Czechoslovak Mathematical Journal

Paul F. Conrad; Michael R. Darnel

Subgroups and hulls of Specker lattice-ordered groups

Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 2, 395–413

Persistent URL: http://dml.cz/dmlcz/127656

Czechoslovak Mathematical Journal, 51 (126) (2001), 395–413

SUBGROUPS AND HULLS OF SPECKER LATTICE-ORDERED GROUPS

Paul F. Conrad, Kansas, and Michael R. Darnel, South Bend

(Received September 7, 1998)

Abstract. In this article, it will be shown that every ℓ-subgroup of a Specker ℓ-group has singular elements and that the class of ℓ-groups that are ℓ-subgroups of Specker ℓ-group form a torsion class. Methods of adjoining units and bases to Specker ℓ-groups are then studied with respect to the generalized Boolean algebra of singular elements, as is the strongly projectable hull of a Specker ℓ-group.

Keywords: lattice-ordered groups, f-rings, Specker groups

MSC 2000: 06F20, 06F25, 46A40, 12J15

1. Introduction and Background Results

This paper continues recent research into the class of Specker ℓ-groups begun by the authors in [4], and continued in [7] and [6]. Earlier work in this series has concentrated on the lattices of such ℓ-groups and especially on the generalized Boolean algebras of singular elements. In this paper, we investigate ℓ-subgroups and various hulls of Specker ℓ-groups, especially the strongly projectable hull.

We review now some of the basic terms and concepts of lattices and lattice-ordered groups. For further reference, the reader should consult [9] (especially Sections 6, 54, and 55). A *lattice* is a partially ordered set L such that for every pair of elements $x, y \in L$, there exists a least upper bound (called the *join* and written $x \vee y$) and a greatest lower bound (called the *meet* and written $x \wedge y$). An *ideal* of a partially ordered set P is a subset S such that if $x \leqslant s \in S$, then $x \in S$, while a *dual ideal* (*filter*) is a subset T such that if $y \geqslant t \in T$, then $y \in T$. An ideal I of a lattice is *prime* if $x \wedge y \in I$ implies either $x \in I$ or $y \in I$. Within a given partially ordered set P, an element a *covers* an element b if $b < a$ and there are no intervening elements.

Since we will be dealing with both groups and rings, additive notation will be used to denote all group operations, even in cases when the groups may not be abelian. A *lattice-ordered group* (written ℓ-group), is a group G whose underlying set is a lattice such that if $g \leqslant h$, then for any $x, y \in G$, $x + g + y \leqslant x + h + y$. For an ℓ-group G, G^+ denotes the set $\{g \in G\colon\ g \geqslant 0\}$. For $g \in G$, the *positive part* of g (written g_+) is $g \vee 0$, while the *negative part* of g (written g_-) is $-g \vee 0$; the *absolute value* of g (written $|g|$) is $g_+ \vee g_- = g_+ + g_- = g \vee -g$. Two elements $a, b \in G$ are *disjoint* if $|a| \wedge |b| = 0$. The underlying lattice of an ℓ-group is necessarily distributive; an ℓ-group G is *completely distributive* if for any sets I, J, $\bigvee_I \bigwedge_J g_{ij} = \bigwedge_J \bigvee_{J^I} g_{f(j),j}$. An ℓ-group G is *archimedean* if for any pair of positive elements $a, b \in G^+$, there exists an integer n such that $na \not\leqslant b$; an archimedean ℓ-group is necessarily abelian. An ℓ-group is *hyperarchimedean* if each ℓ-homomorphic image is archimedean.

An *ℓ-subgroup* A of an ℓ-group G is both a sublattice and a subgroup. A is *dense* if for each $0 < g \in G$, there exists $0 < a \in A$ such that $a \leqslant g$; A is *large* if for each $0 < g \in G$, there exists a positive integer n and $0 < a \in A$ such that $a \leqslant ng$. An ℓ-subgroup C is *convex* if $0 \leqslant x \leqslant c \in C$ implies $x \in C$; $\mathcal{C}(G)$ will denote the lattice of convex ℓ-subgroups of G, partially ordered by inclusion. For $C \in \mathcal{C}(G)$, the (right) cosets of C are ordered by $C + x \leqslant C + y$ if there exists $c \in C$ such that $c + x \leqslant y$. A normal convex ℓ-subgroup L is called an *ℓ-ideal*; under the quotient group operation and coset order, G/L is an ℓ-group. $\mathcal{C}(G)$ is a complete sublattice of the lattice of subgroups of G. Thus for any element $g \in G$, there exists a minimal convex ℓ-subgroup containing g; this is called a *principal* convex ℓ-subgroup and is denoted $G(g)$. $\mathcal{C}_p(G)$ will denote the lattice of principal convex ℓ-subgroups of G, partially ordered by inclusion. An element $g \in G$ is a *strong order unit* for G if $G(g) = G$.

A convex ℓ-subgroup P of an ℓ-group G is *prime* if P^+ is a prime lattice ideal of G^+; a convex ℓ-subgroup is prime if and only if its right cosets are totally ordered under the coset order. Under inclusion, the prime convex ℓ-subgroups of an ℓ-group form a *root system*: i.e., a partially ordered set in which no two incomparable elements have a lower bound. The intersection of a chain of prime subgroups is a prime subgroup and so there exist minimal prime subgroups. For any $0 \neq g \in G$, a convex ℓ-subgroup M maximal with respect to not containing g is called a *value* of g and is called *regular*; if M is a value of some element $g \in G$, then M is a prime subgroup of G. $\Gamma(G)$ will denote the regular subgroups of G, partially ordered by inclusion. A *plenary subset* Δ of $\Gamma(G)$ is a dual ideal of $\Gamma(G)$ such that $\bigcap\{M\colon\ M \in \Delta\} = (0)$. An abelian ℓ-group is completely distributive if and only if $\Gamma(G)$ has a (necessarily unique) minimal plenary subset.

For $X \subseteq G$, the *polar* of X, written $X^\perp$, $= \{g \in G\colon |g| \wedge |x| = 0 \text{ for all } x \in X\}$. $X^\perp \in \mathcal{C}(G)$. If $X = \{g\}$ for some $g \in G$, $\{g\}^\perp$ and $\{g\}^{\perp\perp}$ are simply written $g^\perp$ and $g^{\perp\perp}$, respectively. If $C \in \mathcal{C}(G)$ and G is the (group) direct sum of C and $C^\perp$, we write $G = C \boxplus C^\perp$. In this case, G is called the *cardinal sum* of C and $C^\perp$ and C is a *cardinal summand* of G. More generally, for a set $\{A_\lambda\}_\Lambda$ of ℓ-groups, the *cardinal product*, denoted $\prod_\Lambda A_\lambda$, is the (external) group direct product of $\{A_\lambda\}_\Lambda$ with pointwise order operations. The *cardinal sum*, written $\sum_\Lambda A_\lambda$, is the ℓ-subgroup of $\prod_\Lambda A_\lambda$ of those elements having finite support. An ℓ-group G is *projectable* if for all $g \in G$, $G = g^{\perp\perp} \boxplus g^\perp$. An ℓ-group G is *strongly projectable* if for any polar subgroup $P \subseteq G$, $G = P \boxplus P^\perp$. A characterization of hyperarchimedean ℓ-groups is that an ℓ-group G is hyperarchimedean if and only if for all $g \in G$, $G = G(g) \boxplus g^\perp$. An element $g \in G$ is a *weak order unit* if $g^\perp = (0)$; strong order units are necessarily weak order units. An ℓ-group G is *complete* if any subset of G bounded above has a least upper bound. Complete ℓ-groups are necessarily archimedean and strongly projectable and every archimedean ℓ-group G has a unique minimal complete ℓ-group $G^\wedge$ into which G can be ℓ-embedded as a dense ℓ-subgroup.

An element $0 \leqslant g \in G$ is *basic* if $G(g)$ is a convex totally ordered subgroup of G. G has a *basis* if each positive element exceeds a basic element. For an ℓ-group G with a basis, the *basis subgroup* $B(G)$ is the convex ℓ-subgroup generated by the basis of G.

A *torsion class* $\mathcal{T}$ of ℓ-groups is a class that is closed with respect to containing convex ℓ-subgroups, closed with respect to ℓ-homomorphic images, and also having the following property: for any ℓ-group G, if $\{C_\lambda\}$ is a set of convex ℓ-subgroups of G such that each $C_\lambda \in \mathcal{T}$, then $\bigvee C_\lambda \in \mathcal{T}$.

A *lattice-ordered ring*, denoted ℓ-ring, is a ring R whose underlying set is lattice-ordered such that $(R, \leqslant, +)$ is an ℓ-group and such that if $a \leqslant b$ and $0 \leqslant c$ in R, then $ac \leqslant bc$ and $ca \leqslant cb$. An *o*-ring is a totally ordered ring. An *f*-ring is an ℓ-ring R such that R is a subdirect product of *o*-rings.

Throughout, $\mathbb{Z}$ will denote the integers, $\mathbb{Q}$ the rationals, and $\mathbb{R}$ the reals, all with the usual addition, multiplication, and order.

An element $0 < g$ in an ℓ-group G is *singular* if for all $0 \leqslant h \leqslant g$, $h \wedge (g - h) = 0$. An ℓ-group G is a *Specker* ℓ-group if G is generated as a group by its singular elements [8], while G is a *singular ℓ-group* if for each $0 < g \in G$, there exists a singular element $s \in G$ such that $s \leqslant g$. The following proposition from [2] gives many useful characterizations and properties of Specker ℓ-groups.

Proposition 1.1. *Let G be an ℓ-group and Λ be its set of minimal prime subgroups. The following are equivalent:*

a) *G is a Specker ℓ-group.*

b) *There exists an ℓ-embedding σ of G into $\prod_\Lambda \mathbb{Z}$ such that $G\sigma$ is generated as a group by characteristic functions.*

c) *There exists an ℓ-embedding σ of G into $\prod_\Lambda \mathbb{Z}$ such that for all $g \in G$, the characteristic function of the support of g, $\chi_{\mathrm{supp}(g)}$, is in $G\sigma$.*

d) *There exists an embedding τ of G into $\prod_\Lambda \mathbb{Z}$ such that $G\sigma$ is generated as a group by characteristic functions and is an ℓ-subgroup of the subring of functions of bounded range.*

e) *There exists an embedding τ of G into $\prod_\Lambda \mathbb{Z}$ such that $G\sigma$ is generated as a group by characteristic functions and the set of characteristic functions is closed with respect to meets.*

The following proposition (proven in [2]) gives a useful representation of elements of a Specker ℓ-group.

Proposition 1.2. *Let G be a Specker ℓ-group. Then for any $0 \neq g \in G$, there exists a set of mutually disjoint singular elements $\{s_1, \ldots, s_n\} \subseteq G$ and integers $m_1, \ldots, m_n$ such that $g = m_1 s_1 + \ldots + m_n s_n$.*

We will call $m_1 s_1 + \ldots + m_n s_n$ a *Specker representation* of g. The reader should be aware that such a representation is not necessarily unique, but each g has a representation for which the integers $m_1, \ldots, m_n$ are distinct and *this* representation is unique.

We remark here that every Specker ℓ-group is hyperarchimedean and that the class of Specker ℓ-groups form a torsion class.

If G is a Specker ℓ-group, a multiplication can be placed on G by, for elements $g, h \in G$ with respective Specker representations $g = m_1 s_1 + \ldots + m_k s_k$ and $h = n_1 t_1 + \ldots + n_p t_p$, defining $g \cdot h = \sum_{i=1}^{k} \sum_{j=1}^{p} m_i n_j (s_i \wedge t_j)$ [2]. With this multiplication and viewing G as an ℓ-subgroup of $\prod_\Lambda \mathbb{Z}$, G is then an f-subring of $\prod_\Lambda \mathbb{Z}$ [2]. This is the unique multiplication on G so that $(G, \leqslant, +, \cdot)$ is an f-ring and so that for any two singular elements s and t, $s \cdot t = s \wedge t$. When this natural multiplication is important, we will refer to the Specker ℓ-group G as a *Specker f-ring*.

A *p-disjoint subset* of an ℓ-group G is a subset $P \subseteq G^+$ such that if $p, q \in P$, then $(p \wedge q) \wedge (p - (p \wedge q)) = 0$. If G is a Specker ℓ-group, then its set of singular elements is a maximal p-disjoint subset. Conversely, if P is a p-disjoint subset of G closed with respect to meets, then the subgroup $\langle P \rangle$ of G generated by P is a Specker ℓ-subgroup of G with P as its set of singular elements [7]. An ℓ-group G is said to have a *Specker signature* [7] if G has a Specker ℓ-subgroup H such that H is large in G.

Part of the fascination with Specker ℓ-groups is that they are part of the large class of ℓ-groups having a *unique addition.* Recall [4] that an ℓ-group G has a *unique addition* if, having chosen an element to be the group identity, there exists a unique group operation $+$ on the lattice $(G, \leqslant)$ such that $(G, \leqslant, +)$ is an ℓ-group, and that G has *essentially one addition* if given any two compatible group operations $+$ and $*$, $(G, \leqslant, +) \cong (G, \leqslant, *)$. Any ℓ-group having a unique addition must be archimedean [12] while at the other extreme, $\mathcal{A}(\mathbb{R})$, the ℓ-group of order-preserving permutations of the real line under pointwise order and composition $\circ$, has essentially one addition [10]. Moreover, Holland [11] has shown that $\mathcal{A}(\mathbb{R})$ also has the property that there exists a unique lattice-ordering $\leqslant$ such that $(\mathcal{A}(\mathbb{R}), \leqslant, \circ)$ is an ℓ-group.

No result completely parallel to Holland's result for $\mathcal{A}(\mathbb{R})$ is possible for Specker ℓ-groups in general. Given the usual operation $+$, there exist many partial orders $\leqslant$ such that $(G, \leqslant, +)$ is an ℓ-group. The following is the best result possible.

Proposition 1.3. *Let $(G, \leqslant +)$ be a Specker ℓ-group with $\mathcal{S}$ as its generalized Boolean algebra of singular and identity elements. Then $\leqslant$ is the unique lattice order on $(G, +)$ such that each element of $\mathcal{S}$ is singular.*

Proof. Suppose $\preceq$ is a lattice order on G such that $(G, \preceq, +)$ is an ℓ-group with each element of $\mathcal{S}$ being singular in $(G, \preceq, +)$. Let $\wedge_1$ denote the meet operation in $(G, \leqslant, +)$ and $\wedge_2$ denote the meet operation in $(G, \preceq, +)$.

Let $s, t \in \mathcal{S}$. Then $s \wedge_1 t = 0$ in $(G, \leqslant, +)$ if and only if $s + t \in \mathcal{S}$, and so $s \wedge_2 t = 0$.

Let $g \in G$; g then equals $m_1 s_1 + \ldots + m_n s_n$, where $s_i \wedge_1 s_j = 0$ if $i \neq j$. Since by the above paragraph, disjoint singular elements remain so in $(G, \preceq, +)$, g is positive in $(G, \preceq, +)$ if and only if each $m_i \geqslant 0$, and so g is positive in $(G, \leqslant, +)$ if and only if g is positive in $(G, \preceq, +)$. $\square$

2. Subgroups and subrings of Specker lattice-ordered groups

Recall [9] that a *component* of an element g of an ℓ-group G is an element x such that $|g - x| \wedge |x| = 0$. The components of an element form a generalized Boolean algebra. $S_G(g)$ will denote the subgroup of G generated by the components of g; $S_G(g)$ is then a Specker ℓ-subgroup of G.

Theorem 2.1. *For an element g of an ℓ-group G, $G(g)$ is ℓ-isomorphic to an ℓ-subgroup of a Specker ℓ-group if and only if there exists an integer n such that $nG(g) \subseteq S_G(g)$.*

Proof. Suppose that G is an ℓ-subgroup of a Specker ℓ-group H. Let $0 < g \in G$. It suffices to prove that there exists an integer n such that $nG(g) \subseteq S_G(g)$ only for the case that G itself is Specker, as $nG(g) \subseteq nH(g)$.

$g = n_1s_1 + \ldots + n_ks_k$, where the s_i are pairwise disjoint singular elements and n_i are distinct integers. Let n be the least common multiple of $n_1, n_2, \ldots, n_k$. Then $ns_i \in S_G(g)$ for all i, and, if t is a singular element of $G(g)$ such that $t \leqslant s_i$ for some i, then n_it_i is a componen t of g and $nt \in S(g)$. Thus if x is any singular element of $G(g)$, $x = (x \wedge s_1) + (x \wedge s_2) + \ldots + (x \wedge s_k)$ implies that $nx \in S_G(g)$.

Now let $h \in G(g)$; h can then be written in the form $h = m_1t_1 + \ldots + m_rt_r$, where the t_r are pairwise disjoint singular elements of $G(g)$ and such that for each j, there exists an i such that $t_j \leqslant s_i$, and where the m_j are integers. Above it was shown that $nt_j \in S_G(g)$ for all j, and so $nh = m_1nt_1 + \ldots + m_rnt_r \in S_G(g)$.

Conversely, suppose that G is an ℓ-group such that for all $g \in G$, $nG(g) \subseteq S_G(g)$. Now if $g \gg k$, then $g \gg nk$. But $nk \in nG(g) \subseteq S_G(g)$, which contradicts that $S_G(g)$ is archimedean. So $G(g)$ is archimedean and hence abelian. As $G(g)$ is also torsion-free, the map $x \to nx$ is an ℓ-isomorphism of $G(g)$ into $S_G(g)$, which is Specker. $\square$

It should be remarked here that Lin [13] gave the first proof that if A is an ℓ-subgroup of a Specker ℓ-group, then for any $g \in A$, there exists an integer n such that $nA(g) \subseteq S_A(g)$.

Now an ℓ-subgroup of a Specker ℓ-group need not be a Specker ℓ-group. For let $G = \mathbb{Z}^{EC}$, the ℓ-group of eventually constant integer sequences with pointwise addition and order [5]. Let A be the ℓ-subgroup generated by $\left(\sum\limits_{i=1}^{\infty} \mathbb{Z}\right) \oplus \mathbb{Z}(2, 2, \ldots)$. G is then a Specker ℓ-group while A is not. However, any ℓ-subgroup A of a Specker ℓ-group G must have elements that are singular in A.

Theorem 2.2. *Let G be a Specker ℓ-group and H be an ℓ-subgroup of G. Then H has elements that are singular in H.*

P r o o f. Since G is a Specker ℓ-group, there exists an ℓ-embedding of G into $\prod_\Lambda \mathbb{Z}$ such that each singular element of G is mapped to a characteristic function. In the following, it is presumed that G has such a representation. Moreover, with such a representation, each $g \in G$ has finite range. For every $0 \neq \alpha \in \operatorname{range}(g)$, the characteristic function of $g^{-1}(\alpha)$ is a singular element s of G. Each $g \in G$ thus has a unique representation of the form $g = \alpha_1s_1 + \ldots + \alpha_ns_n$, where $\{\alpha_1, \ldots, \alpha_n\}$ is the nonzero range of g, s_i is the characteristic function of $g^{-1}(\alpha_i)$, and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. This will be called the *standard representation* of g and α_i the *standard coefficient* of s_i. Finally, define $\sigma(g) = \alpha_1 + \ldots + \alpha_n$.

Let $0 < a \in H$ be such that $\sigma(a)$ is minimal over all positive elements of H; let $a = \alpha_1s_1 + \ldots + \alpha_ns_n$ be its standard representation. Note that if $0 < h \in H$ such that $h \in H(a) = a^{\perp\perp}$, then $\operatorname{supp}(h) \cap \operatorname{supp}(s_i) \neq \emptyset$ for all $1 \leqslant i \leqslant n$. For otherwise, let x be the projection of a onto $H(h) = h^{\perp\perp}$; then $0 < \sigma(x) < \sigma(a)$, a contradiction to the choice of a.

Let $0 \leqslant z \leqslant a$ in H and let b be the projection of a onto $z^{\perp\perp}$. Then, since G, and thus H, is hyperarchimedean, $b \in H$, $b \wedge (a-b) = 0$, and $\operatorname{supp}(b) = \operatorname{supp}(z)$. The goal now is to show that $z = b$. Let $z = \beta_1 t_1 + \ldots + \beta_r t_m$ be the standard representation of z; the standard representation of b then is $b = \alpha_1 r_1 + \ldots + \alpha_n r_n$, where $0 < r_i = s_i \wedge (t_1 \vee \ldots \vee t_m)$.

Suppose there exists r_i such that $\left|\{t_j \colon r_i \wedge t_j > 0\}\right| \geqslant 2$. Let $\{\beta_{j_1}, \beta_{j_2}, \ldots\}$ be the standard coefficients of those t_j's such that $r_i \wedge t_j > 0$; assume $\beta_{j_1} < \beta_{j_2} < \ldots$. Now for any $x \in \operatorname{supp}(r_i)$, $\beta_{j_1} b(x) - \alpha_i z(x) \leqslant 0$, and so $\operatorname{supp}[(\beta_{j_1} b - \alpha_i z) \vee 0] \cap \operatorname{supp}(r_i) = 0$. By arguments presented above, this means $(\beta_{j_1} b - \alpha_i z) \vee 0 = 0$, and so $\beta_{j_1} b \leqslant \alpha_i z$. Since $\operatorname{supp}(r_i) \setminus \operatorname{supp}(t_{j_1}) \neq \emptyset$, $\beta_{j_1} b < \alpha_i z$. Let c be the projection of b onto $(\alpha_i z - \beta_{j_1} b)^{\perp}$; since for any $x \in r_i \wedge t_{j_1}$, $\alpha_i z(x) - \beta_{j_1} b(x) = 0$, $c > 0$. Note that if $x \in \operatorname{supp}(c) \cap \operatorname{supp}(r_k)$, then $\alpha_i z(x) = \beta_{j_1} \alpha_k$, implying that $z(x) = \beta_{j_1} \alpha_k / \alpha_i$. Thus for any k such that $\operatorname{supp}(c) \cap \operatorname{supp}(r_k) \neq \emptyset$, there exists a unique t_{j_k} such that $t_{j_k} \wedge r_k > 0$. Since $z \leqslant a$, then $\beta_{j_k} \leqslant \alpha_k$, while $\beta_{j_1} < \alpha_i$. Let w be the projection of z onto $c^{\perp\perp}$; then $\sigma(a) \leqslant \sigma(w) = \sum \beta_{j_k} < \sum \alpha_k = \sigma(c) = \sigma(a)$, a contradiction to the choice of a.

So for any r_i, there exists a unique t_j such that $r_i \wedge t_j > 0$. Since $t_1 \vee \ldots \vee t_m = r_1 \vee \ldots \vee r_n$ and both sets of singular elements of G are pairwise disjoint, we must have that $m \leqslant n$ and that each $t_j = \vee\{r_i \colon r_i \wedge t_j > 0\}$.

Suppose now there exists t_{j_0} such that $\left|\{r_i \colon r_i \wedge t_{j_0} > 0\}\right| \geqslant 2$. Let $R_{j_0} = \{r_i \colon r_i \wedge t_{j_0} > 0\}$ and let $\alpha_{i_1} < \alpha_{i_2} < \ldots$ be the standard coefficients of the r_i's $\in R_{j_0}$. Since $z \leqslant b$, $\beta_{j_0} \leqslant \alpha_i$ for all $r_i \in R_{j_0}$, and so $\beta_{j_0} < \alpha_{i_2}$. So $\sigma(b) \leqslant \sigma(z) = \sum\limits_{j=1}^{m} \beta_j = \Big(\sum\limits_{j \neq j_0} \beta_j\Big) + \beta_{j_0} < \Big(\sum\limits_{j \neq j_0} \beta_j\Big) + 2\beta_{j_0} < \Big(\sum\limits_{i \notin R_{j_0}} \alpha_i\Big) + \alpha_{i_1} + \alpha_{i_2} \leqslant \sum\limits_{i=1}^{n} \alpha_i = \sigma(b)$, which is an obvious contradiction.

So for all t_j, there exists a unique r_i such that $t_j \wedge r_i > 0$. Clearly, then, $t_j = r_i$. Since $z \leqslant b$, $\beta_i \leqslant \alpha_i$ for all i, implying $\sigma(b) \leqslant \sigma(z) = \sum\limits_{i=1}^{n} \beta_i \leqslant \sum\limits_{i=1}^{n} \alpha_i = \sigma(b)$. So for each i, $\beta_i = \alpha_i$, and so $z = b$. $\square$

Corollary 2.3. *Let G be an ℓ-group ℓ-isomorphic to an ℓ-subgroup of a Specker ℓ-group. Then G is a singular ℓ-group.*

Proof. Let $0 < g \in G$ and let $0 < t \in G(g)$ be a singular element of $G(g)$. Then $0 < g \wedge t \leqslant g$ and $s = g \wedge t$ is singular. $\square$

Corollary 2.4. *Let G be an ℓ-group ℓ-isomorphic to an ℓ-subgroup of a Specker ℓ-group with S as the set of singular elements of G. Then $S^{\perp} = (0)$.*

Corollary 2.5. *Let G be an ℓ-group ℓ-isomorphic to an ℓ-subgroup of a Specker ℓ-group. Then G has a Specker signature.*

Corollary 2.6. *Let G be an ℓ-group ℓ-isomorphic to an ℓ-subgroup of a Specker ℓ-group with S as the set of singular elements of G. For $s \in S$ and $0 < g \in G$, let g_s be the projection of g onto $s^{\perp\perp}$. Then $g = \bigvee\limits_{s \in S} g_s$.*

Now if A is an ℓ-subgroup of a Specker ℓ-group G, then a singular element of A may not be singular in G. A will be called a *Specker* ℓ-subgroup* of G if A itself is a Specker ℓ-group and every singular element of A is also singular in G.

Proposition 2.7. *The intersection of Specker* ℓ-subgroups is a Specker* ℓ-subgroup.*

First proof. Let $\{A_\lambda\}$ be Specker* ℓ-subgroups of a Specker ℓ-group G, and let $K = \bigcap A_\lambda$. Let $0 < k \in K$; then in G, $k = k_1 s_1 + \ldots + k_n s_n$, where $0 < k_1 < \ldots < k_n$ and $s_1, \ldots, s_n$ are singular elements of G. Now since for any λ_0, $k \in A_{\lambda_0}$ implies that $k = m_1 t_1 + \ldots + m_p t_p$ for mutually disjoint singular elements $t_1, \ldots, t_p \in A_{\lambda_0}$. Since $s = s_1 \vee \ldots \vee s_n = t_1 \vee \ldots \vee t_p \in K$, $k_1 s_1 = k - [k - k_1 s) \vee 0] \in K$. By induction, $k_i s_i \in K$ for all i. Thus $s_i = s \wedge k_i s_i \in K$ and thus K is Specker*. $\square$

Second proof. Let $\Sigma\mathcal{B}$ be the set of filters $\mathcal{F}$ on the generalized Boolean algebra of singular elements of G maximal with respect to $0 \notin \mathcal{F}$. For any $\mathcal{F}$, $A_{\mathcal{F}}$ is a pure subring of $\prod\limits_{\Sigma\mathcal{B}} \mathbb{Z}$, and so $K = \bigcap A_{\mathcal{F}}$ is also a pure subring of $\prod\limits_{\Sigma\mathcal{B}} \mathbb{Z}$ and thus is a Specker* ℓ-group. $\square$

Now let A be an ℓ-subgroup of a Specker ℓ-group G. By Proposition 2.7, there exists a minimal Specker* ℓ-subgroup $G(A)$ of G containing A. However, $G(A)$ may not be unique in the sense that if A is contained in another Specker ℓ-group R, then $R(A) \cong G(A)$. As an example, $2\mathbb{Z} \subset \mathbb{Z}$ and the minimal Specker* ℓ-subgroup of $\mathbb{Z}$ containing $2\mathbb{Z}$ is $\mathbb{Z}$. However, $2\mathbb{Z}$ is itself a Specker ℓ-group. Thus to get uniqueness, we need an additional hypothesis, which will be presented in Proposition 2.10 below.

Let $\mathscr{S}$ be the class of ℓ-groups G such that G is ℓ-isomorphic to an ℓ-subgroup of a Specker ℓ-group.

Theorem 2.8. *$\mathscr{S}$ is a torsion class of ℓ-groups, closed with respect to ℓ-subgroups.*

Proof. First, if A is an ℓ-subgroup of $G \in \mathscr{S}$, then clearly $A \in \mathscr{S}$. So $\mathscr{S}$ is closed with respect to convex ℓ-subgroups.

Now let $G \in \mathscr{S}$ and K be a convex ℓ-subgroup of G. Let $K+g \in G/K$, and n be an integer such that $nG(g) \subseteq S_G(g)$. Then

$$n\left[\frac{G}{K}(K+g)\right] = n\left[\frac{K+G(g)}{K}\right] \subseteq \frac{K+S_G(g)}{K} \subseteq S_{G/K}(K+g).$$

So $\mathscr{S}$ is closed with respect to ℓ-homomorphic images.

Now let $\mathcal{C}$ be a chain of convex ℓ-subgroups of an ℓ-group G such that for all $C \in \mathcal{C}$, $C \in \mathscr{S}$. Let $0 < g \in \bigcup \mathcal{C}$; there exists $C \in \mathcal{C}$ such that $g \in C$, and so there exists an integer n such that $nC(g) \subseteq S_C(g)$. But since C is convex, $C(g) = G(g) = (\bigcup \mathcal{C})(g)$ and $S_C(g) = S_{\bigcup \mathcal{C}}(g) = S_G(g)$. So $\bigcup \mathcal{C} \in \mathscr{S}$, and thus by Zorn's Lemma, the set of convex ℓ-subgroups of G in $\mathscr{S}$ has maximal elements.

Now let A, B be convex ℓ-subgroups of an ℓ-group G that are in $\mathscr{S}$. Since A, B are also hyperarchimedean, then $A \vee B$ is also hyperarchimedean. Let $0 < g \in A \vee B$. Then $g = a + b$, where $a \in A$ and $b \in B$. Since $A \vee B$ is hyperarchimedean, $a = \bar{a} + a_1 \in (a \wedge b)^{\perp} \boxplus G(a \wedge b)$, and $b = \bar{b} + b_1 \in (a \wedge b)^{\perp} \boxplus G(a \wedge b)$. Note $0 \leqslant \bar{a} \wedge \bar{b} = (a - a_1) \wedge (b - b_1) \leqslant (a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0$. So $g = \bar{a} + (a_1 + b_1) + \bar{b} \in (A \vee B)(\bar{a}) \boxplus (A \vee B)(a_1 + b_1) \boxplus (A \vee B)(\bar{b})$, $(A \vee B)(g) = (A \vee B)(\bar{a}) \boxplus (A \vee B)(a_1 + b_1) \boxplus (A \vee B)(\bar{b})$, and $S_{A \vee B}(g) = S_{A \vee B}(\bar{a}) \boxplus S_{A \vee B}(a_1 + b_1) \boxplus S_{A \vee B}(\bar{b})$.

Since $\bar{a}, a_1 + b_1 \in A$, there exist integers n_1 and n_2 such that $n_1(A \vee B)(\bar{a}) = n_1 A(\bar{a}) \subseteq S_A(\bar{a}) = S_{A \vee B}(\bar{a})$ and $n_2(A \vee B)(a_1 + b_1) = n_2 A(a_1 + b_1) \subseteq S_A(a_1 + b_1) = S_{A \vee B}(a_1 + b_1)$. Likewise, there exists an integer n_3 such that $n_3(A \vee B)(\bar{b}) \subseteq S_{A \vee B}(\bar{b})$. So $n_1 n_2 n_3[(A \vee B)(g)] \subseteq S_{A \vee B}(g)$. So if A, B are convex ℓ-subgroups of G that are also in $\mathscr{S}$, then $A \vee B \in \mathscr{S}$. Thus for any collection $\{C_\lambda\}$ of convex ℓ-subgroups of G such that $\{C_\lambda\} \subseteq \mathscr{S}$, $\bigvee C_\lambda \in \mathscr{S}$.

The closure of $\mathscr{S}$ with respect to ℓ-subgroups is obvous. $\square$

Theorem 2.9. *For an ℓ-group G, $G \in \mathscr{S}$ if and only if G is archimedean and $G^\wedge$ is a Specker ℓ-group*

P r o o f. If G is archimedean and $G^\wedge$ is a Specker ℓ-group, then $G \subseteq G^\wedge$ implies that $G \in \mathscr{S}$.

Conversely, $G \in \mathscr{S}$ implies that G is hyperarchimedean. By Corollary 2.3, G is a singular ℓ-group and so $G^\wedge$ is also singular and hyperarchimedean. But then $G^\wedge$ is Specker. $\square$

Now if $G \in \mathscr{S}$, then $G^\wedge$ is a Specker ℓ-group and G is dense in $G^\wedge$. Let $G^\#$ be the intersection of all Specker* ℓ-subgroups of $G^\wedge$ that contain G.

Proposition 2.10. *$G^\#$ is the unique minimal Specker ℓ-group in which G is dense.*

P r o o f. Suppose H is a Specker ℓ-group and G is dense in H. Then G is dense in $H^\wedge$ as well, and so $G \subseteq G^\wedge \subseteq H^\wedge$. Since G is dense in $H^\wedge$, any singular element of $G^\wedge$ is also singular in $H^\wedge$, and thus $G^\wedge$ is a Specker* ℓ-subgroup of $H^\wedge$.

Now let $\mathcal{A}$ be the set of Specker* ℓ-subgroups of $H^\wedge$ containing G, and let $J = \bigcap\{A : A \in \mathcal{A}\}$. Since $G^\wedge \in \mathcal{A}$, $J = J \cap G^\wedge = \bigcap\limits_{A\in\mathcal{A}} (A \cap G^\wedge) = G^\#$ since if A is a Specker* ℓ-subgroup of $G^\wedge$ containing G, then A is also a Specker* ℓ-subgroup of $H^\wedge$. Since H is also a Specker* ℓ-subgroup of $H^\wedge$, $G^\# \subseteq H$. □

Proposition 2.11. *$G^\#$ is the intersection of all pure subrings of $G^\wedge$ that contain G, and so is the pure subring of $G^\wedge$ generated by G.*

P r o o f. Any Specker* ℓ-subgroup of $G^\wedge$ is also a pure subring of $G^\wedge$. □

In [6], it was shown that if I is a ring ideal of a Specker ℓ-group G, then $(I, +)$ is a saturated subgroup of $(G, +)$. The converse is also true.

Proposition 2.12. *If R is a Specker f-ring, then any saturated subgroup I of $(R, +)$ is a ring ideal.*

P r o o f. Let $h = m_1s_1 + \ldots + m_ps_p \in I$ and $y = n_1t_1 + \ldots + n_rt_r \in R$. Then $h = m_1(s_1 \wedge t_1) + \ldots + m_1(s_1 \wedge t_r) + \ldots + m_r(s_r \wedge t_1) + \ldots + m_r(s_r \wedge t_p)$ while $y = n_1(s_1 \wedge t_1) + \ldots + n_1(s_p \wedge t_1) + \ldots + n_r(s_1 \wedge t_r) + \ldots + n_r(s_p \wedge t_r)$. Since $(s_{i_1} \wedge t_{j_1}) \wedge (s_{i_2} \wedge t_{j_2}) = 0$ if $i_1 \neq i_2$ or $j_1 \neq j_2$, $gh = \sum\limits_{i=1}^{p}\sum\limits_{j=1}^{r} m_in_j(s_i \wedge t_j)$. Since $m_i(s_i \wedge t_j) \in I$ because I is saturated, then $n_jm_i(s_i \wedge t_j) \in I$. So $gh \in I$. □

Corollary 2.13. *The ring ideals of a Specker f-ring R form a distributive lattice with $I \vee J = I + J$ and $I \wedge J = I \cap J$.*

P r o o f. Since ring ideals are identical with saturated subgroups, we must have that $I \vee J = I + J$ and $I \wedge J = I \cap J$.

Now let I, J, and K be ring ideals of R. Let $a \in I \cap (J + K)$; then $a = b + c$ (where $b \in J$ and $c \in K$). Without loss of generality, there exists a pairwise disjoint set of singular elements $\{s_1, \ldots, s_n\}$ and sets of integers $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, $\{c_1, \ldots, c_n\}$ such that $a = a_1s_1 + \ldots + a_ns_n$, $b = b_1s_1 + \ldots + b_ns_n$, and $c_n = c_1s_1 + \ldots + c_ns_n$. We can assume without loss of generality that $a_i \neq 0$ for all $1 \leqslant i \leqslant n$.

Now in $\mathbb{Z}$, $\langle a_i\rangle \cap (\langle b_i\rangle + \langle c_i\rangle) = (\langle a_i\rangle \cap \langle b_i\rangle) + (\langle a_i\rangle \cap \langle c_i\rangle)$. Thus each $a_i = u_i + v_i$, where $u_i \in \langle a_i\rangle \cap \langle b_i\rangle$ and $v_i \in \langle a_i\rangle \cap \langle c_i\rangle$. But $\langle a_i\rangle \cap \langle b_i\rangle = \langle d_i\rangle$, where d_i is the least common multiple of a_i and b_i. Thus u_i is a multiple of both a_i and b_i, and thus

$u_is_i \in I \cap J$ since by saturation, $a_is_i \in I$ and $b_is_i \in J$. Likewise, $v_is_i \in I \cap K$. But then

$$a = (u_1s_1 + \ldots + u_ns_n) + (v_1s_1 + \ldots + v_ns_n) \in (I \cap J) + (I \cap K).$$

So $I \cap (J + K) \subseteq (I \cap J) + (I \cap K)$. The reverse containment is clear. $\square$

Now if R is a Specker f-ring, then each convex ℓ-subgroup is saturated and so by Proposition 2.12 is a ring ideal. The following proposition gives a ring-theoretic characterization of the convex ℓ-subgroups of R.

Proposition 2.14. *For a ring ideal I of a Specker f-ring R, the following are equivalent:*

a) *R/I is torsion free.*
b) *I is pure.*
c) *I is an ℓ-ideal of $(R, \leqslant, +)$.*

P r o o f. Clearly $(a \Leftrightarrow b)$ and $(c \Rightarrow a)$.

Assume that R/I is torsion free. Now let $g \in I$ have Specker representation $m_1s_1 + \ldots + m_ns_n$. Since I is a ring ideal, $m_is_i = s_ig \in I$ for all $1 \leqslant i \leqslant n$. Since R/I is torsion free, $m_is_i \in I$ implies $s_i \in I$ for all i. Thus $u = s_1 \vee \ldots \vee s_n = s_1 + \ldots + s_n \in I$.

Now suppose $0 \leqslant h \leqslant g$ in R. Let $h = k_1t_1 + \ldots + k_pt_p$ be a Specker representation of h. Then for all $1 \leqslant j \leqslant p$, $t_j \leqslant u$ implies that $t_j = t_ju \in I$. So $h \in I$. $\square$

Proposition 2.15. *Let G be a Specker f-ring and S be its set of singular elements. For each $s \in S$, let n_s be an integer. Let H be the subgroup of $(G, +)$ generated by $\{n_ss\colon\ s \in S\}$. Then H is a ring ideal of G if and only if for every component t of s, $n_st \in H$.*

P r o o f. If H is a ring ideal of G, then H is saturated. Thus if t is a component of a singular element $s \in G$, then since n_st is a component of n_ss, $n_st \in H$.

Conversely, let $0 < h \in H$; then $h = k_1s_{s_1}s_1 + \ldots + k_mn_{s_m}s_m$. Let $t \in S$. Then $ht = k_1n_{s_1}(s_1t) + \ldots + k_mh_{s_m}(s_mt) = k_1n_{s_1}(s_1 \wedge t) + \ldots + k_mn_{s_m}(s_m \wedge t)$. Since $s_i \wedge t$ is a component of s_i for all i, $k_in_{s_i}(s_i \wedge t) \in H$, and so $gt \in H$. So for any $g \in G$, $hg = h(\alpha_1t_1 + \ldots + \alpha_pt_p) \in H$. $\square$

3A. Adjoining a unit to a Specker ℓ-group.

Proposition 3.1. *For an ℓ-group G, G is a Specker ℓ-group with an order unit if and only if G has a strong order unit that is singular.*

Let $\mathcal{B}$ be a generalized Boolean algebra; let $\Sigma\mathcal{B}$ be the set of ultrafilters on $\mathcal{B}$. $\mathcal{B}$ can be considered as a set of subsets of $\Sigma\mathcal{B}$, by mapping $b \in \mathcal{B}$ to $\{\mathcal{F} \in \Sigma\mathcal{B}\colon\ b \in \mathcal{F}\}$. Let $\mathcal{C} = \mathcal{B} \cup \{\Sigma\mathcal{B} \setminus b\colon\ b \in \mathcal{B}\}$. $\mathcal{C}$ is then the free Boolean algebra generated by $\mathcal{B}$, and will be denoted $\mathcal{B}^u$. Now if G is a Specker ℓ-group, let $\{s_\lambda\}$ be a maximal pairwise disjoint subset of singular elements of G. G can then be ℓ-embedded into $\prod(s_\lambda)^{\perp\perp}$; in $\prod(s_\lambda)^{\perp\perp}$, let u be the element $(\ldots, s_{\lambda_1}, s_{\lambda_2}, \ldots)$ and G^u be the ℓ-subgroup of $\prod(s_\lambda)^{\perp\perp}$ generated by $G \cup \{u\}$. G^u is then a Specker ℓ-group with singular order unit u. (Equivalently, if G does not have a unit, $G^u = G \oplus \langle u\rangle$, where $g \in G^u$ is singular if g is singular in G or if $u - g$ is singular in G).

Proposition 3.2. *Let $\mathcal{B}$ be a generalized Boolean algebra; let G be the Specker ℓ-group having $\mathcal{B}$ as its generalized Boolean algebra of singular and identity elements; and let H be the Specker ℓ-group having $\mathcal{B}^u$ as its generalized Boolean algebra of singular and identity elements. Then $G^u \cong H$.*

P r o o f. It suffices to show that the generalized Boolean algebras of singular and identity elements are isomorphic [6]. Now s is singular in G^u if either $s \in G$ or $u - s$ is singular in G.

Let $\mathcal{C}$ be the generalized Boolean algebra of singular and identity elements of G^u. Define $\alpha\colon\ \mathcal{B}^u \to \mathcal{C}$ by

$$t\alpha = \begin{cases} t, & t \in \mathcal{B}, \\ u - (\Sigma\mathcal{B} \setminus t), & t \notin \mathcal{B}. \end{cases}$$

α is then easily verified to be a lattice isomorphism. □

Proposition 3.3. *For a Specker ℓ-group G, G^u is the unique Specker ℓ-group with unit containing G as a dense ℓ-subgroup.*

P r o o f. Suppose G is dense in a Specker ℓ-group H with order unit x. Let $\mathcal{S}(G)$ and $\mathcal{S}(H)$ denote the generalized Boolean algebras of singular and identity elements of G and H, respectively.

Then $x = \bigvee\{s\colon\ s \text{ singular in } G\}$. Define

$$\alpha\colon\ \mathcal{S}(G^u) \to \mathcal{S}(H)\colon\ t\alpha = \begin{cases} t, & t \in \mathcal{S}(G), \\ x - (u - t), & t \notin \mathcal{S}(G). \end{cases}$$

α is then easily shown to be a lattice isomorphism of $\mathcal{S}(G^u)$ into $\mathcal{S}(H)$, and so lifts to an ℓ-embedding of G^u into H. $\square$

Proposition 3.4. *Let G be a Specker ℓ-group with a unit u. Then the following are isomorphic as Boolean algebras:*

a) *the principal polars $\{g^{\perp\perp}\}$ of G,*
b) *the principal convex ℓ-subgroups of G,*
c) *the components of u,*
d) *the singular and identity elements of G.*

Proof. Since G is hyperarchimedean, $g^{\perp\perp} = G(g)$, showing the identity of the first two Boolean algebras. Now for any $g \in G$, there exists a unique singular element $s_g \in G$ such that $G(g) = G(s_g)$, and so the map $G(g) \to s_g$ is a Boolean isomorphism of the principal convex ℓ-subgroups of G onto the components of u, which form the Boolean algebra of singular and identity elements. $\square$

We remark here that the first three Boolean algebras listed above are isomorphic for any hyperarchimedean ℓ-group with an order unit u.

Proposition 3.5. *Let G be a dense ℓ-subgroup of a Specker ℓ-group H. Then H can be ℓ-embedded into $(G^u)^\wedge$.*

Proof. Let $\mathcal{S}(G)$ and $\mathcal{S}(H)$ be the generalized Boolean algebras of singular elements of G and H, respectively. Let t be singular in H; then $t = \bigvee\{s \in \mathcal{S}\colon\ s \leqslant t\}$. Now $x = \bigvee\{s \in \mathcal{S}\colon\ s \leqslant t\}$ exists in $(G^u)^\wedge$.

So define $\alpha\colon\ \mathcal{S}(H) \to \mathcal{S}[(G^u)^\wedge]\colon\ t \to x$. α is clearly a lattice homomorphism and if $t_1 \neq t_2$ in $\mathcal{S}(H)$, either $t_1 \setminus t_2 > 0$ or $t_1 \setminus t_2 > 0$, in which case $t_1\alpha \neq t_2\alpha$. So α is injective. $\square$

3B. Adjoining a basis to a Specker ℓ-group.

Let G be a Specker ℓ-group; since G is hyperarchimedean, then for any plenary subset Δ of $\Gamma(G)$, G can be ℓ-embedded into $\prod_\Delta \mathbb{Z}$. Then $G \oplus \sum_\Delta \mathbb{Z}$ is a Specker ℓ-subgroup of $\prod_\Delta \mathbb{Z}$ containing G and having a basis. Moreover, in $G \oplus \sum_\Delta \mathbb{Z}$, $G^{\perp\perp} = G \oplus \sum_\Delta \mathbb{Z}$. This is the only method to ℓ-embed G as a Specker* ℓ-subgroup of a Specker ℓ-group H with a basis such that $G^{\perp\perp} = H$.

Now if $\Gamma(G)$ has a minimal plenary subset Δ, then G is completely distributive and, without loss of generality, $\sum_\Delta \mathbb{Z} \subseteq G \subseteq \prod_\Delta \mathbb{Z}$, showing that G already has a basis. If $\Gamma(G)$ does not have a minimal plenary subset, then for any plenary subset Δ of $\Gamma(G)$, there exists another Λ such that $\Lambda \subset \Delta$. Thus if G is not completely distributive, there is no unique way of adjoining a basis to G. Moreover, as will be shown in

Example 4.4, there is no way of guaranteeing that any two such bases have the same cardinality.

3C. The strongly projectable hull of a Specker ℓ-group G.

Let $\mathcal{B}$ be a generalized Boolean algebra. For $X \subseteq \mathcal{B}$, let $X^{\perp} = \{b \in \mathcal{B}\colon\ b \wedge x = \bar{0}$ for all $x \in X\}$. $\mathcal{B}$ will be called *strongly projectable* if for all $X \subseteq \mathcal{B}$, $\mathcal{B}$ is the convex hull of $X^{\perp\perp} \cup X^{\perp}$.

Proposition 3.6. *Let G be a Specker ℓ-group and $\mathcal{S}$ be its generalized Boolean algebra of singular and identity elements. G is strongly projectable if and only if $\mathcal{S}$ is strongly projectable.*

Now any Specker ℓ-group G has a *strongly projectable hull*; that is, a unique minimal strongly projectable ℓ-group G^{SP} in which G is dense ([1], [2], [3]). It may or may not be clear that any generalized Boolean algebra $\mathcal{B}$ has a strongly projectable hull $\mathcal{B}^{SP}$, but this will be proven shortly.

Theorem 3.7. *Let G be a Specker ℓ-group. Then G^{SP} is a Specker ℓ-group.*

Proof. Since $G^{\wedge}$ is strongly projectable and G is dense in $G^{\wedge}$, $G \subseteq G^{SP} \subseteq G^{\wedge}$. Since G is dense in G^{SP}, any singular element of G is also singular in G^{SP}. Thus $G \subseteq K$ the Specker radical of G^{SP}.

Let P be a polar in K; then $G^{SP} = P^{\perp\perp} \boxplus P^{\perp}$. Since $K \in \mathcal{C}(G)$, $K = (K \cap P^{\perp\perp}) \boxplus (K \cap P^{\perp}) = P \boxplus (K \cap P^{\perp})$. Therefore, K is strongly projectable. Since G is dense in K, $K = G^{SP}$, and thus G^{SP} is Specker. □

Proposition 3.8. *For any generalized Boolean algebra $\mathcal{B}$, there exists a unique minimal strongly projectable generalized Boolean algebra $\mathcal{C}$ in which $\mathcal{B}$ is dense.*

Proof. Let G be the Specker ℓ-group whose generalized Boolean algebra of singular and identity elements is isomorphic to $\mathcal{B}$. Let $\mathcal{C}$ be the generalized Boolean algebra of singular and identity elements of G^{SP}; then $\mathcal{C}$ is strongly projectable.

Now suppose that $\mathcal{D}$ is a strongly projectable generalized Boolean algebra such that $\mathcal{B}$ is dense in $\mathcal{D}$; let H be the Specker ℓ-group generated by $\mathcal{D}$. Then G is dense in H and so $G \subseteq G^{SP} \subseteq H$, showing $\mathcal{C} \subseteq \mathcal{D}$. □

Proposition 3.9. *Let G be a Specker ℓ-group and $\mathcal{S}$ be its generalized Boolean algebra of singular and identity elements. Let $\mathcal{S}^{SP}$ be the strongly projectable hull of $\mathcal{S}$, and let H be the Specker ℓ-group generated by $\mathcal{S}^{SP}$. Then $G^{SP} = H$.*

Proof. By definition, $\mathcal{S}^{Sp}$ is the generalized Boolean algebra of singular elements of G^{SP}. □

Now let G be a completely distributive Specker ℓ-group; then $\Gamma(G)$ has a minimal plenary subset Δ and G can be ℓ-embedded into $\prod_\Delta \mathbb{Z}$ such that $\sum_\Delta \mathbb{Z} \subseteq G \subseteq B(\Delta, \mathbb{Z})$, where $B(\Delta, \mathbb{Z}$ is the ℓ-subgroup of $\prod_\Delta \mathbb{Z}$ of bounded integer functions.

Theorem 3.10. *If G is a completely distributive Specker ℓ-group with an order unit u, then $G^{SP} = G^\wedge = B(\Delta, \mathbb{Z})$.*

Since G has a basis, G is dense in $B(\Delta, \mathbb{Z})$, while $B(\Delta, \mathbb{Z})$ is a complete Specker ℓ-group. So $G^{SP} \subseteq G^\wedge \subseteq B(\Delta, \mathbb{Z})$. Also, $B(\Delta, \mathbb{Z}) = B(\Delta, \mathbb{Z})(u) = (\prod_\Delta \mathbb{Z})(u)$.

Now for any polar P of $B(\Delta, \mathbb{Z})$, there exists $I \subseteq \Delta$ such that $P = \{x \in B(\Delta, \mathbb{Z})\colon \operatorname{supp}(x) \subseteq I\}$. Since G^{SP} is strongly projectable, $u\big|_I \in G^{SP}$. Thus the characteristic functions of all subsets of Δ are in G^{SP}, and so $G^{SP} = B(\Delta, \mathbb{Z})$. □

Corollary 3.11. *If G is a completely distributive Specker ℓ-group, then $G^{SP} = G^\wedge$ is the ℓ-ideal of $B(\Delta, \mathbb{Z})$ generated by G.*

Proof. We can assume, for Δ the minimal plenary subset of $\Gamma(G)$, that $\sum_\Delta \mathbb{Z} \subseteq G \subseteq G^{SP} \subseteq G^\wedge \subseteq B(\Delta, \mathbb{Z})$. For any $g \in G$, $G(g)$ has a basis and an order unit. Thus $[G(g)]^{SP} = G^{SP}(g)$ is a cardinal summand of $B(\Delta, \mathbb{Z})$, and so $G^{SP} = \bigvee\limits_{g\in G} G^{SP}(g)$ is the ℓ-ideal of $B(\Delta, \mathbb{Z})$ generated by G. □

Proposition 3.12. *Let G be a Specker ℓ-group and $\mathcal{S}$ be its generalized Boolean algebra of singular and identity elements. $\mathcal{S}$ is a complete atomic Boolean algebra if and only if G is ℓ-isomorphic to $B(\Lambda, \mathbb{Z})$ for some index set Λ.*

Proof. Suppose that $\mathcal{S}$ is a complete atomic Boolean algebra. Then for any atom $s \in \mathcal{S}$, s is basic in G and so G has a basis. Let Λ be the minimal plenary subset of $\Gamma(G)$; there exists an ℓ-embedding σ of G into $B(\Lambda, \mathbb{Z})$ such that singular elements of G are mapped to characteristic functions. Let $I \subseteq \Lambda$. By the completeness of $\mathcal{S}$, $s = \bigvee\{s_\lambda\colon \lambda \in I\}$ exists in $\mathcal{S}$ and $s\sigma = \chi_I$. So $G\sigma$ contains all characteristic functions of subsets of Λ and thus $G\sigma = B(\Lambda, \mathbb{Z})$.

The converse is clear. □

Theorem 3.13. *Let G be a Specker ℓ-group and $\mathcal{S}$ be its generalized Boolean algebra of singular and idientity elements. The following are equivalent:*

a) *G is complete, completely distributive, and has a unit.*
b) *G is strongly projectable, completely distributive, and has a unit.*
c) *$\mathcal{S}$ is atomic and laterally complete.*
d) *$\mathcal{S}$ is an atomic complete Boolean algebra.*
e) *$G \cong B(\Lambda, \mathbb{Z})$ for some index set Λ.*

P r o o f. (a$\Leftrightarrow$b) is clear from Theorem 3.10, as is $(e \Rightarrow a)$. For $(a \Rightarrow d)$, since G has a basis, $\mathcal{S}$ is atomic. The equivalence of $(c \Leftrightarrow d)$ is well known, and the equivalence of $(d \Leftrightarrow e)$ is Proposition 3.12. □

Corollary 3.14. *Let G be a Specker ℓ-group and $\mathcal{S}$ be its generalized Boolean algebra of singular and identity elements. Let $\mathcal{S}^L$ be the Boolean algebra of joins of all pairwise disjoint subsets of $\mathcal{S}$, and let H be the Specker ℓ-group generated by $\mathcal{S}^L$. Then $(G^u)^\wedge \cong H$.*

P r o o f. The proof is immediate from Theorem 3.13 and so will be omitted. □

Proposition 3.15. *If Λ is infinite, then $B(\Lambda, \mathbb{Z})$ has nonclosed prime subgroups.*

P r o o f. Since Λ is infinite, $\Lambda = \Lambda_1 \cup \Omega_1$, where $\Lambda_1 \cap \Omega_1 = \emptyset$ and Λ_1, Ω_1 are both infinite. Likewise, $\Lambda_1 = \Lambda_2 \cup \Omega_2$, where both Λ_2, Ω_2 are infinite and $\Lambda_2 \cap \Omega_2 = \emptyset$. Continuing in this way, we can find a sequence $\Lambda_1 \supset \Lambda_2 \supset \dots$ of infinite sets such that $\bigcap \Lambda_i = \emptyset$. Since $\{\Lambda_i\}_{i=1}^{\infty}$ has the finite intersection property, there exists an ultrafilter $\mathcal{U}$ on Λ such that $\Lambda_i \in \mathcal{U}$ for all i. Since $\bigcap \Lambda_i = \emptyset$, $\mathcal{U}$ is nonprincipal and thus $K_{\mathcal{U}} = \{g \in B(\Lambda, \mathbb{Z})\colon \operatorname{supp}(g) \in \mathcal{U}\}$ is a nonclosed prime subgroup. □

Now if G is a completely distributive Specker ℓ-group, then since G is archimedean, G must have a basis, and so must have a basis subgroup $B(G)$. G is finite-valued if and only if $G = B(G)$. The following theorem, generalizing an example given by Conrad in [2], characterizes finite-valued Specker ℓ-groups in terms of a-closures. (Recall that an ℓ-group H is an *a-extension* of an ℓ-group G if for each $0 < h \in H$, there exists $0 < g \in G$ and a positive integer n such that $h < ng$ and $g < nh$. An ℓ-group G is a-closed if G admits no proper a-extensions.)

Theorem 3.16. *Let G be a Specker group. G is finite-valued if and only if G has a unique a-closure.*

P r o o f. ($\Rightarrow$) Clear, since $\Gamma(G)$ is trivially ordered [15].

($\Leftarrow$) Suppose that G is not finite-valued. Then there exists a singular element a and pairwise disjoint singular elements $\{b_j\}_{j=1}^{\infty}$ such that $0 < b_j < a$ for all j.

We can assume without loss of generality that $G = G(a)$, as in any event, $G = G(a) \boxplus a^{\perp}$, and so if it can be shown that $G(a)$ has two nonisomorphic a-closures, then G must also.

Let $\Gamma = \Gamma(G)$; G can then be ℓ-embedded into $\prod_\Gamma \mathbb{R}$ such that each singular element is mapped to the characteristic function of a subset of Γ. Define $h \in \prod_\Gamma \mathbb{R}$

by:

$$h(\lambda)=\begin{cases}\pi(1+1/j), & \lambda\in \operatorname{supp}(b_j);\\ \pi, & \lambda\notin \operatorname{supp}(b_j) \text{ for all } j,\end{cases}$$

and let H be the ℓ-subgroup of $\prod_\Gamma \mathbb{R}$ generated by $G\cup\{h\}$.

Now let m be any integer and g be any element of G. The claim is that there exists a singular element $s\in G$ such that $\operatorname{supp}(s)=\operatorname{supp}[(mh+g)\vee 0]$. This is clearly true if $m=0$, and if for all $\lambda\in\Gamma$, $(mh+g)(\lambda)>0$, then $s=a$ is a clear choice. On the other hand, if for all λ, $(mh+g)(\lambda)\leqslant 0$, then $s=0$ is the choice.

So suppose there exist $\lambda_1,\lambda_2\in\Gamma$ such that $(mh+g)(\lambda_1)<0<(mh+g)(\lambda_2)$. There exist pairwise disjoint singular elements $\{t_1,\ldots,t_n\}\subseteq G$ and integers $\{k_1,\ldots,k_n\}$ such that $g=k_1t_1+\ldots+k_nt_n$.

Suppose $m>0$. Let $N=\{i\colon$ there exists $\lambda\in\operatorname{supp}(t_i)$ such that $(mh+g)(\lambda)<0\}$. For $i\in N$, let $K_i=\{j\colon t_i\wedge b_j>0 \text{ and } m\pi(1+1/j)+k_i>0\}$. Then K_i is finite, else $m\pi+k_i=\lim\limits_{j\to\infty}[m\pi(1+1/j)+k_i]\geqslant 0$, and so for all $\lambda\in\operatorname{supp}(t_i)$, $(mh+g)(\lambda)\geqslant 0$. Hence, since K_i is finite, let $s=a-\sum\limits_{i\in N}t_i+\sum\limits_{i\in N}\sum\limits_{j\in K_i}(t_i\wedge b_j)$.

Suppose $m<0$. Then let $N=\{i\colon$ there exists $\lambda\in\operatorname{supp}(t_i)$ such that $(mh+g)(\lambda)<0\}$, and let $K_i=\{j\colon t_i\wedge b_j>0 \text{ and } m\pi(1+1/j+k_i<0\}$. Again, K_i is finite. In this case, let $s=\sum\limits_{i\in N}\Big[t_i-\sum\limits_{j\in K_i}(t_i\wedge b_j)\Big]$.

Now for any $0<x\in H$, there exist finite sets I and J, integers $\{m_{ij}\}_{I\times J}$, and elements $\{g_{ij}\}_{I\times J}\subseteq G$ such that $x=\bigvee\limits_I\bigwedge\limits_J[(m_{ij}h+g_{ij})\wedge 0$. For $(i,j)\in I\times J$, let s_{ij} be a singular element of G such that $\operatorname{supp}(s_{ij})=\operatorname{supp}[(m_{ij}h+g_{ij})\vee 0]$.

Then

$$\begin{aligned}\operatorname{supp}(x)&=\bigcup_I\bigcap_J\operatorname{supp}[(m_{ij}h+g_{ij})\vee 0]\\&=\bigcup_I\bigcap_J\operatorname{supp}(s_{ij})\\&=\operatorname{supp}\Big(\bigvee_I\bigwedge_J s_{ij}\Big)\end{aligned}$$

and so for all $0<x\in H$, there exists a singular element $s\in G$ such that $\operatorname{supp}(x)=\operatorname{supp}(s)$. Since the range of x is bounded and bounded away from 0, x is a-equivalent to s. Thus H is an a-extension of G, but cannot be represented by step functions [2]. Since G does have an a-closure which consists of real-valued step functions, G has more than two nonisomorphic a-closures. $\square$

4. Examples

Example 4.1. $G = \sum_{i=1}^{\infty} \mathbb{Z} \oplus \mathbb{Z}[2,2,2,\ldots]$ is an ℓ-subgroup of the eventually constant integer sequences $\mathbb{Z}^{EC} \subseteq \prod_{i=1}^{\infty} \mathbb{Z}$ that is not Specker, nor a Specker* ℓ-subgroup of $\mathbb{Z}^{EC}$, yet is dense. Note that for any $g \in G$, $2G(g) \subseteq S(g)$.

More generally, let $\{k_i\} \subseteq \mathbb{Z}^+ \backslash \{0\}$ be a bounded set and let $H = \sum_{i=1}^{\infty} \mathbb{Z}[(k_1, k_2, \ldots)]$. Then $(k_1, k_2, \ldots)$ is an order unit for H. Let $n = \bigvee\{k_i\}$. Then $nH \subseteq S((k_1, k_2, \ldots))$.

Example 4.2. An example of an ℓ-subgroup A of a Specker ℓ-group F in which A is not large in the minimal Specker* ℓ-subgroup H of F containing A.

Let $F = B(\omega, \mathbb{Z}) \boxplus \mathbb{Z}$ and let $G = \sum_{i=1}^{\infty} \mathbb{Z} \oplus \mathbb{Z}[((2,2,2,\ldots),1)]$. G is then an ℓ-subgroup of F. Then $H = \left(\sum_{i=1}^{\infty} \mathbb{Z} \boxplus \mathbb{Z}\right) \oplus \mathbb{Z}[((1,1,1,\ldots),0)]$ is the minimal Specker* ℓ-subgroup of F that contains G but G is not dense in H.

Now let K be the ℓ-subgroup of $B(\omega, \mathbb{Z}) \boxplus \mathbb{R}$ generated by $\sum_{i=1}^{\infty} \mathbb{Z}$ and $((1,1,1,\ldots), \frac{1}{2})$. K is then a minimal Specker ℓ-group in which G is dense.

Example 4.3. Nobeling [14] proved that if G is a Specker* ℓ-subgroup of a Specker ℓ-group H, then $H = G \oplus A$, where A is generated by characteristic functions. This example will show that in most cases, A can not be chosen to be an ℓ-subgroup of H.

Let $\omega = \{0,1,2,3,\ldots\}$ and let $1 < p_1 < p_2 < p_3 < \ldots$ be an infinite set of prime integers. For every p_i, let x_i be the characteristic function of $\{p_i, p_i^2, p_i^3, \ldots\}$. Note that if $i \neq j$, then $x_i \wedge x_j = 0$.

Assume that $B(\omega, \mathbb{Z}) = \left(\sum_{i=1}^{\infty} \mathbb{Z}\right) \oplus A$, where A is an ℓ-subgroup of $B(\omega, \mathbb{Z})$ generated by characteristic functions. Now if $c \in B(\omega, \mathbb{Z})$ is a characteristic function of an infinite subset of ω, $c = s + a$, where $x \in \sum_{i=1}^{\infty} \mathbb{Z}$ and $a \in A$. So $a = c - s$; let $N = \max\{i \in \omega\colon\ s(i) \neq 0\}$. Then for all $i > N$, $a(i) = c(i)$, and so $a \vee 0$ has infinite support.

Each $x_i = s_i + a_i$, where $s_i \in \sum_{i=1}^{\infty} \mathbb{Z}$ and $a_i \in A$; let $N_i = \max(\operatorname{supp}(s_i))$ and let n_{ij} be the j^{th} element of $\operatorname{supp}(a_i \vee 0)$ *after* N_i. Note that if $i_1 \neq i_2$, $\{n_{i_1}\} \cap \{n_{i_2}\} = \emptyset$.

Let b be the characteristic function of $\{n_{11}, n_{22}, n_{33}, \ldots\}$. Then $b = s + d$, where $s \in \sum_{i=1}^{\infty} \mathbb{Z}$ and $d \in A$. Again, $d \vee 0$ has infinite support and there exists N such that for all $k > N$, $\operatorname{supp}(d \vee 0)(k) = b(k)$. So for all i such that $n_{ii} > N$, $\operatorname{supp}(d \vee$

0) $\cap$ supp$(a_i \vee 0) = \{n_{ii}\}$. Thus $(d \vee 0) \wedge (a_i \vee 0) \in \sum\limits_{i=1}^{\infty} \mathbb{Z}$, which contradicts that $A \cap \Big(\sum\limits_{i=1}^{\infty} \mathbb{Z}\Big) = (0)$.

Example 4.4. Let $\mathcal{J}$ denote the set of irrational numbers and let G be the ℓ-subgroup of $\prod\limits_{\mathcal{J}} \mathbb{Z}$ generated by characteristic functions of the form $(p,q) \cap \mathcal{J}$, where p, q are rational numbers. Let $\Delta_1 = \mathbb{Q}\pi$ (all rational multiples of π); Δ_1 is then a countable plenary subset of $\Gamma(G)$. On the other hand, $\mathbb{R} = \mathbb{Q} \oplus \mathbb{Q}\pi \oplus D$, where D is uncountable. Then $\Delta_2 = D$ is an uncountable plenary subset of $\Gamma(G)$, disjoint from Δ_1.

References

[1] *P. F. Conrad*: The hulls of representable ℓ-groups and f-rings. J. Austral. Math. Soc. *16* (1973), 385–415.
[2] *P. F. Conrad*: Epi-archimedean ℓ-groups. Czechoslovak Math. J. *24* (1974), 192–218.
[3] *P. F. Conrad*: The hulls of semiprime rings. Czechoslovak Math. J. *28* (1978), 59–86.
[4] *P. F. Conrad and M. R. Darnel*: Lattice-ordered groups whose lattices determine their additions. Trans. Amer. Math. Soc. *330* (1992), 575–598.
[5] *P. F. Conrad and M. R. Darnel*: Countably valued lattice-ordered groups. Algebra Universalis *36* (1996), 81–107.
[6] *P. F. Conrad and M. R. Darnel*: Generalized Boolean algebras in lattice-ordered groups. Order *14* (1998), 295–319.
[7] *P. F. Conrad and J. Martinez*: Signatures and S-discrete lattice-ordered groups. Algebra Universalis *29* (1992), 521–545.
[8] *P. F. Conrad and D. McAlister*: The completion of a ℓ-group. J. Austral. Math. Soc. *9* (1969), 182–208.
[9] *M. R. Darnel*: The Theory of Lattice-ordered Groups. Marcel Dekker, 1995.
[10] *M. R. Darnel, M. Giraudet and S. H. McCleary*: Uniqueness of the group operation on the lattice of order-automorphisms of the real line. Algebra Universalis *33* (1995), 419–427.
[11] *W. C. Holland*: Partial orders of the group of automorphisms of the real line. Proc. International Conf. on Algebra, Part 1 (Novosibirsk, 1989), pp. 197–207; Contemp. Math. Vol. 131. Amer. Math. Soc., 1992.
[12] *J. Jakubík*: Lattice-ordered groups with unique addition must be archimedean. Czechoslovak Math. J. *41(116)* (1991), 559–603.
[13] *S. Lin*: Some Theorems on Lattice-ordered Groups. Dissertation, University of Kansas, 1991.
[14] *C. Nobeling*: Verallgemeinerung eines Satzes von Herrn E. Specker. Invent. Math. *6* (1968), 41–55.
[15] *S. Wolfenstein*: Contribution à l'étude des groupes reticulés: Extensions archimédiennes, Groupes à valeurs normales. Thesis, Sci. Math. Paris, 1970.

Authors' addresses: P. F. Conrad, Department of Mathematics, University of Kansas, USA; M. R. Darnel, Department of Mathematics, Indiana University South Bend, USA, e-mail: mdarnel@iusb.edu.