Czechoslovak Mathematical Journal, 51 (126) (2001), 415–432

CERTAIN PARTIAL ORDERS ON SEMIGROUPS

Mario Petrich, Zagreb

(Received September 23, 1998)

Dedicated to the memory of Professor H. S. Zuckerman

Abstract. Relations introduced by Conrad, Drazin, Hartwig, Mitsch and Nambooripad are discussed on general, regular, completely semisimple and completely regular semigroups. Special properties of these relations as well as possible coincidence of some of them are investigated in some detail. The properties considered are mainly those of being a partial order or compatibility with multiplication. Coincidences of some of these relations are studied mainly on regular and completely regular semigroups.

Keywords: semigroup, regular, completely semisimple, completely regular, band of groups, normal band of groups, partial order, compatible with multiplication, coincidence of relations

MSC 2000: 20M10, 20M17

1. Introduction and summary

A number of relations on arbitrary and special semigroups are often instrumental in the study of the structure and properties of these semigroups. Besides Green's relations, which are actually equivalences, the most notable is the natural partial order on regular semigroups. The latter has been extended to general semigroups and represents the first inkling as to, given two elements of the semigroup, which one should be considered higher and which one lower. For certain special classes of semigroups, other relations may be devised which turn out to be partial orders with possibly additional properties such as compatibility with multiplication. We are thus faced with a variety of opportunities to use these relations as effective tools in studying the structure of some special classes of semigroups. With a choice of such relations on a given semigroup S, we may ask for conditions on S which would ensure that some of them coincide. There is a two-fold profit from this: new classes of semigroups may arise in this way and we may be able to understand the given relations better by comparing them with one another.

Drazin [4] gave a systematic review of some of these relations, see $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{N}$ later, as well as discussed the accompanying conditions of weak and quasi-separativity. We take his exposition as a "useful basis for deeper investigations" in his own words. We add the relation introduced by Mitsch [7] as a generalization of the natural partial order for regular semigroups introduced by Hartwig [5] and Nambooripad [8].

Section 2 contains most of the needed concepts and notation. We collect a number of simple properties of the five relations cited above in Section 3. In Section 4, we follow the same pattern for regular semigroups, the main result characterizing those on which $\mathcal{N}$ is compatible with multiplication. Regular semigroups on which $\mathcal{C} = \mathcal{N}$ are characterized in Section 5 as completely semisimple satisfying $\mathcal{D}$-majorization. For completely regular semigroups in Section 6 we characterize $\mathcal{S}$ and $\mathcal{N}$ in several ways. Completely regular semigroups on which $\mathcal{S} = \mathcal{N}$ are characterized in Section 7 as bands of groups. The main result in Section 8 characterizes normal bands of groups, within regular semigroups, in several ways in terms of relations studied here. In Section 9, we briefly introduce a new relation closely related to $\mathcal{S}$ and $\mathcal{M}$.

2. Terminology and notation

We introduce here the needed relations on arbitrary semigroups and a few concepts and symbols.

We follow the pattern and the definitions adopted by Drazin [4] adding the relation $\mathcal{M}$. Below $\mathcal{C}$ stands for Conrad, $\mathcal{S}$ for Sussman, $\mathcal{N}$ for Nambooripad, $\mathcal{M}$ for Mitsch, $\leqslant$ for the natural relation (in regular semigroups: the natural partial order which crystalized in the process of simplifying relations introduced by Nambooripad). The following definitions are considered on an arbitrary semigroup S. For $a, b \in S$, let

$$\begin{aligned}
a \;\mathcal{C}\; b \quad &\text{if} \quad asa = asb = bsa \text{ for all } s \in S,\\
a \;\mathcal{S}\; b \quad &\text{if} \quad a^2 = ab = ba,\\
a \;\mathcal{N}\; b \quad &\text{if} \quad a = axa = axb = bxa \text{ for some } x \in S,\\
a \;\mathcal{M}\; b \quad &\text{if} \quad a = pa = pb = aq = bq \text{ for some } p, q \in S^1,\\
a \leqslant b \quad &\text{if} \quad a = eb = bf \text{ for some } e, f \in E(S).
\end{aligned}$$

Now and later, S^1 denotes the semigroup S with an identity adjoined unless S already has one and $E(S)$ stands for the set of idempotents of S.

On certain rings R, Conrad [3] defined a relation $\leqslant$ by: $a \leqslant b$ if $ara = bra$ for all $r \in R$ and Sussman [13] defined one by: $a \leqslant b$ if $a^2 = ab$; the above versions were suggested by Burgess and Raphael [1] and Drazin [4], respectively. Nambooripad [8] had several versions for his order on regular semigroups; another one was proposed by Hartwig [5]. The relation proposed by Mitsch [7] for arbitrary semigroups also admits several variants.

Hand in hand with the above relations go several versions of "separativity" conditions on arbitrary semigroups. Indeed, S is said to be *weakly separative* if for any $a, b \in S$,

$$asa = asb = bsa = bsb \text{ for all } s \in S \text{ implies } a = b;$$

S is said to be *quasi-separative* if for any $a, b \in S$,

$$a^2 = ab = ba = b^2 \text{ implies } a = b.$$

The first of these two concepts was introduced by Burgess and Raphael [1] whereas the second one by Drazin [4]. They are obviously designed as necessary and sufficient conditions for antisymmetry of the relations $\mathcal{C}$ and $\mathcal{S}$, respectively. The latter concept represents a generalization of the concept of separativity for commutative semigroups introduced by Hewitt and Zuckerman [6] where "separativity" has a natural interpretation in terms of semicharacters.

Recall that a relation θ on a semigroup S is *compatible with multiplication* if for any $a, b, c \in S$, $a \;\theta\; b$ implies $ac \;\theta\; bc$ and $ca \;\theta\; cb$. A semigroup S is *strongly* π*-regular* if for any $a \in S$, there exist positive integers m and n such that $a^m \in a^{m+1}S$ and $a^n \in Sa^{n+1}$. Let $\mathcal{P}$ be a semigroup property. A semigroup S is locally $\mathcal{P}$ if for every $e \in E(S)$, the semigroup eSe has property $\mathcal{P}$. Elements a and b are inverses of each other if $a = aba$ and $b = bab$. The set of all inverses of a is denoted by $V(a)$. If every element of S has an inverse, S is a *regular semigroup*; if every element of S has an inverse with which it commutes, S is a *completely regular semigroup*.

A semigroup S is *completely semisimple* if each of its principal factors is completely (0-)simple; if in addition, for any $J_x < J_y$ and every $e \in E(J_y)$ there exists a unique $f \in E(J_x)$ such that $e > f$, then S satisfies *$\mathcal{D}$-majorization*. A completely regular semigroup S in which $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is a normal band (i.e. satisfies the identity $axya = ayxa$) is a *normal band of groups*.

On any set, ε denotes the equality relation.

For undefined terms and symbols we follow the terminology and notation of the books [10] and [11].

3. General semigroups

We start with elementary properties of the relations introduced and then concentrate on the question when $\mathcal{S}$ is antisymmetric or a partial order.

Lemma 3.1. *The following statements are true in any semigroup S.*

(i) *$\mathcal{C}$ is reflexive and compatible with multiplication.*

(ii) *$\mathcal{S}$ is reflexive.*

(iii) *$\mathcal{N}$ is antisymmetric and transitive.*

(iv) *$\mathcal{M}$ is a partial order.*

(v) *$\mathcal{N} \subseteq \leqslant \subseteq \mathcal{M}$.*

Proof. (i) The first assertion is obvious while the second is verified easily.

(ii) This is trivial.

(iii) Let $a \mathcal{N} b$ and $b \mathcal{N} a$. Then there exist $x, y \in S$ such that $a = axb$ and $b = byb = ayb = bya$ so that

$$a = axb = ax(byb) = (axb)yb = ayb = b.$$

Therefore $\mathcal{N}$ is antisymmetric. Next let $a \mathcal{N} b$ and $b \mathcal{N} c$. There exist $x, y \in S$ such that

$$a = axa = axb = bxa, \quad b = byb = byc = cyb$$

whence

$$a = axa = (axb)xa = ax(byb)xa = (axb)y(bxa) = aya,$$
$$a = axb = ax(byc) = (axb)yc = ayc,$$

and analogously $a = cya$ which shows that $a \mathcal{N} c$. Consequently $\mathcal{N}$ is transitive.

(iv) This was proved in [7] (Theorem 3 and Corollary to Theorem 4).

(v) If $a = axa = axb = bxa$, then for $e = ax$ and $f = xa$, we get $e, f \in E(S)$ and $a = eb = bf$. Therefore $\mathcal{N} \subseteq \leqslant$. If $a = eb = bf$ where $e, f \in E(S)$, then $a = ea = eb = af = bf$. Hence $\leqslant \subseteq \mathcal{M}$. □

For the relation $\mathcal{C}$ we have the following simple result.

Lemma 3.2. *The following conditions on a semigroup S are equivalent.*

(i) *$\mathcal{C}$ is antisymmetric.*

(ii) *$\mathcal{C}$ is a partial order.*

(iii) *S is weakly separative.*

Proof. This follows easily from [1] (Proposition 3).

For the remainder of this section we consider the properties of antisymmetry and being a partial order for the relation $\mathcal{S}$. We start with known statements. □

Lemma 3.3. *The following statements are true in any semigroup S.*

(i) *$\mathcal{S}$ is antisymmetric if and only if S is quasi-separative.*

(ii) *If S is quasi-separative, then $\mathcal{C}$ is a partial order.*

(iii) *If S is quasi-separative and strongly π-regular, then S is completely regular.*

Proof. (i) This is obvious.

(ii) By [4] (Proposition 2), quasi-separativity implies weak separativity. The assertion now follows by part (i) and Lemma 3.2.

(iii) This follows directly from [4] (Proposition 4). □

In view of triviality of Lemma 3.3 (i), we will use this statement without explicit reference.

Let $S = \mathcal{M}^\circ(I, G, \Lambda; P)$ be a Rees matrix semigroup with zero and $p_{\lambda i} = 0$, $p_{\lambda j} \neq 0$ and $p_{\mu i} \neq 0$. Let $a = (i, g, \lambda)$ and $b = (j, p_{\lambda j}^{-1}, \mu)$. Then $a \mathrel{\mathcal{S}} 0$, $0 \mathrel{\mathcal{S}} a$ and $ab \not\mathrel{\mathcal{S}} 0b$ and $\mathcal{S}$ is neither antisymmetric nor compatible with multiplication on S. If $g, h \in G$, $g \neq h$, then $(i, g, \lambda) \mathrel{\mathcal{S}} (i, h, \lambda)$ so that $\mathcal{S}$ restricted to the $\mathcal{H}$-class of (i, g, λ) is not the equality relation. However, $\mathcal{S}$ restricted to a group $\mathcal{H}$-class in any semigroup is equality. This should be compared with the natural partial order $\leqslant$: in any semigroup S, if $a = eb = bf$ where $e, f \in E(S)$ and $a \mathrel{\mathcal{H}} b$, then $b = ay$ for some $y \in S^1$ so that $a = eb = eay = ay = b$. Therefore $\leqslant$ restricted to any $\mathcal{H}$-class of an arbitrary semigroup is the equality relation. Also note that $\mathcal{S}\big|_{E(S)} = \leqslant\big|_{E(S)}$ for any semigroup S.

It thus appears that $\mathcal{S}$ is a relatively weak relation and the semigroup needs considerable reinforcement in order that $\mathcal{S}$ have the usual properties such as being a partial order or being compatible with multiplication. We address next the question: when is $\mathcal{S}$ antisymmetric? In other words, which semigroups are quasi-separative? The solution will be in the form: which kinds of subsemigroups of a semigroup S are not admissible for this condition to hold. Then we shall provide an answer of the same kind for the query: when is $\mathcal{S}$ a partial order?

4. Regular semigroups

After establishing several statements concerning general regular semigroups, we examine the conditions under which $\mathcal{N}$ is compatible with multiplication or is equal to $\mathcal{C}$.

Lemma 4.1. *The following conditions on a semigroup S are equivalent.*

(i) *$\mathcal{N}$ is reflexive.*

(ii) *$\mathcal{N}$ is a partial order.*

(iii) *S is regular.*
(iv) $\mathcal{N} = \mathcal{M}$.

P r o o f. Parts (i) and (ii) are equivalent in view of Lemma 3.1 (iii).

(i) $\Rightarrow$ (iii). This follows directly from the definition of $\mathcal{N}$.

(iii) $\Rightarrow$ (iv). By Lemma 3.1 (v), we have $\mathcal{N} \subseteq \mathcal{M}$. Let $a\ \mathcal{M}\ b$. Then $a = pa = pb = aq = bq$ for some $p, q \in S^1$ and $a = axa$ for some $x \in S$. Hence

$$a = axa = aqxpa = aqxpb = bqxpa$$

so that $a\ \mathcal{N}\ b$. Therefore $\mathcal{M} \subseteq \mathcal{N}$ and equality prevails.

(iv) $\Rightarrow$ (i). By Lemma 3.1 (iv), $\mathcal{M}$ is reflexive and thus so is $\mathcal{N}$. □

In addition to Lemma 3.1 which is valid for arbitrary semigroups, the next result states what is true for regular semigroups.

Lemma 4.2. *The following statements are true in any regular semigroup S.*
(i) *$\mathcal{C}$ is a partial order.*
(ii) $\mathcal{C} \subseteq \mathcal{S} \cap \mathcal{N}$.
(iii) *$\mathcal{S} \cap \mathcal{N}$ is a partial order.*
(iv) *$\mathcal{N} = \leqslant = \mathcal{M}$ and this is a partial order.*

P r o o f. (i) By [4] (Proposition 7), regularity implies weak separativity which coupled with Lemma 3.2 yields the assertion.

(ii) Let $a\ \mathcal{C}\ b$ and $x \in V(a)$. Then

$$a^2 = a(xa)a = a(xa)b = ab$$

and similarly $a^2 = ba$ so that $a\ \mathcal{S}\ b$. Also

$$a = axa = axb = bxa \tag{1}$$

and thus $a\ \mathcal{N}\ b$. The assertion follows.

(iii) Let $\theta = \mathcal{S} \cap \mathcal{N}$. Then θ is reflexive since both $\mathcal{S}$ and $\mathcal{N}$ are, see Lemma 4.1, and antisymmetric since $\mathcal{N}$ is, see Lemma 3.1 (iii). Let $a\ \theta\ b$ and $b\ \theta\ c$. Then (1) holds for some $x \in S$ so that

$$a^2 = ab = (axb)b = axb^2 = axbc = ac$$

and similarly $a^2 = ca$ whence $a\ \mathcal{S}\ c$. By Lemma 3.1 (iii), we also have $a\ \mathcal{N}\ c$ and thus $a\ \theta\ c$. Therefore θ is transitive and thus it is a partial order.

(iv) This follows directly from Lemmas 4.1 and 3.1 (iv). □

From the proofs of Lemmas 4.1 and 4.2, we conclude that for an arbitrary semigroup S and $a, b \in S$ where a is regular, the following holds:

(i) if $a \mathcal{M} b$, then $a \mathcal{N} b$,

(ii) if $a \mathcal{C} b$, then $a \mathcal{S} b$.

By Lemma 3.1 (v), we have that $\mathcal{N} \subseteq \mathcal{M}$. Hence the difference between $\mathcal{M}$ and $\mathcal{N}$ may occur only when $a \mathcal{M} b$ and a is not regular.

In view of Lemma 4.2 (iv), for a regular semigroup we can write $\mathcal{N}$ or $\leqslant$ or $\mathcal{M}$; we will generally use $\mathcal{N}$.

Most of the next result is due to Nambooripad and can be deduced from [8] (Theorem 3.3) via the equivalence of his definitions of the natural partial order and the one adopted here. However, it seems instructive to give a direct proof.

Theorem 4.3. *The following conditions on a regular semigroup S are equivalent.*

(i) *S is locally inverse.*

(ii) *$\mathcal{N}$ is compatible with multiplication.*

(iii) *$\mathcal{N}$ is locally compatible with multiplication.*

Proof. (i) $\Rightarrow$ (ii). Let $a, b, c \in S$ be such that $a \leqslant b$. Then $a = eb = bf$ for some $e, f \in E(S)$. Let

$$b' \in V(b), \quad u \in V(bfc), \quad v \in V(b'bcubfc),$$
$$g = b'bfcubf, \quad h = b'bcubfcvb'bcubf, \quad t = ubfcvb'bcubfc.$$

Using $eb = bf$, simple verification shows that $cubf \in E(S)$ so that $g = g(cubf)$, $g, h, t \in E(S)$, $g = gb'b$ and $g, h \leqslant b'b$ and hence $gh = hg$. Therefore

$$\begin{aligned} bfc &= (bfc)u(bfc) = b(b'bfcubf)c = bgc = bg(cubf)c \\ &= bg(b'bcubfc) = bg(b'bcubfc)v(b'bcubfc) \\ &= bghc = bhgc = bc(ubfcvb'bcubf)(b'bfcubf)c \\ &= bc(ubfcvb'bcubfc) = bct. \end{aligned}$$

It follows that $ac = ebc = bct$ with $e, t \in E(S)$ and thus $ac \leqslant bc$. A dual argument will show that also $ca \leqslant cb$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Let $e, f, g \in E(S)$ be such that $f, g \in eSe$; hence $e \geqslant f$ and $e \geqslant g$. Then $fg \leqslant eg = g$ so that $fg = gu$ for some $u \in E(S)$ whence $fg = gfg$; similarly $gf \leqslant ge$ implies $gf = gfg$. Therefore $fg = gf$ and eSe is an inverse semigroup. □

A condition possibly stronger than compatibility of $\mathcal{N}$ with multiplication, characterized in Theorem 4.3, is that $\mathcal{C}$ and $\mathcal{N}$ coincide, in view of Lemma 3.1 (i). As a preparation for treating this case, we prove the next lemma which will also find an application in Section 8 and is of some independent interest.

Lemma 5.1. *Let B be a bicyclic semigroup. Then*

$$(m,n)\ \mathcal{S}\ (p,q) \Leftrightarrow \begin{cases} \text{either } m = n \geqslant p = q, \\ \text{or } (m,n) = (p,q), \quad m \neq n, \end{cases}$$
$$(m,n)\ \mathcal{N}\ (p,q) \Leftrightarrow m - n = p - q, \quad p \leqslant m.$$

Moreover, $\varepsilon = \mathcal{C} \subset \mathcal{S} \subset \mathcal{N}$ and $\mathcal{S}$ is not compatible with multiplication.

Proof. Indeed,

$$\begin{aligned}(m,n)\ \mathcal{S}\ (p,q) &\Leftrightarrow (m,n)^2 = (m,n)(p,q) = (p,q)(m,n) \\ &\Leftrightarrow (2m-r, 2n-r) = (m+p-s, n+q-s) = (p+m-t, q+n-t) \\ &\Leftrightarrow m - r = p - s, \ n - r = q - s, \ s = t\end{aligned}$$

where $r = \min\{m,n\}$, $s = \min\{n,p\}$, $t = \min\{q,m\}$. If $m = n$, then $p = q = s$ so that $p \leqslant n$. Let $m > n$. Then $r = n$ so that $q = s$. Also $m - n = p - q$ so that $p > q$. Now $q = s$ implies that $q = n$. But then $m - n = p - q$ implies that also $m = p$. Finally, let $m < n$. Then $r = m$ so that $p = t$. Also $m - n = p - q$ and thus $p < q$. Now $p = t$ implies that $p = m$ whence also $q = n$.

Conversely, the above conditions easily imply that $(m,n)\ \mathcal{S}\ (p,q)$.

Further, B is an inverse semigroup; thus by Lemma 4.2, $\mathcal{N} = \leqslant$, and by [11] (Lemma II.1.6),

$$\begin{aligned}(m,n)\ \mathcal{N}\ (p,q) &\Leftrightarrow (n,m)(m,n) = (n,m)(p,q) \\ &\Leftrightarrow (n,n) = (n+p-r, m+q-r) \ \text{ where } \ r = \min\{m,p\} \\ &\Leftrightarrow m - n = p - q, \ \ p \leqslant m.\end{aligned}$$

Let $(m,n)\ \mathcal{C}\ (p,q)$. Since B has an identity, we get $(m,n)\ \mathcal{S}\ (p,q)$. By the above, we have either $m = n \geqslant p = q$ or $(m,n) = (p,q)$ and $m \neq n$. So it remains to consider only the case $m = n \geqslant p = q$. In particular,

$$(m,m)(m,0)(m,m) = (m,m)(m,0)(p,p)$$

whence $(2m,m) = (m+p,p)$ so that $m = p$ and again $(m,n) = (p,q)$.

Finally, $(2,2)\ \mathcal{S}\ (1,1)$ but $(2,2)(0,1)\ \not\mathcal{S}\ (1,1)(0,1)$ and $\mathcal{S}$ is not compatible with multiplication. $\square$

We are now ready for the main result of this section. Recall that the relation $\mathcal{S} \cap \mathcal{N}$ occurs in Lemma 4.2 (ii).

Theorem 5.2. *The following conditions on a regular semigroup S are equivalent.*

(i) *S is completely semisimple and satisfies $\mathcal{D}$-majorization.*

(ii) $\mathcal{C} = \mathcal{N}$.

(iii) $\mathcal{C} = \mathcal{S} \cap \mathcal{N}$.

Proof. (i) $\Rightarrow$ (ii). By Lemma 4.2 (ii), we have $\mathcal{C} \subseteq \mathcal{N}$.

By [9] (Theorem 3.4), for any $J_a \leqslant J_b$ there exists a function $\varphi_{J_b,J_a} \colon J_b \to J_a$ such that the system of all such functions satisfies some strong conditions relative to the multiplication in S of which we will now make heavy use. Let $a, b \in S$ be such that $a \mathrel{\mathcal{N}} b$. Then $a = axa = axb = bxa$ for some $x \in S$. Hence

$$a = a(x\varphi_{J_x,J_a})a = a(x\varphi_{J_x,J_a})(b\varphi_{J_b,J_a}) = (b\varphi_{J_b,J_a})(x\varphi_{J_x,J_a})a,$$

which in a Rees matrix semigroup easily implies that $a = b\varphi_{J_b,J_a}$. Now for any $s \in S$, letting $x = asa$, we obtain

$$asa = as(a\varphi_{J_a,J_x}) = as(b\varphi_{J_b,J_a}\varphi_{J_a,J_x}) = as(b\varphi_{J_b,J_x}) = asb$$

and analogously $asa = bsa$. Therefore $a \mathrel{\mathcal{C}} b$, which shows that $\mathcal{N} \subseteq \mathcal{C}$ and equality prevails.

(ii) $\Rightarrow$ (iii). By Lemma 4.2 (ii), we have $\mathcal{C} \subseteq \mathcal{S}$. Hence the hypothesis yields $\mathcal{C} = \mathcal{C} \cap \mathcal{S} = \mathcal{N} \cap \mathcal{S}$.

(iii) $\Rightarrow$ (i). Assume first that S has a bicyclic subsemigroup B. If $a, b \in B$ are such that $a \mathrel{\mathcal{C}} b$ in S, then $asa = asb = bsa$ for all $s \in S$ and thus also for all $s \in B$ so that $a \mathrel{\mathcal{C}_B} b$, where $\mathcal{C}_B$ is the relation $\mathcal{C}$ on B. Hence $\mathcal{C}\big|_B \subseteq \mathcal{C}_B$. By Lemma 5.1, we have $\mathcal{C}_B = \varepsilon$. Now let $e, f \in E(B)$ be such that $e < f$. Then $e \not\mathrel{\mathcal{C}} f$ and $e \mathrel{\mathcal{S} \cap \mathcal{N}} f$, contrary to the hypothesis. Therefore S cannot have a bicyclic subsemigroup, which by [2] (Theorem 2.54) implies that S is completely semisimple.

By Lemma 3.1 (i), $\mathcal{C}$ is compatible with multiplication and hence so is $\mathcal{S} \cap \mathcal{N}$. Let $e, f, g \in E(S)$ be such that $f, g \in eSe$. Then $e \geqslant f$ and $e \geqslant g$. Thus $f \mathrel{\mathcal{S} \cap \mathcal{N}} e$ and $g \mathrel{\mathcal{S} \cap \mathcal{N}} e$. Hence $fg \mathrel{\mathcal{S} \cap \mathcal{N}} eg = g$ so that $fg = gfg$; also $gf \mathrel{\mathcal{S} \cap \mathcal{N}} ge = g$ and thus $gf = gfg$. Therefore $fg = gf$. In addition, the hypothesis implies that $f \mathrel{\mathcal{C}} e$ and $g \mathrel{\mathcal{C}} e$ whence, for any $x \in S$, we have

$$fxf = fxe = exf, \quad gxg = gxe = exg,$$

which together with $fg = gf$ yields

$$fxg = (fxe)g = (exf)g = (exg)f = (gxe)f = gxf.$$

Let $x \in eSe$ and let y be an inverse of x. One verifies easily that eye is an inverse of x in eSe. Letting $g = eyx$, we get $g \in E(eSe)$ and $fx = fxg = gxf$ whence $fx = fxf$. Symmetrically one obtains $xf = fxf$ which implies that $fx = xf$. Therefore idempotents are central in eSe so that eSe is a Clifford semigroup.

Now assume also that $f \mathcal{D} g$. We may represent the principal factor of f as $\mathcal{M}^\circ(I, G, \Lambda; P)$ (or without the zero) and set $f = (i, p_{\lambda i}^{-1}, \lambda)$ and $g = (j, p_{\mu j}^{-1}, \mu)$. Then for $a = (i, g, \mu)$ and $a' = (j, p_{\mu j}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda)$, simple computation yields that $f = aa'$, $g = a'a$ and $a' \in V(a)$. It follows that $a = fa = ag$ and $a' = a'f = ga'$ whence $a, a' \in eSe$. We have seen above that eSe is a Clifford semigroup and thus $aa' = a'a$ so that $f = g$. Therefore S satisfies $\mathcal{D}$-majorization. □

Note that by [9] (Theorem 3.4), the semigroups occurring in Theorem 5.2 (i) are precisely regular semigroups which are subdirect products of completely (0)-simple semigroups.

6. Completely regular semigroups

We establish here some general properties of the relations $\mathcal{S}$ and $\mathcal{N}$ on completely regular semigroups.

Lemma 6.1. *In a completely regular semigroup, $\mathcal{S} \subseteq \mathcal{N}$ and $\mathcal{S}$ is a partial order.*

Proof. This was established in [4] (Propositions 6 and 5). □

Corollary 6.2. *The following conditions on a semigroup S are equivalent.*
(i) *S is completely regular.*
(ii) *S is strongly π-regular and $\mathcal{S}$ is a partial order.*
(iii) *S is strongly π-regular and quasi-separative.*

Proof. This follows easily from Lemmas 3.3 and 6.1. □

Recall that completely regular semigroups S may be regarded as having the unary operation $a \to a^{-1}$, where a^{-1} is the inverse of a in the maximal subgroup of S containing a. Since $a^{-1}a = aa^{-1}$ it is convenient to denote $a^\circ = aa^{-1}$ so that a° is the identity of any subgroup of S containing a. Note that by Lemma 4.2, $\mathcal{N} = \leqslant$ on S.

Lemma 6.3. *Let S be a completely regular semigroup. The following conditions on elements $a, b \in S$ are equivalent.*
(i) *$a \mathcal{N} b$.*
(ii) *$a = b^\circ a = ab^{-1}a = ab^\circ$.*

(iii) $a = eb = bf$ *for some* $e, f \in E(D_a)$.
(iv) $a = b^\circ ab^{-1}ab^\circ$.
(v) $a^\circ \leqslant b^\circ,\ \ a = ab^{-1}a$.

P r o o f. (i) $\Rightarrow$ (ii). By hypothesis $a = eb = bf$ for some $e, f \in E(S)$. Then $a = bf$ implies that $a = b^\circ a = af$ and $a = eb$ implies that $a = ab^\circ$ so that

$$a = ab^{-1}b = ab^{-1}(bf) = ab^{-1}a.$$

(ii) $\Rightarrow$ (iii). Indeed,

$$a = ab^\circ = (ab^{-1})a = b^\circ a = b(b^{-1}a)$$

where $ab^{-1} = (ab^{-1}a)b^{-1} \in E(D_a)$ and $b^{-1}a = b^{-1}(ab^{-1}a) \in E(D_a)$.

(iii) $\Rightarrow$ (i). This is trivial.

Evidently, part (ii) is equivalent to both (iv) and (v). □

Lemma 6.3 reveals that the natural partial order on a completely regular semigroup can be expressed in terms of the elements themselves, that is, part (ii) indicates that $a \leqslant b$ can be written in terms of a and b alone (without the existence of suitable idempotents), as in the case of inverse semigroups where $a \leqslant b$ is equivalent to $a = ab^{-1}a$ alone. Part (iii) is often useful in the study of the natural partial order on completely regular semigroups.

Lemma 6.4. *Let S be a completely regular semigroup. The following conditions on elements $a, b \in S$ are equivalent.*
(i) $a\ \mathcal{S}\ b$.
(ii) $a = a^\circ b = ba^\circ$.
(iii) $a = eb = be$ *for some* $e \in E(S)$.
(iv) $a\ \mathcal{N}\ b$, $ab = ba$.
(v) $a^\circ\ \mathcal{N}\ b^\circ$, $a^\circ b = ab^\circ$, $b^\circ a = ba^\circ$.
(vi) $a = ab^{-1}a = ba^{-1}b$.

Moreover, if these conditions are satisfied, then $ab^{-1} = b^{-1}a \in E(D_a)$ *and* $ab^{-1}\ \mathcal{N}\ e$*, where e is as in part* (iii).

P r o o f. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (iv). Obviously $a\ \mathcal{N}\ b$ and $ab = (be)b = b(eb) = ba$.

(iv) $\Rightarrow$ (v). By Lemma 6.3, we have $a^\circ\ \mathcal{N}\ b^\circ$. Also $a = eb$ for some $e \in E(S)$ so that

$$a^2 = (eb)a = e(ba) = e(ab) = (ea)b = ab = ba,$$

which implies that $a = a^\circ b = ba^\circ$. Hence $a^\circ b = a = ab^\circ$ and $b^\circ a = a = ba^\circ$.

(v) $\Rightarrow$ (vi). Indeed, $a^\circ = a^\circ b^\circ = b^\circ a^\circ$ and hence $a = ab^\circ = b^\circ a$ so that $a = a^\circ b = ba^\circ$ and thus $a \mathcal{N} b$. By Lemma 6.3, we have $a = ab^{-1}a$. Also

$$a = a^\circ b = aa^{-1}b = (ba^\circ)a^{-1}b = ba^{-1}b.$$

(vi) $\Rightarrow$ (i). The hypothesis implies that $a = ab^\circ = ba^\circ$ so that $a = (ab^{-1})b = b(b^{-1}a)$ where $ab^{-1} = (ab^{-1}a)b^{-1} = (ab^{-1})^2$ and similarly

$$b^{-1}a = b^{-1}(ab^{-1}a) = (b^{-1}a)^2.$$

Letting $e = ab^{-1}$ and $f = b^{-1}a$, we get $a = eb = bf$ with $e, f \in E(S)$. We show next that $e = f$. Indeed, $ef = (ab^{-1})(b^{-1}a) = ab^{-2}a$ and

$$\begin{aligned} fe &= (b^{-1}a)(ab^{-1}) = b^{-1}(ba^{-1}b)(ba^{-1})b^{-1} = b^\circ a^{-1}b^2a^{-1}b^\circ \\ &= (b^\circ a)a^{-2}b^2a^{-2}(ab^\circ) = aa^{-2}b^2a^{-2}a, \end{aligned}$$

which implies that $ef \mathcal{H} fe$. Since $e \mathcal{D} f$, it follows that $e \mathcal{H} f$ whence $e = f$. Therefore $a = eb = be$ where $e \in E(S)$. Hence $a^2 = a(eb) = (ae)b = ab$ and $a^2 = (be)a = b(ea) = ba$ so that $a \mathcal{S} b$.

Assume that the above conditions are satisfied. Then

$$\begin{aligned} ab^{-1} &= (b^\circ a)b^{-1} = b^{-1}(ba)b^{-1} = b^{-1}(ab)b^{-1} = b^{-1}(ab^\circ) = b^{-1}a, \\ ab^{-1} &= (ab^{-1}a)b^{-1} = (ab^{-1})^2, \\ ab^{-1} &= (ea)b^{-1} = e(ab^{-1}) = b^{-1}a = b^{-1}(ae) = (b^{-1}a)e = (ab^{-1})e, \end{aligned}$$

which establishes all assertions. □

Lemma 6.4 illustrates the nature of the relation $\mathcal{S}$ and clarifies its relationship with the natural order on completely regular semigroups.

7. Bands of groups

The result of this section asserts that, within completely regular semigroups, bands of groups characterize those on which $\mathcal{S}$ and $\mathcal{N}$ coincide. The proof is broken into two lemmas.

Here completely regular semigroups are regarded as algebras with multiplication and inversion. They form a variety denoted by $\mathcal{CR}$ whose lattice of subvarieties is denoted by $\mathcal{L}(\mathcal{CR})$. Bands of groups, that is completely regular semigroups in which $\mathcal{H}$ is a congruence, form a subvariety $\mathcal{BG}$ of $\mathcal{CR}$.

Lemma 7.1. *The variety $\mathcal{BG}$ is the largest subvariety $\mathcal{V}$ of $\mathcal{CR}$ with the property that $\mathcal{S} = \mathcal{N}$ on every member of $\mathcal{V}$.*

Proof. Call a member $\mathcal{V}$ of $\mathcal{L}(\mathcal{CR})$ *good* if $\mathcal{S} = \mathcal{N}$ on all members of $\mathcal{V}$. We must show that $\mathcal{BG}$ is good and for any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{V} \subseteq \mathcal{BG}$ if and only if $\mathcal{V}$ is good. The first assertion obviously implies the direct part of the second assertion.

Hence let $S \in \mathcal{BG}$ and let $a, b \in S$ be such that $a \mathcal{N} b$, say $a = eb = bf$ for some $e, f \in E(S)$. By Lemma 6.3, we may suppose that $e, f \in E(D_a)$. In addition, since S is a band of groups,

$$a = eb \,\mathcal{H}\, eb^2 = (eb)b = ab$$

and similarly $a \,\mathcal{H}\, ba$. Now $a^2 = e(ba) = (ab)f$ in the completely simple semigroup D_a evidently implies that $a^2 = ba = ab$ and hence $a \,\mathcal{S}\, b$. Therefore $\mathcal{N} \subseteq \mathcal{S}$ and by Lemma 6.4 we have $\mathcal{S} \subseteq \mathcal{N}$; so equality prevails.

We must show next that for $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, if $\mathcal{V}$ is good, then $\mathcal{V} \subseteq \mathcal{BG}$. By contrapositive, assume that $\mathcal{V} \nsubseteq \mathcal{BG}$. For $n \geqslant 2$, let LG_n be the ideal extension of the left zero semigroup $I = \{0, 1, \ldots, n-1\}$ by the cyclic group $Z_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ with multiplication: for $\overline{i} \in Z_n$, $j \in I$ we have

$$\begin{aligned} &\overline{i} * j = l \quad \text{where } i + j \equiv l \pmod{n}, \ \ 0 \leqslant l < n, \\ &j * \overline{i} = j. \end{aligned}$$

Then LG_n is a semigroup; let RG_n be the semigroup obtained from LG_n by reversing the multiplication. According to ([12], Theorem 1), $\mathcal{V} \nsubseteq \mathcal{BG}$ implies that either $LG_n \in \mathcal{V}$ or $RG_n \in \mathcal{V}$ for some $n \geqslant 2$. By symmetry, we may assume that $LG_n \in \mathcal{V}$. Then

$$1^2 = 1 * \overline{1} = 1, \ \ \overline{1} * 1 = 2, \ \ 1 = 1 * \overline{1} = \overline{0} * 1$$

so that $1 \not\mathcal{S}\, \overline{1}$ and $1 \,\mathcal{N}\, \overline{1}$ which shows that $\mathcal{S} \neq \mathcal{N}$. Therefore $\mathcal{V}$ is not good, as required. $\square$

Lemma 7.2. *Let S be a completely regular semigroup. Then $\mathcal{S} = \mathcal{N}$ on S if and only if S satisfies the identity $(xy)^\circ x^2 = x(xy)^\circ x$.*

Proof. Necessity. Let $x, y \in S$, $e = (xy)^\circ$, $f = y(xy)^{-1}x$ and $a = (xy)^\circ x$. Then $e, f \in E(S)$ and $a = ex = xf$ so that $a \,\mathcal{N}\, x$. The hypothesis implies that $ax = xa$ so that $(xy)^\circ x^2 = x(xy)^\circ x$.

Sufficiency. Let $a, b \in S$ be such that $a \,\mathcal{N}\, b$. By Lemma 6.3 we have $a = eb = bf$ for some $e, f \in E(D_a)$. We let $D_a = \mathcal{M}(I, G, \Lambda; P)$ be a Rees matrix semigroup without zero. We may write $a = (i, g, \lambda)$, $e = (i, p_{\mu i}^{-1}, \mu)$, $f = (j, p_{\lambda j}^{-1}, \lambda)$ since $a = ea = af$. Also let $x = (j, h, \mu)$. Then

$$(bx)^\circ = (bfx)^\circ = (ax)^\circ = ((i, g, \lambda)(j, h, \mu))^\circ = e$$

and by the given identity, we have $(bx)^\circ b^2 = b(bx)^\circ b$ so that $eb^2 = beb$. Therefore $(eb)^2 = (eb)b = b(eb)$, that is $a^2 = ab = ba$ and $a\ \mathcal{S}\ b$. We have proved that $\mathcal{N} \subseteq \mathcal{S}$; Lemma 6.4 gives the opposite inclusion. $\square$

Theorem 7.3. *Let S be a completely regular semigroup. Then $\mathcal{S} = \mathcal{N}$ on S if and only if S is a band of groups.*

Proof. By Lemma 7.2 the variety $\mathcal{V} = [(xy)^\circ x^2 = x(xy)^\circ x]$ consists precisely of those completely regular semigroups on which $\mathcal{S} = \mathcal{N}$. By Lemma 7.1, we must have $\mathcal{V} \subseteq \mathcal{BG}$ by the maximality of the latter. But also $\mathcal{BG} \subseteq \mathcal{V}$ so that we have the equality $\mathcal{V} = \mathcal{BG}$. $\square$

8. Normal bands of groups

In the main result of this section we characterize normal bands of groups, within regular semigroups, in various ways by means of the relations under study here.

Theorem 8.1. *The following conditions on a regular semigroup S are equivalent.*

(i) *S is a normal band of groups.*
(ii) *S is quasi-separative, completely semisimple and satisfies $\mathcal{D}$-majorization.*
(iii) *$\mathcal{C} = \mathcal{S}$.*
(iv) *$\mathcal{S}$ is compatible with multiplication.*
(v) *$\mathcal{C} = \mathcal{N}$ and $\mathcal{S}$ is antisymmetric.*
(vi) *$\mathcal{S} \cap \mathcal{N}$ is compatible with multiplication and $\mathcal{S}$ is antisymmetric.*

Proof. (i) $\Rightarrow$ (ii). Trivially, every completely regular semigroup is quasi-separative and completely semisimple. By [9] (Theorem 4.1), a normal band of groups satisfies $\mathcal{D}$-majorization.

(ii) $\Rightarrow$ (i). By hypothesis every principal factor of S is completely (0)-simple and by [9] (Theorem 3.4), for any $\mathcal{J}$-classes $J_a < J_b$ there is a function $\varphi_{J_b,J_a}\colon J_b \to J_a$ with some strong properties relative to the multiplication in S. Free use of some of these properties will be made below.

We show first that S is completely regular. Equivalently, we must prove that every $\mathcal{J}$-class is closed under multiplication, for in such a case, every $\mathcal{J}$-class of S is completely simple. By contradiction, assume that there exist $a, b \in S$ for which $J_a < J_b$ and $b^2 \in J_a$. Letting $\varphi = \varphi_{J_b,J_a}$, we obtain

$$b^2 = b(b\varphi) = (b\varphi)b = (b\varphi)^2$$

where evidently $b \neq b\varphi$, which contradicts the hypothesis that S is quasi-separative. Therefore S is completely regular.

Now [9] (Theorems 3.4 and 4.1) implies that S is a normal band of groups.

(i) $\Rightarrow$ (iii). By Lemma 4.2, we have $\mathcal{C} \subseteq \mathcal{S}$. For the opposite inclusion, using [10] (Theorem IV.4.3), we may set $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$, a strong semilattice of completely simple semigroups. Let $a\ \mathcal{S}\ b$ with $a \in S_\alpha$ and $b \in S_\beta$. Then $\alpha \leqslant \beta$ and $a = a(b\varphi_{\beta,\alpha}) = (b\varphi_{\beta,\alpha})a$, which in the completely simple semigroup S_α implies that $a = b\varphi_{\beta,\alpha}$. For any $s \in S_\gamma$ we obtain

$$asa = (a\varphi_{\alpha,\alpha\gamma})(s\varphi_{\gamma,\alpha\gamma})(a\varphi_{\alpha,\alpha\gamma}) = (a\varphi_{\alpha,\alpha\gamma})(s\varphi_{\gamma,\alpha\gamma})(b\varphi_{\beta,\alpha\gamma}) = asb$$

and similarly $asa = bsa$ so that $a\ \mathcal{C}\ b$. Consequently $\mathcal{S} \subseteq \mathcal{C}$ and equality prevails.

(iii) $\Rightarrow$ (iv). By Lemma 3.1 (i), $\mathcal{C}$ is compatible with multiplication and thus so is $\mathcal{S}$.

(iv) $\Rightarrow$ (ii). Let $a\ \mathcal{S}\ b$ and $a' \in V(a)$. Then $aa'\ \mathcal{S}\ ba'$ and $a'a\ \mathcal{S}\ a'b$ so that $aa' = ba'aa' = ba'$ and $a'a = a'aa'b = a'b$. Hence $a = aa'a = aa'b = ba'a$ so that $a\ \mathcal{N}\ b$. It follows that $\mathcal{S} \subseteq \mathcal{N}$ and since $\mathcal{N}$ is antisymmetric, see Lemma 3.1 (iv), so is $\mathcal{S}$, which by Lemma 3.3 implies that S is quasi-separative.

Assume next that S has a bicyclic subsemigroup B. Trivially, $\mathcal{S}\big|_B$ coincides with the $\mathcal{S}$-relation on B. It follows from Lemma 5.1 that $\mathcal{S}\big|_B$ is not compatible with multiplication, which contradicts the hypothesis. Therefore by ([2], Theorem 2.54), S is completely semisimple.

Let $e, f \in E(S)$ be such that $f\ \mathcal{N}\ e$ and let $x \in eSe$. Then $f\ \mathcal{S}\ e$ and hence $fx\ \mathcal{S}\ ex = x$ so that $(fx)^2 = x(fx)$. For $x' \in V(x)$ in eSe and $f = x'x$ we obtain $(x'x)x(x'x)x = x(x'x)x$ so that $x'x^3 = x^2$. Symmetrically, we can show that $x^3x' = x^2$. It follows that eSe is strongly π-regular. We have seen above that S is quasi-separative, which now by Lemma 3.3 (iii) yields that eSe is completely regular.

Next let $e, f, g \in E(S)$ be such that $f\ \mathcal{N}\ e$, $g\ \mathcal{N}\ e$ and $f\ \mathcal{D}\ g$. Then $f\ \mathcal{S}\ e$ so that $fg\ \mathcal{S}\ eg = g$ whence $(fg)g = g(fg)$; also $gf\ \mathcal{S}\ ge = g$ whence $g(gf) = (gf)g$. Therefore $fg = gfg = gf$. We now have $f\ \mathcal{D}\ g$ and $fg = gf$. By the preceding paragraph, eSe is completely regular so that D_f is a completely simple semigroup. But then $f\ \mathcal{H}\ g$ and finally $f = g$. Consequently S satisfies $\mathcal{D}$-majorization.

(ii) and (v) are equivalent. This is a direct consequence of Theorem 5.2.

(v) $\Rightarrow$ (vi). From the equivalence of parts (iii) and (v) we obtain that $\mathcal{S} \cap \mathcal{N} = \mathcal{C}$, which is compatible with multiplication by Lemma 3.1 (i). Also by hypothesis, $\mathcal{S}$ is antisymmetric.

(vi) $\Rightarrow$ (ii). Assume that S has a bicyclic subsemigroup B. Trivially, $\mathcal{S}\big|_B = \mathcal{S}_B$, the $\mathcal{S}$-relation on B. Let $a\ \mathcal{N}\ b$ in S with $a, b \in B$. Then $a = be$ for some $e \in E(S)$, which implies that

$$a = aa^{-1}a = aa^{-1}be = aa^{-1}bb^{-1}be = bb^{-1}aa^{-1}be = bb^{-1}a$$

so that $a\ \mathcal{N}_B\ b$ where $\mathcal{N}_B$ is the $\mathcal{N}$-relation B. Hence $\mathcal{N}\big|_B = \mathcal{N}_B$ and using Lemma 5.1, we obtain

$$(\mathcal{S} \cap \mathcal{N})\big|_B = \mathcal{S}\big|_B \cap \mathcal{N}\big|_B = \mathcal{S}_B \cap \mathcal{N}_B = \mathcal{S}_B,$$

which, again by Lemma 5.1, is not compatible with multiplication. This contradicts the hypothesis. Therefore [2] (Theorem 2.54) implies that S is completely semisimple.

In the third paragraph of the proof that (iv) implies (ii) above, the argument remains valid in the present case if we replace $\mathcal{S}$ by $\mathcal{S} \cap \mathcal{N}$ and use the hypothesis that S is quasi-separative. Therefore eSe is completely regular. The fourth paragraph of that proof also remains valid in this case if we replace $\mathcal{S}$ by $\mathcal{S} \cap \mathcal{N}$. Therefore S satisfies $\mathcal{D}$-majorization. $\square$

Similarly as in the remark after Theorem 5.2, we have that semigroups occuring in Theorem 8.1 (i), according to [9] (Theorem 4.1), are precisely regular semigroups which are subdirect products of completely simple semigroups with a zero possibly adjoined.

Corollary 8.2. *In a normal band of groups, we have* $\mathcal{C} = \mathcal{S} = \mathcal{N} = \mathcal{M} = \leqslant$ *and this is a compatible partial order.*

Proof. This follows directly from Lemmas 3.1 (iv), (v) and 4.1 and Theorem 8.1. $\square$

Corollary 8.3. *A quasi-separative strict inverse semigroup S is a Clifford semigroup and conversely.*

Proof. For the background on these semigroups, see [11] (Section II.4). In fact, S is completely semisimple and satisfies $\mathcal{D}$-majorization, so by Theorem 8.1, S is a normal band of groups. But it is also an inverse semigroup so it must be a Clifford semigroup. The converse is trivial. $\square$

9. A new partial order

On a completely regular semigroup S we evidently have the equivalence

$$a\ \mathcal{S}\ b \Leftrightarrow a = a^\circ b = b a^\circ.$$

Inspired by this and the definition of $\mathcal{M}$, we introduce a relation $\mathcal{P}$ by

$$a\ \mathcal{P}\ b \ \text{ if } \ a = pa = pb = ap = bp \ \text{ for some } \ p \in S^1.$$

We now list some simple properties of $\mathcal{P}$.

Lemma 9.1. *The following statements are true in any semigroup S:*

(i) $\mathcal{P} \subseteq \mathcal{S} \cap \mathcal{M}$.

(ii) $\mathcal{P}$ *is a partial order.*

P r o o f. (i) Let $a, b \in S$ and $p \in S^1$ be such that $a = pa = pb = ap = bp$. Then

$$a^2 = a(pb) = (ap)b = ab$$

and similarly $a^2 = ba$ so that $\mathcal{P} \subseteq \mathcal{S}$. Trivially $\mathcal{P} \subseteq \mathcal{M}$.

(ii) Evidently $\mathcal{P}$ is reflexive and it is antisymmetric since $\mathcal{P} \subseteq \mathcal{M}$ and $\mathcal{M}$ is antisymmetric by Lemma 3.1 (iv). Let $a, b, c \in S$ and $p, q \in S^1$ be such that

$$a = pa = pb = ap = bp, \quad b = qb = qc = bq = cq.$$

Then $a = pb = pqc$ which together with $a = bp = qbp = qa$ implies that $a = qpqc$. Similarly, $a = bp = cqp$ which together with $a = pb = pbq = aq$ implies that $a = aq = cqpq$ where $qpq \in S^1$. Finally, $a = qa = aq$ implies that $qpqa = qpa = qa = a$ and similarly, $aqpq = a$. Therefore $a \mathcal{P} c$ and $\mathcal{P}$ is transitive. Consequently $\mathcal{P}$ is a partial order. $\square$

We now briefly compare the relations $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{N}$ on a regular semigroup S. Recall from Lemma 4.2 (iv) that on S we have $\mathcal{N} = \leqslant = \mathcal{M}$ and clearly $\mathcal{P}\big|_{E(S)} = \mathcal{N}\big|_{E(S)}$. For completely regular semigroups we have a complete answer.

Proposition 9.2. *On a completely regular semigroups S, we have $\mathcal{P} = \mathcal{S}$; moreover, $\mathcal{P} = \mathcal{N}$ if and only if S is a band of groups.*

P r o o f. By Lemma 9.1 (i) we have that $\mathcal{P} \subseteq \mathcal{S}$; the opposite inclusion follows from the remark at the beginning of this section. Hence $\mathcal{P} = \mathcal{S}$. The second assertion now follows by Theorem 7.3. $\square$

The situation in inverse semigroups is somewhat more complex. In a bicyclic semigroup, Lemma 5.1 implies that $\mathcal{P} = \mathcal{S} \subset \mathcal{N}$. For a Brandt semigroup, we have the following simple result. For any set X, we denote by $|X|$ its cardinality.

Proposition 9.3. *On a Brandt semigroup $S = B(G, I)$ we have $\mathcal{P} = \mathcal{N}$; moreover, $\mathcal{P} = \mathcal{S}$ if and only if either $|I| = 1$ or $|I| = 2$ and $|G| = 1$.*

P r o o f. It is well known that for $a, b \in S$ we have

$$a \leqslant b \Leftrightarrow \text{either } a = 0 \text{ or } a = b$$

and thus $\mathcal{P} = \mathcal{N}$.

First let $|I| = 1$. Then $S = G^\circ$, a group with zero. Let $a, b \in S$ be such that $a\ \mathcal{S}\ b$. If $a = 0$, then clearly $a\ \mathcal{P}\ b$. Otherwise $a, b \in G$ and $a = b$.

Next let $|I| = 2$ and $|G| = 1$, say $I = \{1, 2\}$ and $G = \{e\}$, and let $a\ \mathcal{S}\ b$. Again if $a = 0$, then $a\ \mathcal{P}\ b$. Otherwise $a \neq 0$ whence $b \neq 0$, say $a = (i, e, j)$ and $b = (k, e, l)$. Recall that $a^2 = ab = ba$. If $i = j$, then $i = j = k = l$ and thus $a = b$. Otherwise, we have $i \neq j$, $j \neq k$ and $l \neq i$. We may set $i = 1$ and $j = 2$, which then implies that $k = 1$ and $l = 2$ so again $a = b$.

Now let $|I| = 2$ and $|G| > 1$, say $I = \{1, 2\}$ and $g, h \in G$, $g \neq h$. Then $(1, g, 2)\ \mathcal{S}\ (1, h, 2)$ and $(1, g, 2)\ \not{\mathcal{P}}\ (1, h, 2)$. It remains to deal with the case $|I| > 2$, say $1, 2, 3 \in I$. Then $(1, e, 2)\ \mathcal{S}\ (1, e, 3)$ and $(1, e, 2)\ \not{\mathcal{P}}\ (1, e, 3)$, where $e \in G$ is any element. $\square$

References

[1] *W. D. Burgess and R. Raphael*: On Conrad's partial order relation on semiprime rings and on semigroups. Semigroup Forum *16* (1978), 133–140.

[2] *A. H. Clifford and G. B. Preston*: The algebraic theory of semigroups, Vol I. Math. Surveys No. 7, Amer. Math. Soc., Providence, 1961.

[3] *P. F. Conrad*: The hulls of semiprime rings. Bull. Austral. Math. Soc. *12* (1975), 311–314.

[4] *M. P. Drazin*: A partial order in completely regular semigroups. J. Algebra *98* (1986), 362–374.

[5] *R. E. Hartwig*: How to partially order regular elements. Math. Japon. *25* (1980), 1–13.

[6] *E. Hewitt and H. S. Zuckerman*: The l_1-algebra of a commutative semigroup. Trans. Amer. Math. Soc. *83* (1956), 70–97.

[7] *H. Mitsch*: A natural partial order for semigroups. Proc. Amer. Math. Soc. *97* (1986), 384–388.

[8] *K. S. S. Nambooripad*: The natural partial order on a regular semigroup. Proc. Edinburgh Math. Soc. *23* (1980), 249–260.

[9] *M. Petrich*: Regular semigroups satisfying certain conditions on idempotents and ideals. Trans. Amer. Math. Soc. *170* (1972), 245–269.

[10] *M. Petrich*: Introduction to Semigroups. Merrill. Columbus, Ohio, 1973.

[11] *M. Petrich*: Inverse Semigroups. Wiley, New York, 1984.

[12] *V. V. Rasin*: On the variety of Cliffordean semigroups. Semigroup Forum *23* (1981), 201–220.

[13] *I. Sussman*: A generalization of Boolean rings. Math. Ann. *136* (1958), 326–338.

Author's address: Department of Mathematics, Univesity of Zagreb, Croatia.