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# FINITELY VALUED $f$-MODULES, AN ADDENDUM

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*Abstract.* In an $\ell$-group $M$ with an appropriate operator set $\Omega$ it is shown that the $\Omega$-value set $\Gamma_\Omega(M)$ can be embedded in the value set $\Gamma(M)$. This embedding is an isomorphism if and only if each convex $\ell$-subgroup is an $\Omega$-subgroup. If $\Gamma(M)$ has a.c.c. and $M$ is either representable or finitely valued, then the two value sets are identical. More generally, these results hold for two related operator sets $\Omega_1$ and $\Omega_2$ and the corresponding $\Omega$-value sets $\Gamma_{\Omega_1}(M)$ and $\Gamma_{\Omega_2}(M)$. If $R$ is a unital $\ell$-ring, then each unital $\ell$-module over $R$ is an $f$-module and has $\Gamma(M) = \Gamma_R(M)$ exactly when $R$ is an $f$-ring in which 1 is a strong order unit.

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Let $M$ be a lattice-ordered group ($\ell$-group) and let $x$ be an element of $M$. A convex $\ell$-subgroup of $M$ that is maximal with respect to not containing $x$ is called a *value* of $x$. With respect to inclusion the set $\Gamma$ of all of the values of the elements in $M$ is a *rooted* partially ordered set (poset); that is, the elements in $\Gamma$ that exceed a given element form a chain. $\Gamma$ is called the *value set* of $M$. A good portion of the theory of $l$-groups is intimately connected with values—see [5], [6], [8] and [1]. Now, suppose that $M$ is also an $\Omega$-group; thus, $\Omega$ is a set and there is a function $\Omega \to \mathrm{End(M)}$ from $\Omega$ to the set of group endomorphisms of $M$. We will occasionally identify $\Omega$ with its image in $\mathrm{End}(M)$. $M$ is called an $\Omega$-$\ell$-*group* if $M^+\omega \subseteq M^+$ for each $\omega \in \Omega$ where $M^+ = \{x \in M\colon x \geqslant 0\}$ is the positive cone of $M$, and $M$ is called an $\Omega$-$d$-*group* if each $\omega \in \Omega$ induces a lattice endomorphism of $M$. $M$ is an $\Omega$-$f$-*group* if

$$\forall \omega \in \Omega,\ \ \forall x, y \in M,\ \ x \wedge y = 0 \Rightarrow x\omega \wedge y = 0.$$

In [7] an endomorphism $\omega$ of this type is called a $p$-endomorphism and in [2] it is called a positive orthomorphism. For example, if $M$ is a right $R$-module over the partially ordered ring $R$ and $M$ is also an $\ell$-group, then $M$ is an $\ell$-module over $R$ iff it is an $R^+$-$l$-group; and it is an $f$-module iff it is an $R^+$-$f$-group. It is known [10, Theorem 1.1) that when $R$ is directed (i.e., $R = R^+ - R^+$) the $\ell$-module $M_R$ is an $f$-module precisely when it can be embedded into the product of a family of totally ordered $R$-$\ell$-modules. More generally, recall that an $\ell$-group is *representable* if it can be embedded into a product of totally ordered groups. In the $\ell$-group $M$, let $N = \{a \in M\colon\ x \wedge y = 0 \Rightarrow (-a + x + a) \wedge y = 0\}$. Then $N$ is a normal $l$-subgroup of $M$, and $M$ is always an $N$-$f$-group and an $M$-$d$-group (the action is by conjugation). $M$ is representable if and only if $N = M$ [1, Theorem 4.1.1]. Now suppose that $M$ is an $\Omega$-$\ell$-group. Then $M$ is a representable $l$-group and an $\Omega$-$f$-group if and only if it can be embedded (as an $\Omega$-$\ell$-group) into the product of a family of totally ordered $\Omega$-$\ell$-groups.

Perhaps it should be reiterated here that, as stated in the introduction to [10], almost all of the results and the arguments presented in the first two sections of that paper are valid for an $\Omega$-$f$-group even though they are presented in the context of an $f$-module. The only change that needs to be taken into account, aside from the loss of normality of a subgroup, is the technical description of the convex $\ell$-$\Omega$-subgroup generated by an element (or a subset). If $M$ is an $\Omega$-group and $\omega_1, \ldots, \omega_k \in \Omega$ and $a \in M$ we denote $a\omega_1 \ldots \omega_k$ by $aW$ and say $W = \omega_1 \ldots \omega_k \in \Omega^\infty$; if $k = 0$ then $aW = a$. Now if $M$ is an $\Omega$-$d$-group then the convex $\ell$-$\Omega$-subgroup generated by $a \in M$ is

$$C_\Omega(a) = \{x \in M\colon\ |x| \leqslant |a|W_1 + \ldots + |a|W_n,\ \ W_i \in \Omega^\infty\}.$$

We denote the lattice of convex $\ell$-$\Omega$-subgroups of $M$ by $\mathcal{C}_\Omega(M)$.

Recall that the function $\alpha\colon\ X \longrightarrow Y$ between two posets $X$ and $Y$ is *isotone* if $a \leqslant b$ in $X$ implies that $\alpha(a) \leqslant \alpha(b)$ in $Y$; and if $a \leqslant b$ iff $\alpha(a) \leqslant \alpha(b)$ then $\alpha$ is an embedding of $X$ into $Y$.

An $\Omega$-*value* of the element $x$ in the $\Omega$-$\ell$-group $M$ is a convex $\ell$-$\Omega$-subgroup that is maximal with respect to excluding $x$. In [10, Theorem 2.2] it is shown that for each $x$ in the $\Omega$-$f$-group $M$ there is a bijection between the set $\Gamma_\Omega(x)$ of the $\Omega$-values of $x$ and the set $\Gamma(x)$ of its values. At the time that [10] was written we had thought that the rooted poset $\Gamma_\Omega$ of $\Omega$-values of the $\Omega$-$f$-group $M$ could be embedded naturally in $\Gamma$, but we couldn't verify it. Several years ago we saw the simple verification. However, since this fact still does not seem to be known we present this embedding (actually, these embeddings) in this addendum. We also show that this embedding is an isomorphism if and only if each convex $\ell$-subgroup is an $\Omega$-subgroup. This is the case if $\Gamma$ satisfies the ascending chain condition (a.c.c.) and $M$ is either representable or finitely valued.

A convex $\ell$-subgroup $P$ of an $\ell$-group $M$ is called *prime* if $a \wedge b = 0$ implies that $a \in P$ or $b \in P$. It is well-known that $P$ is prime iff the poset of convex $\ell$-subgroups that contain $P$ is a chain iff $P$ is finitely meet irreducible in the lattice $\mathcal{C}(M)$ of all convex $\ell$-subgroups of $M$ [1, Theorem 1.2.10]; also, it is a consequence of [10, Corollary 1.3] that each $\Omega$-value in an $\Omega$-$f$-group is prime.

Let $M$ be an $\Omega$-po-group; that is, $M$ is an $\Omega$-group, a po-group and $M^+\omega \subseteq M^+$ for each $\omega \in \Omega$. If $A$ is a subgroup of $M$ let $A_\Omega$ denote the subgroup of $A$ that is generated by the family of all those $\Omega$-subgroups of $M$ that are contained in $A$. It is obvious that $A_\Omega$ is the largest $\Omega$-subgroup of $M$ that is contained in $A$. Moreover, we have the following.

(i) $(\cap_i A_i)_\Omega = \cap_i (A_i)_\Omega$.

(ii) If $B$ is an $\Omega$-subgroup of $M$ then the convex subgroup $C(B)$ generated by $B$ is an $\Omega$-subgroup; if $B$ is an $\Omega$-$\ell$-subgroup of the $\Omega$-$\ell$-group $M$ then $C(B)$ is an $\Omega$-$\ell$-subgroup.

(iii) If $A$ is an $\ell$-subgroup (respectively, convex $\ell$-subgroup) of the $\Omega$-$d$-group $M$ then $A_\Omega$ is an $\Omega$-$\ell$-subgroup (respectively, convex $\ell$-$\Omega$-subgroup) of $M$.

(iv) If $A$ is a prime subgroup of the $\Omega$-$f$-group $M$ then $A_\Omega$ is a prime subgroup. (This follows from [10, Corollary 1.3].)

(v) For an element $x$ in the $\Omega$-$f$-group $M$ the bijection of value sets $\Gamma(x) \longrightarrow \Gamma_\Omega(x)$ is given by $A \mapsto A_\Omega$ ([10, Theorem 2.2]).

Now, suppose that $M$ is both an $\Omega_1$-$f$-group and an $\Omega_2$-$f$-group; more briefly, $M$ is an $\Omega_1$-$\Omega_2$-$f$-group. We are interested in the relation between the value sets $\Gamma_{\Omega_1}(M)$ and $\Gamma_{\Omega_2}(M)$. We write $\Omega_1 \leqslant_M \Omega_2$ if $A_{\Omega_2} \subseteq A_{\Omega_1}$ for each convex $\ell$-subgroup $A$ of $M$. For example, if $\Omega_1 = \{1\}$, or if $M$ is abelian and $\Omega_1 \subseteq \mathbb{N}$, then $\Omega_1 \leqslant_M \Omega_2$ for any $\Omega_2$; or if $\Omega_1 \subseteq \Omega_2$ then $\Omega_1 \leqslant_M \Omega_2$.

**Lemma 1.** *Let $M$ be an $\Omega_1$-$f$-group and an $\Omega_2$-$f$-group. The following statements are equivalent.*

(a) $\Omega_1 \leqslant_M \Omega_2$.

(b) *Each convex $\ell$-$\Omega_2$-subgroup is an $\Omega_1$-subgroup.*

(c) *If $A \in \Gamma(M)$, then $A_{\Omega_2} \subseteq A_{\Omega_1}$.*

(d) *For each $a \in M$, $C_{\Omega_1}(a) \subseteq C_{\Omega_2}(a)$.*

(e) *If $A \in \Gamma(M)$ then $(A_{\Omega_1})_{\Omega_2} = A_{\Omega_2}$.*

(f) *If $A$ is a convex $\ell$-subgroup of $M$ then $(A_{\Omega_1})_{\Omega_2} = A_{\Omega_2}$.*

(g) *For each $x \in M$, if $A \in \Gamma_{\Omega_1}(x)$ then $A_{\Omega_2} \in \Gamma_{\Omega_2}(x)$. (Thus, the mapping $\Gamma_{\Omega_1}(x) \longrightarrow \Gamma_{\Omega_2}(x)$ given by $A \mapsto A_{\Omega_2}$ is a bijection.)*

Proof. For (a) $\Rightarrow$ (b), if $A \in \mathcal{C}_{\Omega_2}(M)$ then $A = A_{\Omega_2} \subseteq A_{\Omega_1} \subseteq A$; so $A = A_{\Omega_1}$. The equivalences of (b) with (d) and of (c) with (e) and the implication (b) $\Rightarrow$ (c) are equally obvious. That (c) $\Rightarrow$ (a) and that (e) and (f) are equivalent follow from (i) and the fact that each convex $\ell$-subgroup is an intersection of a set of values. To see that (e) $\Rightarrow$ (g), let $\alpha_i\colon \Gamma(x) \longrightarrow \Gamma_{\Omega_i}(x)$ be the bijection given by (v) and let $\beta = \alpha_2\alpha_1^{-1}\colon \Gamma_{\Omega_1}(x) \longrightarrow \Gamma_{\Omega_2}(x)$. If $A \in \Gamma_{\Omega_1}(x)$ and $\alpha_1(B) = B_{\Omega_1} = A$, then $\beta(A) = \alpha_2(B) = B_{\Omega_2} = (B_{\Omega_1})_{\Omega_2} = A_{\Omega_2}$ by (e). To see that (g) $\Rightarrow$ (e) take $B \in \Gamma(M)$; so $B \in \Gamma(x)$ for some $x \in M$. Then $A = B_{\Omega_1} \in \Gamma_{\Omega_1}(x)$ and hence $(B_{\Omega_1})_{\Omega_2} = A_{\Omega_2} \in \Gamma_{\Omega_2}(x)$. But then $(B_{\Omega_1})_{\Omega_2} = B_{\Omega_2}$ since they are comparable $\Omega_2$-values of $x$. $\square$

The embedding of $\Gamma_\Omega(M)$ into $\Gamma(M)$ will follow from

**Lemma 2.** *Let $X$ and $Y$ be rooted subsets of some poset and suppose that $X \xrightarrow{\varrho} Y$ is an order preserving onto mapping such that $\varrho(x) \leqslant x$ for each $x$ in $X$. Then each of the right inverses of $\varrho$ is an embedding of $Y$ into $X$.*

Proof. If $\varrho(x_1) < \varrho(x_2)$ then since $\varrho(x_i) \leqslant x_i$, $x_1$ and $x_2$ are comparable. But then $x_1 < x_2$ since $\varrho$ is isotone. Thus, each right inverse $\varphi$ of $\varrho$ is also isotone. If $\varphi(y_1) < \varphi(y_2)$, then $y_1 = \varrho\varphi(y_1) < \varrho\varphi(y_2) = y_2$; so $\varphi$ is an embedding. $\square$

**Theorem 3.** *Let $M$ be an $\Omega_1$-$\Omega_2$-$f$-group and suppose that $\Omega_1 \leqslant_M \Omega_2$. Then there is a natural embedding of posets $\Gamma_{\Omega_2}(M) \longrightarrow \Gamma_{\Omega_1}(M)$ given by: for each $B \in \Gamma_{\Omega_2}(M)$ choose $A \in \Gamma_{\Omega_1}(M)$ with $B = A_{\Omega_2}$.*

Proof. By (g) of Lemma 1 the assignment $A \mapsto A_{\Omega_2}$ is an isotone map of $\Gamma_{\Omega_1}(M) = \cup_x \Gamma_{\Omega_1}(x)$ onto $\Gamma_{\Omega_2}(M) = \cup_x \Gamma_{\Omega_2}(x)$. By Lemma 2 each right inverse of this mapping embeds $\Gamma_{\Omega_2}(M)$ into $\Gamma_{\Omega_1}(M)$. $\square$

Since each convex $\ell$-$\Omega$-subgroup is the intersection of a set of $\Omega$-values it is clear that $\Gamma_{\Omega_1}(M) = \Gamma_{\Omega_2}(M)$ if and only if $\mathcal{C}_{\Omega_1}(M) = \mathcal{C}_{\Omega_2}(M)$ for an $\Omega_1$-$\Omega_2$-$f$-group $M$. If $\Gamma_{\Omega_1} = \Gamma_{\Omega_2}$ then the embedding given in Theorem 3 is the identity. The same conclusion holds if this embedding is an isomorphism, as we will now show. If $K$ is an $\Omega$-value (or a value) in $M$ we will denote its cover in the lattice $\mathcal{C}_\Omega(M)$ (or $\mathcal{C}(M)$) by $K^*$. Of course, $K \in \Gamma_\Omega(x)$ (or $K \in \Gamma(x)$) exactly when $x \in K^* \setminus K$.

**Lemma 4.** *Suppose that $M$ is an $\Omega_1$-$\Omega_2$-$f$-group with $\Omega_1 \leqslant_M \Omega_2$ and $K \in \Gamma_{\Omega_1}(M)$. Then*
(a) $K^* \subseteq (K_{\Omega_2})^*$.
(b) *Either $(K^*)_{\Omega_2} = K_{\Omega_2}$ or $(K^*)_{\Omega_2} = (K_{\Omega_2})^* = K^*$.*

Proof. (a) If $x \in K^* \setminus K$ then $x \in (K_{\Omega_2})^* \setminus K_{\Omega_2}$, by (g) of Lemma 1, and hence $x \in (K_{\Omega_2})^* \setminus K$. Now, $K$ and $(K_{\Omega_2})^*$ are comparable since they both contain $K_{\Omega_2}$; so $K \subset (K_{\Omega_2})^*$ and $K^* \subseteq (K_{\Omega_2})^*$.

(b) If the containment $K_{\Omega_2} \subseteq (K^*)_{\Omega_2}$ is proper then $(K_{\Omega_2})^* \subseteq (K^*)_{\Omega_2}$. Thus, $(K^*)_{\Omega_2} \subseteq K^* \subseteq (K_{\Omega_2})^* \subseteq (K^*)_{\Omega_2}$ and (b) follows. $\square$

**Corollary 5.** *The following statements are equivalent for the $\Omega_1$-$\Omega_2$-f-group $M$ with $\Omega_1 \leqslant_M \Omega_2$.*

(a) *The mapping $\Gamma_{\Omega_1}(M) \longrightarrow \Gamma_{\Omega_2}(M)$ given by $K \mapsto K_{\Omega_2}$ is an isomorphism.*

(b) *$K^* \setminus K = (K_{\Omega_2})^* \setminus K_{\Omega_2}$ for each $K \in \Gamma_{\Omega_1}(M)$.*

(c) *$\Gamma_{\Omega_1}(M) = \Gamma_{\Omega_2}(M)$.*

(d) *$(K_{\Omega_2})^* = (K^*)_{\Omega_2}$ for each $K$ in $\Gamma_{\Omega_1}(M)$.*

Proof. (a) $\Rightarrow$ (b). That $K^* \setminus K \subseteq (K_{\Omega_2})^* \setminus K_{\Omega_2}$ follows from (g) of Lemma 1 (or (a) of Lemma (4)). Let $x \in (K_{\Omega_2})^* \setminus K_{\Omega_2}$; then $x$ has an $\Omega_1$-value $L$ which contains $K_{\Omega_2}$ since $K_{\Omega_2}$ is an $\Omega_1$-subgroup. Since $L_{\Omega_2} \supseteq K_{\Omega_2}$ and $x \notin L_{\Omega_2}$, necessarily $L_{\Omega_2} = K_{\Omega_2}$. Thus $L = K$ and $x \in K^* \setminus K$.

(b) $\Rightarrow$ (c). Suppose that $K \in \Gamma_{\Omega_1}$ and $x \in K \setminus K_{\Omega_2}$. Then $x$ has an $\Omega_2$-value $A$ that contains $K_{\Omega_2}$. Since $A$ and $K$ are comparable $A \subset K$ and hence $A = K_{\Omega_2}$. So $x \in (K_{\Omega_2})^* \setminus K_{\Omega_2} = K^* \setminus K$. This contradiction gives that $K = K_{\Omega_2}$. Thus, $\mathcal{C}_{\Omega_1}(M) = \mathcal{C}_{\Omega_2}(M)$.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (a). Suppose that $K, L \in \Gamma_{\Omega_1}$ with $K_{\Omega_2} = L_{\Omega_2}$. If $K \subset L$ then $K^* \subseteq L$. But then $L^* = (L_{\Omega_2})^* = (K_{\Omega_2})^* = K^* \subseteq L$ by Lemma 4. Since $K$ and $L$ are comparable, necessarily $K = L$. $\square$

**Corollary 6.** *Let $M$ be an $\Omega$-$f$-group. There is a natural embedding of posets $\Gamma_\Omega(M) \longrightarrow \Gamma(M)$ given by: for each $B \in \Gamma_\Omega(M)$ choose $A \in \Gamma(M)$ with $B = A_\Omega$. This embedding is an isomorphism iff it is the identity iff $(A^*)_\Omega = (A_\Omega)^*$ for each $A \in \Gamma(M)$.*

We next consider a finite condition on an $\Omega$-$f$-group. Recall that the $\ell$-group $M$ is *finitely valued* if $\Gamma(x)$ is finite for each $x$ in $M$. If $M$ is finitely valued or is representable then $A$ is normal in $A^*$ for each $A$ in $\Gamma(M)$; that is, $M$ is *normal-valued* [1, p. 29 and Proposition 10.11].

**Theorem 7.** *Let $M$ be an $\Omega_1$-$\Omega_2$-$f$-group which satisfies the following conditions.*

(a) *$\Omega_1 \leqslant_M \Omega_2$.*

(b) *For each $\omega_1 \in \Omega_2$ and each $\omega_2 \in \Omega_1$ there is an $\bar{\omega}_1 \in \Omega_1$ such that $\omega_2\omega_1 \leqslant \bar{\omega}_1\omega_2$; that is $x\omega_2\omega_1 \leqslant x\bar{\omega}_1\omega_2$ for each $x \in M^+$.*

(c) *If $A \in \Gamma_{\Omega_1}$ then $A \triangleleft A^*$.*

(d) *$M$ is representable or finitely valued.*

*If $\Gamma_{\Omega_1}(M)$ satisfies a.c.c. then $\Gamma_{\Omega_1}(M) = \Gamma_{\Omega_2}(M)$.*

P r o o f. We first show that if $0 < a \in M$ has a single value (i.e., $a$ is *special*), then $C_{\Omega_1}(a) = C_{\Omega_2}(a)$. Suppose that $a\omega_2 \notin C_{\Omega_1}(a)$ for some $\omega_2 \in \Omega_2$. Then $C(a) \subseteq C_{\Omega_1}(a) \subset C_{\Omega_2}(a)$ and hence $a\omega_2 > C_{\Omega_1}(a)$ by [10, Corollary 2.8]. We claim that $a\omega_2^n \geqslant C_{\Omega_1}(a\omega_2^{n-1})$ for each $n \in \mathbb{N}$. First, note that if $W = \alpha_1 \ldots \alpha_t \in \Omega_1^\infty$ then there exists $\overline{W} = \beta_1 \ldots \beta_t \in \Omega_1^\infty$ such that $\omega_2 W \leqslant \overline{W}\omega_2$. Now if, inductively, $a\omega_2^n \geqslant C_{\Omega_1}(a\omega_2^{n-1})$, and $W_1, \ldots, W_t \in \Omega^\infty$ then

$$a\omega_2^n W_1 + \ldots + a\omega_2^n W_t \leqslant (a\omega_2^{n-1}\overline{W}_1 + \ldots + a\omega_2^{n-1}\overline{W}_t)\omega_2 \leqslant a\omega_2^{n+1}.$$

So $a\omega_2^{n+1} \geqslant C_{\Omega_1}(a\omega_2^n)$. The chain

$$0 < a < a\omega < a\omega^2 < \ldots$$

gives rise to an increasing sequence in $\Gamma_{\Omega_1}$. Let $A_0$ be an $\Omega_1$-value of $a$; let $A_1$ be an $\Omega_1$-value of $a\omega$ which contains $A_0$, and, in general, let $A_{n+1}$ be an $\Omega_1$-value of $a\omega^{n+1}$ which contains $A_n$. Then for some $n$ $A_n = A_{n+1} = \ldots$. But $A_n^*/A_n = \big(C_{\Omega_2}(a\omega_2^n) + A_n\big)/A_n$ is a nonzero totally ordered $\Omega_1$-$f$-group with the upper bound $a\omega_2^{n+1} + A_n$. Thus, $a\omega_2 \in C_{\Omega_1}(a)$ for each $\omega_2 \in \Omega_2$ and $C_{\Omega_1}(a) = C_{\Omega_2}(a)$.

If $M$ is finitely valued, then, for each $a$ in $M$, there is a finite set of special elements $\{a_1, \ldots, a_n\}$ such that $C_{\Omega_1}(a) = C_{\Omega_1}(a_1) \oplus \ldots \oplus C_{\Omega_1}(a_n)$ by [5, Theorem 3.7] and [10, Theorem 2.9]. Thus, $C_{\Omega_1}(a) = C_{\Omega_2}(a)$ and $\mathcal{C}_{\Omega_1}(M) = \mathcal{C}_{\Omega_2}(M)$. If $M$ is representable and $A \in \Gamma_{\Omega_1}$ let $P$ be a minimal prime subgroup contained in $A$. Then $P$ is a normal $\Omega_1$-$\Omega_2$-subgroup of $M$ by [1, Theorem 4.1.1] and [2, Proposition 1.1] or [7, 2.1]. Since $M/P$ is totally ordered, and hence finitely valued, $A$ is an $\Omega_2$-subgroup of $M$; but then $\mathcal{C}_{\Omega_1}(M) = \mathcal{C}_{\Omega_2}(M)$. □

The following corollary generalizes [9, Lemma 1.6 and its corollary].

**Corollary 8.** *Let $M$ be an $\Omega$-$f$-group for which $\Gamma(M)$ satisfies a.c.c. If $M$ is either finitely valued or representable, then $\Gamma(M) = \Gamma_\Omega(M)$.*

Let $R$ and $F$ be directed po-rings with $F$ commutative and suppose that $R$ is an algebra over $F$. Recall that $R$ is a po-algebra over $F$ if it is an $F$-po-module, that is, $F^+R^+ \subseteq R^+$; and $M$ is an algebra $R$-module if it is both an $R$-module and an $F$-module and $(x\alpha)r = (xr)\alpha = x(\alpha r)$ for each $x \in M$, $\alpha \in F$ and $r \in R$. An $f$-module $M$ over $R$ is understood to also be an $f$-module over $F$.

**Corollary 9.** *Let $M$ be a unital algebra $f$-module over the directed po-$F$-algebra $R$. If $\Gamma_F(M)$ has a.c.c. then $\Gamma_F(M) = \Gamma_R(M)$.*

If $\Gamma$ has d.c.c. instead of a.c.c. then Corollary 8 is no longer true. For example, let $R = \mathbb{Q}[x]$ be the totally ordered polynomial ring over the rationals with positive cone $R^+ = \{p_0 + p_1x + \ldots + p_nx^n\colon\ p_n > 0\}$, and let $M = R_R$. Then the proper convex subgroups of $R$ are $0 \subset P_0 \subset P_1 \subset \ldots \subset P_n \subset \ldots$ where $P_n$ is the subgroup of all polynomials of degree at most $n$. Hence, $\Gamma$ is countable and satisfies d.c.c. but $\Gamma_R = \{0\}$.

If the elements of $\Omega$ only induce $\ell$-endomorphisms of $M$, then Corollary 8 need not hold. As an example let $M = \mathbb{Z} \oplus \mathbb{Z}$ be the direct sum of two copies of the integers with positive cone $M^+ = \{(x, y)\colon\ x \geqslant 0 \text{ and } y \geqslant 0\}$, and let $\Omega = \{\omega\}$ where $(x, y)\omega = (y, x)$. Then $\Gamma$ is the two element trivially ordered poset and $\Gamma_\Omega$ is a singleton.

It is easy to give examples of archimedean $\Omega$-$f$-groups $M$ for which $\Gamma(M) \neq \Gamma_\Omega(M)$. For example, let $M$ be the direct product of infinitely many copies of the reals with $\Omega = M^+$ acting by multiplication. It is well-known (and easy to see) that if $M$ is hyperarchimedean, that is each $\ell$-homomorphic image of $M$ is archimedean, then $\Gamma(M) = \Gamma_\Omega(M)$ if $M$ is an $\Omega$-$f$-group (see [7, Lemma 2.7]).

Let $R$ be an $\ell$-ring which has the property that each $\ell$-module $M_R$ has $\Gamma(M) = \Gamma(M_R)$. If $S$ is the $\ell$-ring obtained by freely adjoining $\mathbb{Z}$ to $R$ ($S = R \oplus \mathbb{Z}$ as $\ell$-groups), then $\Gamma(S) = \Gamma_R(S_R)$ yields that $R = 0$. However, the situation is different for unital modules. The $\ell$-module $M$ over the $\ell$-ring $R$ is called a *strong* $l$-module if for each $x \in M^+$ and all $r, s \in R$, $x(r \vee s) = xr \vee xs$. Each $\ell$-module over a totally ordered ring is strong, and if $R$ is the direct sum of a finite number of nonzero totally ordered unital rings then each unital $\ell$-module over $R$ is strong. $M$ is a *strict* $\ell$-module if $0 < xr$ whenever $0 < x \in M$ and $0 < r \in R$. If $R$ has a nonzero strict $\ell$-module then $R$ is itself strict; that is, $R$ is an $\ell$-domain.

**Theorem 10.** *The following statements are equivalent for the unital $\ell$-ring $R$.*

(a) *$1 \in R^+$ and $R = C(1)$.*

(b) *$\Gamma(R) = \Gamma_R(R_R)$.*

(c) *Each unital right $\ell$-module $M$ has $\Gamma(M) = \Gamma_R(M)$.*

*Moreover, if $R$ is an $\ell$-domain then each of these statements is equivalent to*

(d) *$R$ has a non-zero strict and strong right $\ell$-module $M$ for which $\Gamma(M) = \Gamma_R(M)$.*

P r o o f. (a) $\Rightarrow$ (c). If $0 \leqslant y \in C(x) \subseteq M$ and $r \in R^+$, then $y \leqslant n|x|$ and $r \leqslant m$ give $yr \leqslant nm|x|$. So $C(x) = C_R(x)$ is a submodule.

(b) $\Rightarrow$ (a). Since each convex $\ell$-subgroup is a right ideal $R = C(1)$ and $R_R$ is an $f$-module. By considering the totally ordered homomorphic images of the $f$-module $R_R$ it can easily be seen and is well-known that $1 \in R^+$.

(d) $\Rightarrow$ (a). If $0 < a \in M$ and $r \in R^+$, then $0 \leqslant ar \leqslant na$ for some integer $n$; so $0 = (na - ar)^- = a(n - r)^-$ and $0 \leqslant r \leqslant n$. Thus $R = C(1)$ is a totally ordered domain. $\square$

Clearly, "right" may be replaced by "left" in (b), (c) and (d). The $\ell$-rings in Theorem 10 are $f$-rings and have the property that each of their unital $\ell$-modules is an $f$-module, but not every such $f$-ring has $R = C(1)$.

*References*


[1] *M. Anderson and T. Feil*: Lattice-Ordered Groups. D. Reidel, Dordrecht, 1988.

[2] *A. Bigard and K. Keimel*: Sur les endomorphismes conservant les polaires d' un groupe réticulé archimédien. Bull. Soc. Math. France *97* (1970), 81–96.

[3] *A. Bigard, K. Keimel, S. Wolfenstein*: Groupes Et Anneaux Réticulés. Springer-Verlag, Berlin, 1977.

[4] *G. Birkhoff and R. S. Pierce*: Lattice-ordered rings. Am. Acad. Brasil. Ci. *28* (1956), 41–69.

[5] *P. Conrad*: The lattice of all convex $\ell$-subgroups of a lattice-ordered group. Czechoslovak Math. J. *15* (1965), 101–123.

[6] *P. Conrad*: Lattice-Ordered Groups. Tulane Lecture Notes, New Orleans, 1970.

[7] *P. Conrad and J. Diem*: The ring of polar preserving endomorphisms of an abelian lattice-ordered group. Illinois J. Math. *15* (1971), 222–240.

[8] *P. Conrad, J. Harvey and C. Holland*: The Hahn embedding theorem for lattice-ordered groups. Trans. Amer. Math. Soc. *108* (1963), 143–169.

[9] *P. Conrad and P. McCarthy*: The structure of $f$-algebras. Math. Nachr. *58* (1973), 169–191.

[10] *S. A. Steinberg*: Finitely-valued $f$-modules. Pacific J. Math. *40* (1972), 723–737.



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