Others can be obtained by means of stochastic equations of the form

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}Y_t$$

where b is smooth with bounded derivatives and Y is one of the previous processes. We do not discuss the details.

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Authors' addresses: F. Flandoli, Dipartimento di Matematica Applicata, Università di Pisa, Via Bonanno 25B, 56126 Pisa, Italy, e-mail: `flandoli@dma.unipi.it`; I. Minelli, Dipartimento di Matematica Pura e Applicata, Università di Padova, Via G. Belzoni 7, 35131 Padova, Italy, e-mail: `minelli@mail.dm.unipi.it`.