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MULTI-FAITHFUL SPANNING TREES OF INFINITE GRAPHS

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Abstract. For an end τ and a tree T of a graph G we denote respectively by $m(\tau)$ and $m_T(\tau)$ the maximum numbers of pairwise disjoint rays of G and T belonging to τ, and we define $\operatorname{tm}(\tau) := \min\{m_T(\tau)\colon T \text{ is a spanning tree of } G\}$. In this paper we give partial answers—affirmative and negative ones—to the general problem of determining if, for a function f mapping every end τ of G to a cardinal $f(\tau)$ such that $\operatorname{tm}(\tau) \leqslant f(\tau) \leqslant m(\tau)$, there exists a spanning tree T of G such that $m_T(\tau) = f(\tau)$ for every end τ of G.

Keywords: infinite graph, end, end-faithful, spanning tree, multiplicity

MSC 2000: 05C99

1. Introduction

In 1964 Halin [4] introduced the concept of an *end-faithful* subgraph (i.e., a subgraph H of a graph G such that each end of G contains exactly one end of H as a subset), and stated his well-known problem of determining if any connected infinite graph contains an end-faithful spanning tree. This problem, which we showed [11] to be closely related to the one of characterizing the connected graphs which have a rayless spanning tree, has been answered for one-ended graphs by the negative by Seymour and Thomas [15], and later but independently by Thomassen [17].

On account of these negative results it is quite natural to ask if any connected infinite graph G has a spanning tree T such that, for each end τ of G, the maximum number $m_T(\tau)$ of pairwise disjoint rays of T belonging to τ is minimal in the sense that $m_T(\tau) \leqslant m_{T'}(\tau)$ for every spanning tree T' of G. This minimum number will be called the *tree-multiplicity* of τ and denoted by $\operatorname{tm}(\tau)$, the *multiplicity* $m(\tau)$ of τ being the maximum number of pairwise disjoint rays of G belonging to τ.

Another aspect of Halin's problem was considered by Zelinka [18] who conjectured that if G is a connected infinite locally finite graph and τ an end of G, then, for any cardinal k with $1 \leqslant k \leqslant m(\tau)$, there is a spanning tree of G having exactly k ends included in τ. He proved it in the particular case when $m(\tau)$ is finite and τ can be separated from all other ends by a finite set of vertices. Later [12] we completely proved the conjecture and even got more general results of the same type.

In this paper we combine these two variations of Halin's problem by studying the following general problem: *Let G be a connected infinite graph and f a function which maps every end τ of G to a cardinal $f(\tau)$ such that* $\mathrm{tm}(\tau) \leqslant f(\tau) \leqslant m(\tau)$. *Does there exist a spanning tree T of G such that $m_T(\tau) = f(\tau)$ for every end τ of G?*

2. Preliminaries

The terminology will be for the most part that of [13] and [14]. Moreover, in order to get a more self-contained paper, we will recall the results of [10], [12], [13], [14] that we will need. In particular, throughout this paper, by a *countable* set we will mean a set whose cardinality is at most $\aleph_0$, that is a set which is either finite or countably infinite.

Graphs considered in this paper are undirected and contain neither loops nor multiple edges. For a set A of vertices of a graph G we denote by $G[A]$ the subgraph of G induced by A. If A is any set of vertices and H any graph, we define $G - A := G[V(G) - A]$ and $G - H := G - V(H)$. If A is a set of edges of G, we will denote by $G \setminus A$ the spanning subgraph of G whose edge set is $E(G) - A$. The *union* of a family $(G_i)_{i\in I}$ of graphs is the graph $\bigcup_{i\in I} G_i$ given by $V(\bigcup_{i\in I} G_i) = \bigcup_{i\in I} V(G_i)$ and $E(\bigcup_{i\in I} G_i) = \bigcup_{i\in I} E(G_i)$. The *intersection* is defined analogously. If $(G_i)_{i\in I}$ is a family of subgraphs of a graph G, the subgraph induced by the union of this family will be denoted by $\bigvee_{i\in I} G_i$. For $x \in V(G)$ the set $N_G(x) := \{y \in V(G)\colon \{x, y\} \in E(G)\}$ is the *neighbourhood* of x in G. If H is a subgraph of G and X a nonempty subgraph of $G - H$, the boundary of H with X is the set $\mathcal{B}(H, X) := \{x \in V(H)\colon\ N_G(x) \cap V(X) \neq \emptyset\}$. The set of components of G is denoted by $\mathcal{C}_G$, and if x is a vertex, then $\mathcal{C}_G(x)$ is the component of G containing x. If H is an induced subgraph of a graph G and N an induced subgraph of a component X of $G - H$, then we set $N + (H) := N \vee G[\mathcal{B}(H, X)]$. A *path* $P = \langle x_0, \ldots, x_n\rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{\{x_i, x_{i+1}\}\colon\ 0 \leqslant i < n\}$. A *ray* is a one-way infinite path $\langle x_0, x_1, \ldots\rangle$. A subray of a ray R is called a *tail* of R.

The ends of a graph G (a concept introduced by Freudenthal [2] and Hopf [7] to study discrete groups, and independently by Halin [4]) are the classes of the equivalence relation $\sim_G$ defined on the set of all rays of G by: $R \sim_G R'$ if and only if there is a ray R'' whose intersections with R and R' are infinite; or equivalently if and only if $\mathcal{C}_{G-S}(R) = \mathcal{C}_{G-S}(R')$ for each finite $S \subseteq V(G)$ (where $\mathcal{C}_{G-S}(R)$ denotes the component of $G - S$ containing a tail of R). We will denote by $[R]_G$ the class of a ray R of G modulo $\sim_G$, by $\mathfrak{T}(G)$ the set of all ends of G, and for $\tau \in \mathfrak{T}(G)$ and any finite $S \subseteq V(G)$, by $\mathcal{C}_{G-S}(\tau)$ the component of $G - S$ which contains some ray belonging to τ. Notice that if G is a tree, then two rays of G are equivalent modulo $\sim_G$ if and only if they have a common tail; hence two disjoint rays of a tree correspond to different ends of this tree.

A subgraph H of G is *end-respecting* (or *end-faithful* (or *coterminal* in [10])) if the map $\varepsilon_{HG}\colon \mathfrak{T}(H) \to \mathfrak{T}(G)$ given by $\varepsilon_{HG}([R]_H) = [R]_G$ for every ray R of H, is injective (bijective, respectively). We denote by $\mathfrak{T}_H(G)$ the image of ε_{HG}, i.e. the set of ends of G having rays of H as elements. Furthermore, for $\mathcal{A} \subseteq \mathfrak{T}(G)$, we set $\mathcal{A}(H) := \mathcal{A} \cap \mathfrak{T}_H(G)$.

Throughout this paper, we will assume that the end set $\mathfrak{T}(G)$ of a graph G is endowed with the topology introduced by Jung [8], called the *end topology*, for which the closure of a subset $\mathcal{A}$ of $\mathfrak{T}(G)$ is the set

$$\begin{aligned}\overline{\mathcal{A}} := \{\tau \in \mathfrak{T}(G)\colon &\text{ for each finite } S \in V(G) \text{ there is } \tau' \in \mathcal{A}\\ &\text{ such that } \mathcal{C}_{G-S}(\tau) = \mathcal{C}_{G-S}(\tau')\},\end{aligned}$$

i.e., is the set of all ends which cannot be separated by a finite $S \subseteq V(G)$ from $\mathcal{A}$.

By [13, Theorem 4.8] the end space $\mathfrak{T}(G)$ of a graph G is scattered (i.e., contains no non-empty subset which is dense in itself) if and only if G has no subdivision of the binary tree as an end-respecting subgraph. Furthermore, by [13, Proposition 4.7], the end space of the binary tree is homeomorphic with the Cantor space 2^ω. Therefore, the cardinality of the end set of a countable graph G is at most $\aleph_0$ or exactly $2^{\aleph_0}$ if $\mathfrak{T}(G)$ is scattered or not, respectively.

For $\mathcal{A} \subseteq \mathfrak{T}(G)$ we define

$$m(\mathcal{A}) := \sup\bigl\{|\mathcal{R}|\colon \mathcal{R} \text{ is a set of pairwise disjoint elements of } \bigcup\mathcal{A}\bigr\}.$$

For $\tau \in \mathfrak{T}(G)$ we write $m(\tau)$ for $m(\{\tau\})$, and call it the *multiplicity* (or *thickness*) of τ. By [11, 11.5] the supremum is attained, i.e. there is a set of pairwise disjoint rays in $\bigcup\mathcal{A}$ of cardinality $m(\mathcal{A})$. This was already proved by Halin [5, Satz 1] and [6, Satz 1] when $\mathcal{A} = \mathfrak{T}(G)$ and $|\mathcal{A}| = 1$, respectively.

For a subgraph H of G, we set $m_H(\tau) := m(\varepsilon_{HG}^{-1}(\tau))$. By the remark we made about ends of trees, we can note that if H is a tree, then H is end-respecting (end-faithful) if and only if $m_H(\tau) \leqslant 1$ ($= 1$, respectively), for every end τ of G.

We will denote by $\mathcal{D}$ (or by $\mathcal{D}_G$ if necessary) the relation between $V(G)$ and $\mathfrak{T}(G)$ defined by $x \mathrel{\mathcal{D}} \tau$ if $x \in V(\mathcal{C}_{G-S}(\tau))$ for every finite $S \subseteq V(G - x)$, or equivalently if there exists an infinite set of paths joining x to the vertex set of some ray $R \in \tau$ and having pairwise only x in common. If $x \mathcal{D} \tau$ then we will say that the vertex x *dominates* the end τ (or is a *neighbour* of τ in [10]), or that τ is *dominated* by x. For $x \in V(G)$ (or $\tau \in \mathfrak{T}(G)$) we will denote by $\mathcal{D}(x)$ (or $\mathcal{D}^{-1}(\tau)$) the set of all ends τ (all vertices x, respectively) such that $x \mathcal{D} \tau$.

An infinite subset S of $V(G)$ is *concentrated* in G if there is an end τ such that $S - V(\mathcal{C}_{G-F}(\tau))$ is finite for each finite $F \subseteq V(G)$ (we also say that S is *"concentrated in τ"*). For example, the vertex set of any ray of a graph G is concentrated in G. Note that every infinite subset of a concentrated set is also concentrated.

A set S of vertices of G is *dispersed* if it has no concentrated subset. Clearly, any finite set of vertices is dispersed, and every subset of a dispersed set is dispersed as well.

An induced subgraph M of a graph G is called a *multi-ending* of G if it possesses the following properties:

M1. M is connected.

M2. The boundary of M with every component of $G - M$ is finite.

M3. Any infinite subset of $V(M)$ which is concentrated in G is also concentrated in M.

M4. $\mathcal{D}_M^{-1}(\tau) = \mathcal{D}_G^{-1}(\varepsilon_{MG}(\tau))$ for each end τ of M.

M5. For any family $(R_i)_{i \in I}$ of pairwise disjoint rays of G such that $\{[R_i]_G \colon i \in I\} \subseteq \mathfrak{T}_M(G)$ there is a family $(R_i')_{i \in I}$ of pairwise disjoint rays of M such that $R_i \cap R_i'$ is infinite for every $i \in I$.

By M3, a multi-ending of G is an end-respecting subgraph of G. By M5, $m(\tau) = m(\varepsilon_{MG}(\tau))$ for every end τ of M. A multi-ending which is rayless is called a 0-*ending*. A 0-ending M is then a connected induced subgraph of G, whose vertex set is dispersed and whose boundary with each component of $G - M$ is finite. A multi-ending M is an *ending* if $|\mathfrak{T}(M)| = 1$; it is a *discrete* multi-ending if $\mathfrak{T}_M(G)$ is a discrete subspace of $\mathfrak{T}(G)$.

For any subset $\mathcal{A}$ of $\mathfrak{T}(G)$ we denote by $\mathbb{M}(a)$ the set of all multi-endings M of G such that $\mathcal{A} = \mathfrak{T}_M(G)$.

Lemma 2.1 ([14, 6.5 (ii) and 7.9])**.** *$\mathbb{M}(a) \neq \emptyset$ if and only if $\mathcal{A}$ is a closed set.*

In particular, $\mathbb{M}(\tau) \neq \emptyset$ for every end τ, since the end topology is Hausdorff.

Lemma 2.2 ([13, 4.15] and [14, 6.11])**.** *Let G be a graph. For any closed discrete subspace Ω of $\mathfrak{T}(G)$ there exists a 0-ending M of G which pairwise separates the elements of Ω, i.e., $\mathcal{C}_{G-S}(\tau') \neq \mathcal{C}_{G-S}(\tau)$ for every pair $\{\tau, \tau'\}$ of distinct elements of Ω.*

Lemma 2.3 ([14, 6.10 and 6.15]). *For every induced subgraph H of G satisfying* M3 *there exists a multi-ending M of G which contains H and satisfies $\mathfrak{T}_M(G) = \mathfrak{T}_H(G)$.*

An immediate consequence of this result and the fact that, if some cofinite subset of a set S is concentrated, then S is concentrated as well, is the following assertion.

Corollary 2.4. *For every multi-ending N of G and every finite $A \subseteq V(G)$ there exists a multi-ending M of G such that $A \cup V(N) \subseteq V(M)$ and $\mathfrak{T}_M(G) = \mathfrak{T}_N(G)$.*

Lemma 2.5 ([14, 6.17]). *Let H be a connected induced subgraph of a graph G whose boundary with each component of $G - H$ is finite. Then any multi-ending of H is a multi-ending of G.*

Lemma 2.6 ([14, 6.19]). *Let M be a multi-ending of a graph G and X a component of $G - M$. Then any induced subgraph N of X satisfying Axiom* M3 *can be extended to a multi-ending N' of X with the following properties:*
(i) *N' contains a neighbour of each element of $\mathcal{B}(M, X)$;*
(ii) *$\mathfrak{T}_{N'}(G) = \mathfrak{T}_N(G)$;*
(iii) *$N' + (M)$ is a multi-ending of $X + (M)$.*

Lemma 2.7 ([14, 6.18]). *Let N be a multi-ending of G and, for every component X of $G - N$, let N_X be a multi-ending of $X + (N)$ containing $\mathcal{B}(N, X)$. Then $M := N \vee \bigcup_{X \in \mathcal{C}_{G-N}} N_X$ is a multi-ending of G such that $\mathfrak{T}_M(G) = \mathfrak{T}_N(G) \cup \bigcup_{X \in \mathcal{C}_{G-N}} \mathfrak{T}_{N_X}(G)$.*

3. Tree-multiplicity

Definition 3.1. Let G be a one-ended graph. The *tree-multiplicity* of G is the cardinal

$$\mathrm{tm}(G) := \min\{m(T) \colon T \text{ is a spanning tree of } G\}.$$

Seymour and Thomas [15, 1.5] and Thomassen [17] proved that there is a one-ended graph G such that $\aleph_1 \leqslant \mathrm{tm}(G) \leqslant 2^{\aleph_0}$. The next result shows that this example of a one-ended graph having no end-faithful spanning tree is, in a certain sense, the simplest possible such example when assuming the Continuum Hypothesis.

Proposition 3.2. *Let G be one-ended. Then* $\mathrm{tm}(G) > 1$ *implies* $\mathrm{tm}(G) > \aleph_0$.

Proof. Suppose $\mathrm{tm}(G)$ countable, and let T be a spanning tree of G such that $m(T) = \mathrm{tm}(G)$. Let $(R_n)_{n<\mathrm{tm}(G)}$ be a family of pairwise disjoint rays of T which is maximal with respect to inclusion. For all positive integers $n < p < \mathrm{tm}(G)$, since $R_n \sim_G R_P$, there is a ray R_{np} of G which meets R_n and R_p in infinitely many vertices. Let $H := T \cup \bigcup_{0 \leqslant n < p < \mathrm{tm}(G)} R_{np}$. This graph is a one-ended spanning subgraph of G with $m(H) \leqslant \aleph_0$, thus, by [10, 3.4], it contains an end-faithful spanning tree T_H. This tree T_H is then a one-ended spanning tree of G, thus end-faithful with G. $\square$

Lemma 3.3 ([11, 10.1]). *If G is one-ended, then $\mathrm{tm}(G) = 0$, i.e., G has a rayless spanning tree if and only if it has an end-faithful spanning tree and its end is dominated.*

Definition 3.4. Let τ be an end of a graph G. Then the tree-multiplicity of τ is the cardinal

$$\mathrm{tm}_G(\tau) := \min\{m_T(\tau)\colon\ T \text{ is a spanning tree of } G\}.$$

We will write $\mathrm{tm}(\tau)$ for $\mathrm{tm}_G(\tau)$ if no confusion is likely.

Proposition 3.5. *Let τ be an end of a graph G, and let $M \subseteq \mathbb{M}(\tau)$. The following assertions hold:*
(i) $\mathrm{tm}_G(\tau) \leqslant \mathrm{tm}(M)$.
(ii) *There is $M' \in \mathbb{M}(\tau)$ such that $M \subseteq M'$ and $\mathrm{tm}_G(\tau) = \mathrm{tm}(M')$.*

Proof. (i) Let T be a spanning tree of M such that $m(T) = \mathrm{tm}(M)$. Extend T to a spanning tree T' of G. Then, since no component of $G - M$ contains a ray belonging to the end τ, it follows that $\mathrm{tm}(\tau) \leqslant m_{T'}(\tau) = m_T(\tau) = \mathrm{tm}(M)$.

(ii) Let T be a spanning tree of G such that $m_T(\tau) = \mathrm{tm}(\tau)$, and x a vertex of M. Let T' be the least subtree of T such that $V(M) \subseteq V(T')$ and that it contains all the rays of T belonging to τ and originating at x. Clearly $\mathfrak{T}_{T'}(G) = \{\tau\}$. Besides, by the minimality of T', the graph $X \cap T'$ is finite for each component X of $G - M$. Thus, by Lemma 2.7, $M' := M \vee T' \in \mathbb{M}(\tau)$, and T' is a spanning tree of M' with $\mathrm{tm}(\tau) = m_{T'}(\tau) \geqslant \mathrm{tm}(M')$. Therefore $\mathrm{tm}(M') = \mathrm{tm}(\tau)$ by (i). $\square$

Definition 3.6. A multi-ending M of G will be said to be *G-perfect* if there is a spanning tree T of M such that $m_T(\tau) = \mathrm{tm}_G(\tau)$ for every $\tau \in \mathfrak{T}_M(G)$.

Note that, if M is G-perfect, then $\mathrm{tm}_M(\varepsilon_{MG}^{-1})) = \mathrm{tm}_G(\tau)$ for each $\tau \in \mathfrak{T}_M(G)$.

Proposition 3.7. *For any discrete multi-ending M of G there exists a G-perfect multi-ending M' of G such that $M \subseteq M'$ and $\mathfrak{T}_M(G) = \mathfrak{T}_{M'}(G)$.*

Notice that, since the boundary of M' with every component of $G - M$ is finite, we can always suppose that every component of $G - M$ contains a ray.

P r o o f. This is Proposition 3.5 (ii) if M is an ending. Assume that $\mathcal{A} := \mathfrak{T}_M(G)$ has more than one element. By Lemma 2.2, since $\mathcal{A}$ is closed and has only isolated points, there is a 0-ending H of G which pairwise separates the elements of $\mathcal{A}$. Denote by Γ the set of components of $G - H$ which are non-disjoint from M, and let $X \in \Gamma$ and $B_X := B(H, X)$. Since M satisfies Axiom M3 of multi-endings and since B_X is finite, the subgraph $M \cap X$ also satisfies M3. Hence, by Corollary 2.4, there is a multi-ending N_X of X which contains $M \cap X$, and with the property that $\mathfrak{T}_{N_X}(G) = \mathfrak{T}_{M\cap X}(G)$. By Lemma 2.6 and Proposition 3.5 (ii), N_X can be extended to a multi-ending M_X of X which contains a neighbour of each element of B_X, with the properties that $M_X + (H)$ is a multi-ending of $X + (H)$, and that $\mathrm{tm}(M_X) = \mathrm{tm}_X(\tau)$ $(= \mathrm{tm}_G(\tau))$ if $\mathfrak{T}_{M\cap X}(G) = \{\tau\}$.

Now let $M' := H \vee \bigcup_{X\in\Gamma} M_X$. This graph contains M by construction. Furthermore, by Lemma 2.7, M' is a multi-ending of $H \vee \bigcup_{X\in\Gamma} X$, hence of G by Lemma 2.5, such that $\mathfrak{T}_{M'}(G) = \mathfrak{T}_H(G) \cup \bigcup_{X\in\Gamma} \mathfrak{T}_{M_X}(G) = \mathcal{A}$. We claim that M' is G-perfect. Let T_H be a spanning tree of H. This tree T_H is rayless since $V(H)$ is dispersed. For $X \in \Gamma$, let T_X be a spanning tree of M_X such that $m_{T_X}(\tau) = \mathrm{tm}(\tau)$ if $X = \mathcal{C}_{G-H}(\tau)$ for some $\tau \in \mathcal{A}$, and which is rayless otherwise. Denote by E_X a subset of the set of edges of T_X which are incident with both B_X and $V(X)$ so that, for each component C of $T_X - B_X$ there is exactly one edge in E_X which is incident with C. Let F_X be the spanning forest of M_X whose edge set is $E(T_X - B_X) \cup E_X$. Then $T := T_H \cup \bigcup_{X\in\Gamma} F_X$ is a spanning tree of M' such that $m_T(\tau) = \mathrm{tm}(\tau)$ for every end $\tau \in \mathcal{A}$. $\square$

4. f-faithful spanning trees

4.1. Definitions and main results.

Definition.

(i) An *end-function* of a graph G is a function f which maps every end τ of G to a cardinal $f(\tau)$ such that $\mathrm{tm}(\tau) \leqslant f(\tau) \leqslant m(\tau)$.

(ii) A spanning tree T of G is said to be *f-faithful* for an end-function f of G, if $m_T(\tau) = f(\tau)$ for every end τ of G.

If $\mathrm{tm}(\tau) \leqslant 1$ (or $= 0$) for every end τ, and if f is the constant end-function mapping every end to 1 (or 0), then an f-faithful spanning tree is an end-faithful (or rayless, respectively) spanning tree.

End-functions of particular interest are tm and tm^*, where tm^* is defined so that $\mathrm{tm}^*(\tau) := \max\{1, \mathrm{tm}(\tau)\}$ for every end τ. Since both results of Seymour and Thomas [15] and Thomassen [17] prove the existence of connected one-ended graphs containing no rayless spanning trees, thus no end-faithful ones, it would be interesting to consider the following problem:

Problem 4.2. Does any connected infinite graph have a tm-faithful spanning tree, and a tm*-faithful spanning tree?

This is obviously meaningful for multi-ended graphs only, since the answer is trivial for one-ended ones. In this paper we will give a partial answer to this problem. We begin with a simple extension of Theorem 2.4 of [12].

Lemma 4.3 ([12, 2.3]). *Let T be a spanning tree of a connected infinite graph G. Let τ_0 be an end of G, and k a cardinal such that $m_T(\tau_0) \leqslant k \leqslant m(\tau_0)$. Then G has a spanning tree T_0 such that $m_{T_0}(\tau_0) = k$ and $m_{T_0}(\tau) = m_T(\tau)$ for every end $\tau \neq \tau_0$.*

We get immediately by induction:

Theorem 4.4. *Let f and f' be two end-functions of a graph G which differ only on finitely many ends. Then G has an f-faithful spanning tree if and only if it has an f'-faithful spanning tree.*

In particular, a graph having an end-faithful spanning tree, such as any countable connected graph, can have an f-faithful spanning tree for some given end-function f with $f(\tau) \neq 1$ for finitely many ends τ.

Definition 4.5. We will say:

(i) A graph G is *end-scattered* if its end space $\mathfrak{T}(G)$ is scattered (see 2).

(ii) A subset of $\mathfrak{T}(G)$ is *countably scattered coverable* if it has a countable cover by closed scattered sets.

(iii) A graph G is *countably end-scattered* if $\mathfrak{T}(G)$ has a countably scattered coverable subset which is dense.

We recall that the cardinality of the end set of the binary tree is $2^{\aleph_0}$; hence any graph whose end set has a cardinality less than $2^{\aleph_0}$ is end-scattered. Notice that the end space of a graph is countably scattered coverable if and only if it is scattered. Moreover, a countable graph is a fortiori countably end-scattered, but we have more general results:

Proposition 4.6 ([14, 8.19]). *A graph G is countably end-scattered whenever it satisfies one of the following conditions:*

(i) $|\mathcal{D}(x)| \leqslant \aleph_0$ *for every* $x \in V(G)$.

(ii) $\{\tau \in \mathfrak{T}(G) \colon |\mathcal{D}^{-1}(\tau)| \geqslant \aleph_0\}$ *is countable.*

(iii) $\{\tau \in \mathfrak{T}(G) \colon m(\tau) \geqslant \aleph_0\}$ *is countable.*

We will now state our main result.

Theorem 4.7. *Let G be a countably end-scattered connected graph, f an end-function of G, $\mathcal{F}_0 := \{\tau \in \mathfrak{T}(G)\colon\ f(\tau) > 0\}$ and $\mathcal{F}_1 := \{\tau \in \mathfrak{T}(G)\colon\ f(\tau) < \aleph_0$ or $f(\tau) = \aleph_0 = m(\tau)\}$. If $\mathcal{F}_0$ or $\mathcal{F}_1$ is countably scattered coverable, then G has an f-faithful spanning tree.*

This generalizes Theorem 2.13 of [12]. We will see (Proposition 5.4) that there may be no f-faithful spanning tree if the hypotheses of Theorem 4.7 are not satisfied. As an obvious consequence of Theorem 4.7 we have the following result.

Corollary 4.8. *Let G be an end-scattered connected graph and f an end-function of G. Then G has an f-faithful spanning tree.*

This last result gives a positive answer to Problem 4.2 for end-scattered connected graphs.

4.2. G-perfect DM-expansions.

To prove Theorem 4.7 we need some concepts and results from [14]. We will only give partial but sufficient statements of these.

Definition 4.9. A *(partial) discrete expansion* of a topological space T is a sequence $(T_n)_{n\geqslant 0}$ satisfying the following conditions. For every $n \geqslant 0$,

DE1. $T_n \subseteq T_{n+1}$,
DE2. T_n is a non-empty closed sets of T,
DE3. $T_n - T_{n-1}$ (with $T_{-1} := \emptyset$) has only isolated points,
DE4. $T = \overline{\bigcup_{n\geqslant 0} T_n}$.

Lemma 4.10 ([14, 8.11])**.** *Any scattered space T has a discrete expansion $(T_n)_{n\geqslant 0}$ such that $T = \bigcup_{n\geqslant 0} T_n$.*

Definition 4.11. A *(partial) expansion of a connected graph G by discrete multi-endings* (*DM-expansion* for short) is a sequence $(G_n)_{n\geqslant 0}$ of subgraphs of G satisfying the following conditions. For every $n \geqslant 0$,

DME1. $G_n \subseteq G_{n+1}$,
DME2. G_n is a multi-ending of G,
DME3. G_0 is discrete and, for each component X of $G - G_n$, the subgraph $M := G_{n+1} \cap X$ is a discrete multi-ending of X which contains a neighbour of each element of $\mathcal{B}(G_n, X)$ and has the property that $M + (G_n)$ is a multi-ending of $X + (G_n)$,
DME4. $\mathfrak{T}(G) = \overline{\bigcup_{n\geqslant 0} \mathfrak{T}_{G_n}(G)}$.

Lemma 4.12 ([14, 7.8]). *If $(G_n)_{n\geqslant 0}$ is a DM-expansion of a connected graph G, then $(\mathfrak{T}_{G_n}(G))_{n\geqslant 0}$ is a discrete expansion of $\mathfrak{T}(G)$. Conversely, if $(\mathcal{A}_n)_{n\geqslant 0}$ is a discrete expansion of $\mathfrak{T}(G)$, then there is a DM-expansion $(G_n)_{n\geqslant 0}$ of G such that $\mathfrak{T}_{G_n}(G) = \mathcal{A}_n$ for every $n \geqslant 0$.*

Lemma 4.13 ([14, 8.12]). *A connected graph has a DM-expansion if and only if it is countably end-scattered.*

Lemma 4.14 ([14, 8.11]). *If G is a connected graph, then the following conditions are equivalent:*
(i) *G is end-scattered;*
(ii) *G has a DM-expansion $(G_n)_{n\geqslant 0}$ such that $\mathfrak{T}(G) = \bigcup_{n\geqslant 0} \mathfrak{T}_{G_n}(G)$;*
(iii) *G has a DM-expansion $(G_n)_{n\geqslant 0}$ such that $\mathfrak{T}(G) = \bigcup_{n\geqslant 0} \mathfrak{T}_{G_n}(G)$ and $G = \bigcup_{n\geqslant 0} G_n$.*

Lemma 4.15 ([14, 7.6.5]). *If $(G_n)_{n\geqslant 0}$ is a DM-expansion of a connected graph G, then*
(i) *Every component of $G - \bigcup_{n\geqslant 0} G_n$ contains an element of $\tau \cup \mathcal{D}^{-1}(\tau)$ for some $\tau \in \mathfrak{T}(G) - \bigcup_{n\geqslant 0} \mathfrak{T}_{G_n}(G)$.*
(ii) *For every distinct $\tau, \tau' \in \mathfrak{T}(G)$, there is a finite $S \subseteq V(\bigcup_{n\geqslant 0} G_n)$ such that $\mathcal{C}_{G-S}(\tau) \neq \mathcal{C}_{G-S}(\tau')$.*

Definition 4.16. A DM-expansion $(G_n)_{n\geqslant 0}$ of G is said to be *G-perfect* if G_n is G-perfect for every $n \geqslant 0$.

Lemma 4.17. *Let $(G_n)_{n\geqslant 0}$ be a DM-expansion of a connected graph G and $(\mathcal{A}_n)_{n\geqslant 0}$ a discrete expansion of $\mathfrak{T}(G)$. Then there is a G-perfect DM-expansion $(G'_n)_{n\geqslant 0}$ of G such that $G_n \subseteq G'_n$, $\mathfrak{T}_{G'_n}(G) = \mathfrak{T}_{G_n}(G) \cup \mathcal{A}_n$ for each $n \geqslant 0$ and every component of $G - G'_n$ contains a ray.*

P r o o f. By DE3 and DME3 the set $\mathfrak{T}_{G_0}(G) \cup \mathcal{A}_0$ is closed and has only isolated points. By Lemma 2.1, there exists a discrete multi-ending $M \in \mathbb{M}(\mathfrak{T}_{G_0}(G) \cup \mathcal{A}_0)$. Let X be a component of $G - M$. The set $V(X \cap G_0)$ is dispersed since $\mathcal{B}(M, X)$ is finite, and $\mathfrak{T}_{G_0}(G) \subseteq \mathfrak{T}_M(G)$. Hence $X \cap G_0$ satisfies Axiom M3. Therefore, by Lemma 2.3, there exists a 0-ending of X which contains $X \cap G_0$ and which, by Lemma 2.6, can be extended to a 0-ending N_X of X which contains a neighbour of each element of $\mathcal{B}(M, X)$, with the property that $N_X + (M)$ is a 0-ending of $X + (M)$. Hence, by Lemma 2.7, $N := M \vee \bigcup_{X\in\mathcal{C}_{G-M}} N_X \in \mathbb{M}(\mathfrak{T}_{G_0}(G) \cup \mathcal{A}_0)$. By Proposition 3.7 there is a G-perfect multi-ending $G'_0 \in \mathbb{M}(\mathfrak{T}_{G_0}(G) \cup \mathcal{A}_0)$ containing N which, by Lemma 2.7, we can choose so that every component of $G - G'_0$ contains a ray.

Suppose that G'_n has already been constructed for some $n \geqslant 0$, so that no component of $G - G'_n$ is rayless. Denote by Γ_n the set of components of $G - G'_n$ which are non-disjoint from G_{n+1} or equal to $\mathcal{C}_{G-G'_n}(\tau)$ for some $\tau \in \mathcal{A}_{n+1}$. Let $X \in \Gamma_n$, $B_X := \mathcal{B}(G'_n, X)$ and $\mathcal{E}_X := \mathfrak{T}_X(G) \cap (\mathfrak{T}_{G_{n+1}}(G) \cup \mathcal{A}_{n+1})$. By DE3 and DME3 the set $\mathcal{E}_X$ is closed and has only isolated points. Then, as for the case $n = 0$, using Lemma 2.3 and Proposition 3.7 we can construct an X-perfect multi-ending N_X of X containing $G_{n+1} \cap X$, with the properties that $\mathfrak{T}_{M_X}(G) = \mathcal{E}_X$ and $N_X + (G'_n)$ is a multi-ending of $X + (G'_n)$, and in addition, by Lemma 2.7, we can choose it so that every component of $G - G'_n$ contains a ray. By Lemma 2.7, $G'_{n+1} := G'_n \vee \bigcup_{X \in \Gamma_n} N_X$ belongs to $\mathbb{M}(\mathfrak{T}_{G_{n+1}}(G) \cup \mathcal{A}_{n+1})$ and, by construction, it is such that no component of $G - G'_{n+1}$ is rayless. It remains to prove that G'_{n+1} is G-perfect.

Since G'_n is G-perfect by the induction hypothesis, it has a spanning tree T_n such that $m_{T_n}(\tau) = \mathrm{tm}(\tau)$ for each $\tau \in \mathfrak{T}_{G'_n}(G)$. Let $X \in \Gamma_n$. Because of the finiteness of B_X, the subgraph $M_X := N_X + (G'_n)$ is a multi-ending of G, and moreover it is G-perfect since N_X is X-perfect by construction. Denote by T_X a spanning tree of M_X such that $m_{T_X}(\tau) = \mathrm{tm}(\tau)$ for each $\tau \in \mathfrak{T}_{M_X}(G)$ if this set is non-empty. Such a set exists since, in this case, M_X is G-perfect. Denote by E_X a subset of the edge set of T_X which is incident with both B_X and $V(X)$ so that, for each component C of $T_X - B_X$, there is exactly one edge in E_X which is incident with C. Let F_X be the spanning forest of M_X whose edge set is $E(T_X - B_X) \cup E_X$. Then $T_{n+1} := T_n \cup \bigcup_{X \in \Gamma_n} F_X$ is a spanning tree of G'_{n+1} such that, for every $\tau \in \mathfrak{T}_{G'_{n+1}}(G)$, $m_{T_{n+1}}(\tau) = m_{T_n}(\tau)$ or $m_{T_X}(\tau)$ according to whether $\tau \in \mathfrak{T}_{G'_n}(G)$ or $X = \mathcal{C}_{G-G'_n}(\tau)$. Thus in both cases $m_{T_{n+1}}(\tau) = \mathrm{tm}(\tau)$. Therefore $(G'_n)_{n \geqslant 0}$ is a G-perfect DM-expansion of G with the required properties. $\square$

We will also need the following two results.

Lemma 4.18 ([10, 3.1]). *Let G be a connected graph, T a spanning tree of G, T_0 any tree of G, a a vertex of T_0, and $\leqslant_a$ the partial order on $V(G)$ such that $x \leqslant_a y$ if and only if x is a vertex of the unique ay-path of T. Then*

$$T_1 := T_0 \cup (T \setminus \{\{x, y\} \in E(T) \colon\ y \in V(T_0) \text{ and } x \leqslant_a y\})$$

is a spanning tree of G.

Lemma 4.19. *If the vertex set of a graph G has a countable cover by dispersed sets (and if in addition every end of G is dominated), then G has an end-faithful (a rayless, respectively) spanning tree.*

P r o o f. If $V(G)$ has a countable cover by dispersed sets, then, by Jung [8, Theorem 5] G has an end-faithful spanning tree (which has particular topological properties with respect to the end-topology). Note that this property is shared by every induced connected subgraph H of G, since $V(H)$ also has a countable cover by dispersed sets (the intersection with $V(H)$ of any set which is dispersed in G is clearly dispersed in H). On the other hand, by a characterization [14, 9.4] of such graphs, G has an end-degree less than or equal to $\omega+1$ (see [14]). Therefore, if in addition all ends of G are dominated, then, by [11, 10.3], the graph G has a rayless spanning tree. □

4.3. Proof of Theorem 4.7.

Case 1. G is one-ended.

Denote by τ the only end of G. Let T be a tm-faithful spanning tree of G and let $\mathcal{R}$ be a set of pairwise disjoint rays of G such that $m(T\cup\bigcup\mathcal{R})=f(\tau)$. This is possible since $m(T)\leqslant f(\tau)\leqslant m(G)$. Now let $\mathcal{R}'$ be a set of cardinality $f(\tau)$ of pairwise disjoint rays of $T\cup\bigcup\mathcal{R}$. Denote by T' a tree of G containing $\bigcup\mathcal{R}'$ which is minimal with respect to inclusion. By the minimality of T', $m(T')=f(\tau)$, and furthermore $T\cup\bigcup\mathcal{R}'$ is finite if $f(\tau)$ is finite. We claim that $m(T\cup T')=f(\tau)$. Indeed, this is obvious if $f(\tau)$ is infinite. If $f(\tau)$ is finite, then this is a consequence of the facts that $m(T\cup\bigcup\mathcal{R}')=f(\tau)$ and that $T\cup\bigcup\mathcal{R}'$ is finite. Then, by Lemma 4.18, for $a\in V(T')$ the tree $T_0:=T'\cup(T\setminus\{\{x,y\}\in E(T)\colon\ y\in V(T')\text{ and }x\leqslant_a y\})$ is a spanning tree of G such that $m(T_0)=f(\tau)$ since $f(\tau)=m(T')\leqslant m(T_0)\leqslant m(T\cup T')=f(\tau)$. Note that T_0 contains a tail of each element of $\mathcal{R}'$.

Case 2. $\mathfrak{T}(G)$ is discrete.

By Lemma 2.2 there exists a 0-ending H of G that we can choose such that each component of $G-H$ contains a ray. Denote by T_H a spanning tree of H. This tree T_H is rayless since $V(H)$ is dispersed.

Let $\tau\in\mathfrak{T}(G)$. Since $B_\tau:=\mathcal{B}(H,\mathcal{C}_{G-H}(\tau))$ is finite, $M_\tau:=\mathcal{C}_{G-H}(\tau)+(H)$ is clearly a G-perfect element of $\mathbb{M}(\tau)$. By Case 1, M_τ has a spanning tree T_τ such that $m(T_\tau)=f(\tau)$. Now denote by E_τ the subset of the set of edges of T_τ which are incident with both B_τ and $V(\mathcal{C}_{G-H}(\tau))$, so that for each component C of $T_\tau-B_\tau$ there is exactly one edge in E_τ which is incident with C. Let F_τ be the spanning forest of M_τ whose edge set is $E(T_\tau-B_\tau)\cup E_\tau$. Then $T:=T_H\cup\bigcup_{\tau\in\mathfrak{T}(G)}F_\tau$ is a spanning tree of G such that $m_T(\tau)=m_{T_\tau}(\tau)=f(\tau)$ for every end τ.

Case 3. $\mathfrak{T}(G)$ is not discrete.

(a) Let $\mathcal{F}$ be either $\mathcal{F}_0$ or $\mathcal{F}_1$ and assume that $\mathcal{F}$ is countably scattered coverable, i.e., $\mathcal{F}\subseteq\bigcup_{n\geqslant 0}\mathcal{A}_n$ where $\mathcal{A}_n$ is scattered and closed. Besides, $\mathfrak{T}(G)=\overline{\bigcup_{n\geqslant 0}\mathcal{B}_n}$ where $\mathcal{B}_n$ is scattered and closed, since G is countably end-scattered. Clearly $\mathcal{A}_n\cup\mathcal{B}_n$ is scattered and closed for every n, and $\mathfrak{T}(G)=\overline{\bigcup_{n\geqslant 0}(\mathcal{A}_n\cup\mathcal{B}_n)}$. By Lemma 4.10, $\mathcal{A}_n\cup\mathcal{B}_n$ has a discrete expansion $(\mathcal{C}^n_p)_{p\geqslant 0}$ such that $\mathcal{A}_n\cup\mathcal{B}_n=\bigcup_{p\geqslant 0}\mathcal{C}^n_p$. For every $n\geqslant 0$ let $\mathcal{D}_n:=\bigcup_{i+j\leqslant n}\mathcal{C}^i_j$. We claim that $(\mathcal{D}_n)_{n\geqslant 0}$ is a discrete expansion of $\mathfrak{T}(G)$. This sequence clearly satisfies the axioms DE1, DE2 and DE4. To prove that it also satisfies DE3, note that $\mathcal{D}_{n+1} - \mathcal{D}_n := \bigcup_{i+j=n}(\mathcal{C}^i_{j+1} - \mathcal{C}^i_j)$. Then, because the sets $\mathcal{C}^i_{j+1}$'s are closed and the subspaces $\mathcal{C}^i_{j+1} - \mathcal{C}^i_j$'s have only isolated points by DE3, clearly $\mathcal{D}_{n+1} - \mathcal{D}_n$ has only isolated points as well.

(b) Let $(H_n)_{n\geqslant 0}$ be a DM-expansion of G. Such a DM-expansion exists by Lemma 4.12, and can be chosen, by this result, so that $\mathfrak{T}_{H_n}(G) = \mathcal{D}_n$ for each $n \geqslant 0$. Moreover, if in addition G is end-scattered, then, by Lemma 4.14, such a DM-expansion $(H_n)_{n\geqslant 0}$ of G can be chosen such that $\mathfrak{T}(G) = \bigcup_{n\geqslant 0} \mathfrak{T}_{H_n}(G)$ and $G = \bigcup_{n\geqslant 0} H_n$. Therefore, by Lemma 4.17, G has a G-perfect DM-expansion $(G_n)_{n\geqslant 0}$ such that $\mathfrak{T}_{G_n}(G) = \mathfrak{T}_{H_n}(G) \cup \mathcal{D}_n$ for each $n \geqslant 0$, $H_n \subseteq G_n$ and no component of $G - G_n$ is rayless, and with the properties that $G = \bigcup_{n\geqslant 0} G_n$ and $\mathfrak{T}(G) = \bigcup_{n\geqslant 0} \mathfrak{T}_{G_n}(G)$ if in addition G is end-scattered. Put $H := \bigcup_{n\geqslant 0} G_n$ and $\mathcal{E} := \bigcup_{n\geqslant 0} \mathfrak{T}_{G_n}(G)$. Then $\mathfrak{T}G) = \overline{\mathcal{E}}$ by Axiom DME4 of Definition 4.11.

In the sequel we will use the notation and properties from the proof of Lemma 4.17. In particular, for each $n \geqslant 0$, Γ_n will be the set of components of $G - G_n$ which are non-disjoint from G_{n+1}, and for $X \in \Gamma_n$, $B_X := \mathcal{B}(G_n, X)$, $\mathcal{E}_X := \mathfrak{T}_{G_{n+1}\cap X}(G)$ and $M_X := G_{n+1} \cap (X + (G_n))$. Note that M_X is a G-perfect discrete multi-ending of G such that $M_X \cap X$ is a multi-ending of X which contains a neighbour of each element of B_X. Furthermore we will set $\Gamma_{-1} := \{G_0\}$, $M_{G_0} := G_0$, $B_{G_0} := \emptyset$ and $\mathcal{E}_{G_0} := \mathfrak{T}_{G_0}(G)$.

(c) We will now define a spanning tree T of H such that $m_T(\tau) = f(\tau)$ for every τ belonging to $\overline{\mathcal{E}}$ or $\mathcal{E}$ according to whether $\mathcal{F}$ is equal to $\mathcal{F}_0$ or $\mathcal{F}_1$. We first construct a spanning forest F of H. Let $n \geqslant -1$ and $X \in \Gamma_n$. Since M_X is G-perfect and discrete, it follows that M_X has a spanning tree T_X such that $m_{T_X}(\tau) = f(\tau)$ for every $\tau \in \mathcal{E}_X$. Then $F_X := T_X - B_X$ is a spanning forest of M_X such that, by the finiteness of B_X, $m_{F_X}(\tau) = f(\tau)$ for all $\tau \in \mathcal{E}_X$. Therefore $F := \bigcup\{F_X : X \in \Gamma_n \text{ and } n \geqslant -1\}$ is a spanning forest of H with the desired properties.

(c.1) $\mathcal{F} = \mathcal{F}_0$.

For every $n \geqslant -1$ and $X \in \Gamma_n$, contract each component of F_X to one of its own vertices. Denote by γ this contraction and let $H^* := \gamma(H)$. It is easy to verify that, by the definition of $\mathcal{F}_0$, $\mathrm{tm}_{H^*}(\tau) = 0$ for each $\tau \in \mathfrak{T}(H^*)$. Besides, if $h_0 := \gamma(V(G_0))$, then, for every $n \geqslant 0$, the set $\{h \in V(H^*)\colon\ d_{H^*}(h_0, h) \leqslant n\}$ is dispersed, where $d_{H^*}(h_0, h)$ denotes the usual distance in H^* between h_0 and h (i.e., the length of a shortest h_0h-path in H^*). Hence, by Lemma 4.18, H^* has a rayless spanning tree T^*. Now, for each edge $e^* = \{h, h'\}$ of T^*, let e be an edge of G joining a vertex of $\gamma^{-1}(h)$ with a vertex of $\gamma^{-1}(h')$. Finally denote by T the graph whose vertex set is $V(H)$ and edge set is $E(F) \cup \{e\colon\ e^* \in E(T^*)\}$. Clearly T is a spanning tree of H such that, each ray contains a subray included in F_X for some $X \in \Gamma_n$ and $n \geqslant -1$. This implies that $m_T(\tau) = f(\tau)$ for every $\tau \in \overline{\mathcal{E}} = \mathfrak{T}(G)$.

(c.2) $\mathcal{F} = \mathcal{F}_1$.

For every $n \geqslant 0$ and $X \in \Gamma_n$, denote by E_X the subset of the set of edges of T_X which are incident with both B_X and $V(X)$ so that, for each component C of F_X, there is exactly one edge in E_X which is incident with C. Then the graph T whose vertex set is $V(H)$ and edge set is $E(F) \cup \bigcup\{E_X\colon\ X \in \Gamma_n \text{ and } n \geqslant 0\}$ clearly is a spanning tree of H such that $m_T(\tau) = f(\tau)$ for every $\tau \in \mathcal{E}$.

(d) If $\mathcal{E} = \mathfrak{T}(G)$, then G is end-scattered by Lemma 4.14, hence $H = G$ by the choice of the DM-expansion $(G_n)_{n\geqslant 0}$, and we are done.

Suppose $\mathcal{E} \neq \mathfrak{T}(G)$. Let $\tau \in \overline{\mathcal{E}} - \mathcal{E}$ and let M_τ be a G-perfect element of $\mathbb{M}(\tau)$. Notice that $M_\tau \cap H \neq \emptyset$, because otherwise τ would not belong to $\overline{\mathcal{E}}$ since it would be separated from $\mathcal{E}$ by a finite subset of $V(M_\tau)$ (by Axiom M2 of the definition of a multi-ending). Denote by Z_τ the union of the components of $G - H$ containing elements of $\tau \cup \mathcal{D}^{-1}(\tau)$. This graph may be empty but, by Lemma 4.15 (i), each component of $G - H$ contains an element of $\tau \cup \mathcal{D}^{-1}(\tau)$ for some $\tau \in \overline{\mathcal{E}} - \mathcal{E}$. Besides, by Lemma 4.15 (ii), $(Z_\tau)_{\tau\in\overline{\mathcal{E}}-\mathcal{E}}$ is a partition of $G - H$. Thus without loss of generality we can assume that $Z_\tau \subseteq M_\tau$ for every $\tau \in \overline{\mathcal{E}} - \mathcal{E}$.

If $\mathcal{F} = \mathcal{F}_0$, then $\mathcal{F}_0 \subseteq \mathcal{E}$ implies that $f(\tau) = 0$, thus M_τ has a rayless spanning tree T_τ. Suppose $\mathcal{F} = \mathcal{F}_1$. Since $\mathcal{B}(G_n, X)$ is finite for every $n \geqslant 0$, any family of pairwise disjoint rays in τ, each of them meeting H, is countable, thus in particular $m_H(\tau) \leqslant \aleph_0$. Hence $m(\tau) > \aleph_0$ since $\tau \notin \mathcal{F}_1$. Therefore $Z_\tau \neq \emptyset$ and $m(Z_\tau) = m_{G-H}(\tau) = m_G(\tau)$. Therefore, by Case 1 and because M_τ is G-perfect, M_τ has a spanning tree T_τ such that $m(T_\tau) = m(T_\tau - H) = f(\tau)$. Now, denote by E_τ the subset of the set of edges of T_τ which are incident with both H and Z_τ so that, for each component C of $T_\tau - H$, there is exactly one edge in E_τ which is incident with C. Let F_τ be the spanning forest of Z_τ whose edge set is $E(T_\tau - H) \cup E_\tau$. It is then straightforward to check that $T' := T \cup \bigcup_{\tau\in\overline{\mathcal{E}}-\mathcal{E}} F_\tau$ is an f-faithful spanning tree of G.

5. Countable graphs

We recall that the cardinality of the end set of a countable connected graph G is at most $\aleph_0$ or exactly $2^{\aleph_0}$ according to whether G is end-scattered or not, and that the end space of the binary tree is homeomorphic with the Cantor set. Moreover, any countable graph has an end-faithful spanning tree (Halin [4, Satz 3]). For countable graphs Theorem [4.7] gives

Theorem 5.1. *Let G be a countable connected graph and f an end-function of G such that $\mathcal{F} := \{\tau \in \mathfrak{T}(G)\colon\ f(\tau) > 0\}$ is countable. Then G has an f-faithful spanning tree.*

We will now see that f-faithful spanning trees do not exist for some end-functions f.

Proposition 5.2. *Let G be a connected countable graph having an f-faithful spanning tree for an end-function f. Let $\mathcal{F} := \{\tau \in \mathfrak{T}(G)\colon\ f(\tau) > 0\}$ and let $\mathcal{C}$ be a closed set of $\mathfrak{T}(G)$. Then*

(i) *$\mathcal{F} \cap \mathcal{C}$ is countable or of cardinality $2^{\aleph_0}$.*

(ii) *If $|\mathcal{F} \cap \mathcal{C}| = 2^{\aleph_0}$, then $\mathcal{F} \cap \mathcal{C}$ contains a non-empty perfect set.*

P r o o f. Let T be an f-faithful spanning tree of G. Since $\mathcal{C}$ is closed, T contains a subtree T' such that $\mathfrak{T}_{T'}(G) = \mathcal{F} \cap \mathcal{C}$.

(i) T' is countable, hence $|\mathcal{F} \cap \mathcal{C}| = |\mathfrak{T}(T')|$ is countable or equal to $2^{\aleph_0}$.

(ii) $|\mathcal{F} \cap \mathcal{C}| = 2^{\aleph_0}$, then T' contains a subdivision of the binary tree, and this implies that $\mathcal{F} \cap \mathcal{C} = \mathfrak{T}_{T'}(G)$ contains a non-empty perfect set. □

An f-faithful spanning tree such that $f(\tau) \leqslant 1$ for every end τ and $f^{-1}(1) =: \mathcal{F}$ was called by Širáň [16] an *$\mathcal{F}$-faithful spanning tree.* He proved [16, Corollary 15] the existence of an $\mathcal{F}$-faithful spanning tree in any countable connected graph G when $\mathcal{F}$ is countable and contains all non-dominated ends of G, and he asked if this result can be extended to uncountable $\mathcal{F}$. Later Hahn and Širáň [3] proved the following

Proposition 5.3 ([3, Theorem 2])**.** *Let $\mathcal{F} := \{\tau \in \mathfrak{T}(G)\colon\ \mathrm{tm}(\tau) \neq \emptyset\}$. If $\mathfrak{T}(G) - \mathcal{F}$ is a discrete subspace of $\mathfrak{T}(G)$, then G has an f-faithful spanning tree.*

Note that on the one hand Širáň's result is generalized by Theorem 5.1, and on the other hand, because of Proposition 5.2, it is not extendable to uncountable $\mathcal{F}$. In particular:

Proposition 5.4. *There is no $\mathcal{F}$-faithful spanning tree if $\aleph_0 < |\mathcal{F}| < 2^{\aleph_0}$. Furthermore, even if G is a connected countable graph such that $|\mathfrak{T}(G)| = 2^{\aleph_0}$ and $\mathrm{tm}(\tau) = 0$ for every end τ, there exists a set $\mathcal{F}$ of ends of G with $|\mathcal{F}| = |\mathfrak{T}(G) - \mathcal{F}| = 2^{\aleph_0}$ such that G has no $\mathcal{F}$-faithful spanning tree.*

P r o o f. The first part is an obvious consequence of Proposition 5.2 (i). As for the second part, since $|\mathfrak{T}(G)| = 2^{\aleph_0}$ and G is countable, it follows that G contains a subdivision D of the binary tree as an end-respecting subgraph. The set $\mathcal{D} = \mathfrak{T}_D(G)$ is then homeomorphic to the Cantor set. We know (see [1, Ch. 9, § 5, exerc. 18 d)]) that there is a subset $\mathcal{F}$ of $\mathcal{D}$ such that $|\mathcal{F}| = |\mathcal{D} - \mathcal{F}| = 2^{\aleph_0}$ and neither $\mathcal{F}$ nor $\mathcal{D} - \mathcal{F}$ contains a non-empty perfect set of $\mathcal{D}$. Therefore, by Proposition 5.2 (ii), G contains no $\mathcal{F}$-faithful spanning tree. □

We do not know whether conditions (i) and (ii) of Proposition 5.2, which are necessary for the existence of an $\mathcal{F}$-faithful spanning tree, are also sufficient. Recently Laviolette and Polat [9] have generalized Hahn and Širáň's result (Proposition 5.3) by giving several sufficient conditions that guarantee the existence of such trees. In particular, they proved that the three properties of a spanning tree: to be connected, acyclic and spanning, are irrelevant for the study of the existence of an $\mathcal{F}$-faithful spanning tree in a graph. In fact, one can dispense with trees altogether.

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