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IVO VRKOČ SEPTUAGENARIAN

Jan Seidler, Praha

Ten years ago we recorded in this journal [FKM] Ivo Vrkoč's sixtieth birthday (born June 10, 1931). With some exaggeration we may say that nothing substantial has changed: Ivo Vrkoč's life remains devoted to mathematics and his office in the institute remains the most likely place he may be found any day of the week. Albeit he retired from the Mathematical Institute of the Academy of Sciences in January 1997, he has still been tied with it by the grant projects he participates in.

Simultaneously, I. Vrkoč was with the Institute of Entomology of the Academy of Sciences in České Budějovice in southern Bohemia in the years 1998–2000. In this way he might resume his cooperation with his former Ph.D. student Vlastimil Křivan, with whom he had already published two joint papers on differential inclusions ([57] being the second of them). Their recent research was motivated by mathematical ecology and problems of biodiversity and resulted in three papers [65], [69] and [70].

The fact that Ivo Vrkoč was able to start successfully a study of a problem so new for him, as animal distribution in heterogeneous environment is, cannot surprise anybody who knows him well. He has never been much interested in developing systematically abstract theories, he enjoys solving concrete problems requiring more a new bright insight than a huge preliminary knowledge. Much of his work stemmed from his discussions with colleagues and from problems posed by them. The unusually wide range of problems he addressed in his papers is closely connected with this basic attitude of his.

Therefore, it is not easy to describe in few words the results Ivo Vrkoč has obtained in the last ten years. We would like to treat now in some detail three of the topics that have attracted his attention, but first let us at least mention the other ones. Besides the work inspired by biology, I. Vrkoč continued his long-lasting collaboration with the physicist Jan Fischer and they coauthored two papers [63], [64] on operator-product expansions in quantum chromodynamics. Continuous dependence of solutions to stochastic partial differential equations on coefficients and the related averaging theorems were studied in the papers [56] and [59], which complemented a series of papers on the same problem I. Vrkoč published in the eighties (cf. [FKM]).

The paper [58] is formally devoted to ordinary differential equations in Hilbert spaces, but its importance becomes clear only if we realize its close relation to ergodic theory of stochastic partial differential equations. Let us consider a Markov process $(X, \boldsymbol{P}_x)$ in $\mathbb{R}^d$ defined by a stochastic differential equation

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}w_t \tag{1}$$

driven by a Wiener process w. A standard procedure, based on an argument due to N. N. Bogolyubov and N. M. Krylov, may be used to show that there exists an invariant probability measure for X if the process X is Feller and the equation (1) admits at least one solution bounded in probability. Since the proof uses relative compactness of balls in $\mathbb{R}^d$ it cannot be extended immediately to stochastic PDEs (modelled as stochastic equations in an infinite dimensional state space). A suitable generalization in this direction was obtained by G. Da Prato, D. Gątarek and J. Zabczyk in [DGZ]. It was the first sufficiently general theorem on invariant measures for stochastic PDEs with non-additive noise; however, we recall here their result only in the very particular deterministic case: Let H be a separable Hilbert space, $A\colon \mathrm{dom}(A) \longrightarrow H$ an infinitesimal generator of a strongly continuous semigroup (e^{At}) on H and $f\colon H \longrightarrow H$ a Lipschitz mapping. Then the dynamical system defined by an equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x) \tag{2}$$

in H has an invariant probability measure, provided there is a bounded solution to (2) and the semigroup (e^{At}) is compact. It is quite natural to ask whether this assertion remains true if the compactness hypothesis on (e^{At}) is relaxed. I. Vrkoč in [58] showed that in the case $A = 0$ everything may go wrong. In any infinite dimensional Hilbert space H there exists a bounded Lipschitz continuous function $f\colon H \longrightarrow H$ such that $f(0) \neq 0$, nonetheless, all solutions to the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{3}$$

converge weakly to 0 as $t \to \infty$. It follows that every solution to (3) has an empty Ω-set, consequently, there is no (nontrivial) finite Borel Radon measure invariant for (3). If H is separable, (3) has an additional interesting property: there exists a sequence of finite dimensional Galerkin approximations to (3), each of this approximations has an invariant probability measure, these invariant measures converge weakly, but the limit measure is of course not invariant. The function f with the above properties is constructed explicitly in [58], the construction using only elementary tools being, however, quite complicated from the technical point of view.

To sketch the results contained in the paper [60] we have to introduce basic definitions concerning the Kurzweil (or Kurzweil-Henstock) integral. We say that $D = \{(\alpha_i), (\tau_i)\}$ is a (tagged) partition of an interval $[a, b] \subseteq \mathbb{R}$ if $a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$ and $\tau_i \in [\alpha_{i-1}, \alpha_i]$, $i = 1, \ldots, k$. Given a strictly positive function $\delta\colon [a, b] \longrightarrow]0, \infty[$ (henceforward, we shall call any such function a gauge) we say that the partition D is δ-fine provided that

$$[\alpha_{i-1}, \alpha_i] \subseteq]\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)[, \quad i = 1, \ldots, k.$$

For a real function $f\colon [a, b] \longrightarrow \mathbb{R}$ and a tagged partition $D = \{(\alpha_i), (\tau_i)\}$ we set

$$S(f, D) = \sum_{i=1}^{k} f(\tau_i)(\alpha_i - \alpha_{i-1})$$

and we call the function f Kurzweil integrable if $I \in \mathbb{R}$ may be found satisfying: for any $\varepsilon > 0$ there is a gauge δ such that

$$|I - S(f, D)| < \varepsilon$$

whenever D is a δ-fine partition of $[a, b]$. We put

$$I = (\mathscr{K})\text{-}\int_a^b f(x)\,\mathrm{d}x,$$

the Kurzweil integral of f. It is known that the Kurzweil integral is equivalent to the Perron one, its distinctive feature being its similarity to the classical Riemann's approach to integration (it suffices to replace the gauge δ in the above definition by a constant $\delta > 0$ to obtain the Riemann integral). A basic problem of any integration theory is to establish good convergence theorems: if f_n are integrable and $f_n \to f$ pointwise on $[a, b]$, find hypotheses implying that f is integrable and

$$(\mathscr{K})\text{-}\int_a^b f_n(x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} (\mathscr{K})\text{-}\int_a^b f(x)\,\mathrm{d}x \tag{4}$$

holds true. The definition of the Kurzweil integral indicates that to prove (4) it is necessary to interchange two limit passages: the convergence of f_n's to f and the convergence of the integral sums $S(f_n, D)$ to the integral of f_n. Usually, to interchange two limit processes some type of uniform convergence for one of them is needed. Indeed, if $f_n \to f$ uniformly on $[a,b]$ then (4) holds, unfortunately, such a theorem is of a little use. On the other hand, it was noted by D. Preiss and Š. Schwabik that if the functions f_n are supposed to be uniformly Kurzweil integrable, one gets a strong convergence theorem yielding also the dominated and monotone convergence theorems (see [Ku], Kapitel 5). More precisely: we say that a set H of Kurzweil integrable functions is equiintegrable if for any $\varepsilon > 0$ there exists a gauge δ such that

$$\left|(\mathscr{K})\text{-}\int_a^b f(x)\,\mathrm{d}x - S(f,D)\right| < \varepsilon$$

for all $f \in H$ and all δ-fine partitions D. Assume that f_n are Kurzweil integrable functions, $f_n \to f$ pointwise on $[a,b]$ and the set $\{f_n;\ n \geqslant 1\}$ is equiintegrable. Then f is Kurzweil integrable and (4) holds.

In the paper [60], the necessity of the equiintegrability assumption is thoroughly studied. In particular, it is proven that if a sequence $\{f_n\}$ of nonnegative Kurzweil integrable functions satisfies $f_n \to 0$ pointwise on $[a,b]$ and

$$(\mathscr{K})\text{-}\int_a^b f_n(x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} 0$$

then $\{f_n\}$ is equiintegrable. It follows that if g_n are absolutely Kurzweil integrable functions (i.e., Lebesgue integrable) and $g_n \to g$ in $L^1([a,b])$ then $\{g_n\}_{n\geqslant 1}$ is again equiintegrable. Therefore, equiintegrability plays essentially the same rôle in the theory of the Kurzweil integral as Vitali's notion of uniform integrability plays in that of the Lebesgue one.

Finally, we would like to mention briefly the papers [66]–[68], concerning nonlinear periodic boundary value problems of the form

$$\ddot{u} = f(t,u,\dot{u}), \quad u(a) = u(b),\ \ \dot{u}(a) = w(\dot{u}(b)), \tag{5}$$

where $w\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous nondecreasing function and $f\colon [a,b]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions. Since the work of G. Scorza Dragoni in the early thirties of the last century it has been known that existence of solutions to (5) may be investigated by means of upper and lower functions. Let us recall that a pair of smooth functions (σ,ϱ) is a lower function (or a lower solution) of (5) if

$$\dot{\sigma}(t) = \varrho(t),\ \ \dot{\varrho}(t) \geqslant f(t,\sigma(t),\varrho(t)), \quad t \in [a,b]; \tag{6}$$

an upper function is defined analogously. Recently, I. Rachůnková and M. Tvrdý succeeded in showing (see [RT]) that the Leray-Schauder degree theory may be used to study existence and multiplicity of (Carathéodory) solutions to (5) under the hypothesis that suitable *nonsmooth* upper and lower functions exist. Namely, they consider lower functions (σ, ϱ) such that σ is absolutely continuous, ϱ is a function of bounded variation whose singular part is nondecreasing and (6) holds almost everywhere on $[a, b]$.

Ivo Vrkoč got acquainted with this theory through a series of lectures Irena Rachůnková and Milan Tvrdý delivered in Prague and he immediately joined this line of research. In the paper [66], accompanying [RT], two seemingly very different definitions of lower and upper functions are proven to be equivalent. In the papers [67] and [68], the general theory is applied to particular cases of the problem (5), equations with rather singular right hand sides f being treated. To state here any of the theorems established in [67] or [68] would require too many preliminaries, so we are forced to content ourselves with quoting an examples covered by the theory developed in [67]. Let $g\colon\,]0,\infty[\, \longrightarrow \mathbb{R}$ be a continuous function and $e\colon [0,1] \longrightarrow \mathbb{R}$ an integrable function. Suppose that

$$\int_0^1 e(s)\,\mathrm{d}s + \limsup_{x\to\infty} g(x) < 0$$

and

$$e(t) + g(x) + \pi^2 x \geqslant \eta \text{ for some } \eta > 0, \text{ almost every } t \in [0,1] \text{ and every } x \geqslant \eta/\pi^2,$$

then the problem

$$\ddot{u} - g(u) = e(t), \quad u(0) = u(1), \;\; \dot{u}(0) = \dot{u}(1)$$

has a positive solution u satisfying $u \geqslant \eta/\pi^2$ on $[0,1]$. In particular, let us consider a function g given by

$$g(x) = kx - \frac{1}{x^\lambda}, \quad k > 0, \;\; \lambda > 0.$$

The quoted result yields existence of a solution even in the case of a weak singularity (i.e. $\lambda \in\,]0,1[$) and for the critical value $k = \pi^2$ of the parameter k; these cases not having been covered by previously known theorems (due to A. C. Lazer and S. Solemini, J. Mawhin and others).

Ivo Vrkoč is a very modest man, indifferent towards worldly success. His aim in mathematics has always been to achieve deep understanding, not to produce a stream of papers. He did not publish all his results, only those he found really interesting and enjoyable. But he has been always prepared to help his colleagues and friends and many of us have profited considerably from his nice ideas and his skill in finding surprising examples and counterexamples. Both his publications and his influence on his colleagues and students turn Ivo Vrkoč into one of the main figures of the Czechoslovak mathematics of the second half of the twentieth century.

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[1] The first part of the list may be found in [FKM], pp. 748–750.