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A UNIFIED APPROACH TO SEVERAL INEQUALITIES INVOLVING FUNCTIONS AND DERIVATIVES

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Abstract. There are many inequalities measuring the deviation of the average of a function over an interval from a linear combination of values of the function and some of its derivatives. A general setting is given from which the desired inequalities are obtained using Hölder's inequality. Moreover, sharpness of the constants is usually easy to prove by studying the equality cases of Hölder's inequality. Comparison of averages, extension to weighted integrals and n-dimensional results are also given.

Keywords: inequalities, averages of functions, quadrature

MSC 2000: 26D10, 26D15

1. Introduction

Let $[a,b]$ be an interval of the real line and $t_0 = a < t_1 < \ldots < t_N = b$ a partition of the interval. Assume that Φ is a function defined on $[a,b]$ except maybe at the points t_j, of class $\mathcal{C}^m$ on (t_{j-1}, t_j), $j = 1, \ldots, N$, and such that $\Phi^{(k)}(t_j-)$ and $\Phi^{(k)}(t_j+)$ exist for $k = 0, \ldots, m-1$ (with the natural exceptions $\Phi^{(k)}(t_0-)$ and $\Phi^{(k)}(t_N+)$). Then if f is a function of class $\mathcal{C}^m$ on $[a,b]$ the following formula holds:

$$\int_a^b \Phi f^{(m)} = \sum_{k=0}^{m-1} \sum_{j=0}^{N} c_{jk} f^{(k)}(t_j) + (-1)^m \int_a^b \Phi^{(m)} f, \tag{1}$$

where c_{jk} is (possibly with a minus sign) the jump of $\Phi^{(m-1-k)}$ at t_j. In the last integral $\Phi^{(m)}$ is intended to be the m-th classical derivative of Φ over the interval $[a,b]$ except possibly at the points defining the partition.

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Formula (1) is elementary. It is obtained by integrating by parts on each interval (t_{j-1}, t_j) and summing over j. Alternatively, Φ can be viewed as a distribution on the real line (extending the function as zero outside $[a, b]$) in such a way that its m-th distributional derivative contains a linear combination of Dirac deltas and their derivatives at the points t_j together with a classical function (the previous $\Phi^{(m)}$). The action of the Dirac deltas and their derivatives on a test function gives the discrete part of the right-hand side of (1). Nevertheless, we feel that using the language of distributions makes things appear harder than they actually are.

Applying Hölder's inequality to the left-hand side of (1) we get an inequality. It says that the absolute value of the right-hand side is bounded by $C\|f^{(m)}\|_p$ where $\|\cdot\|_p$ denotes the L^p-norm and C is an appropriate constant, independent of f. Any choice of Φ will provide an inequality but a remarkable feature of our approach is that one can proceed backwards and determine Φ through the values of the coefficients involved in (1) and the function $\Phi^{(m)}$. We show in Section 2 that many inequalities appearing in Chapter XV of [4] can be obtained in this way, using as Φ a function which coincides with a polynomial on each interval (t_{j-1}, t_j). Particular cases of these inequalities are also the quadrature formulae of Section 3. Moreover, to show that an inequality is sharp only requires to consider the equality cases in Hölder's inequality; either we get an extremal function in the class under consideration or, if the extremal function is not allowed, it can be approximated by admissible functions.

Sections 4 and 5 contain variants of the same theme: in Section 4, Φ is chosen so that we compare averages over two intervals and in Section 5 some of the previous results are extended to weighted integrals. We end the paper with an application of the weighted inequalities to n-dimensional results.

2. Inequalities for the average of the function on an interval

2.1. Our first inequality will be given by the function

$$\Phi(t) = \begin{cases} t - a & \text{if } t \in (a, x), \\ t - b & \text{if } t \in (x, b). \end{cases} \tag{2}$$

Then

$$-\int_a^b \Phi f' = \int_a^b f - (b-a)f(x) \tag{3}$$

from which we get

$$\left| \frac{1}{b-a} \int_a^b f - f(x) \right| \leqslant \frac{1}{b-a} \|\Phi\|_{p'} \|f'\|_p. \tag{4}$$

An easy computation gives

$$C(p,a,b,x) = \|\Phi\|_{p'} = \begin{cases} \left(\dfrac{(x-a)^{p'+1}+(b-x)^{p'+1}}{p'+1}\right)^{1/p'} & \text{if } p>1, \\ \max(x-a,b-x) & \text{if } p=1. \end{cases} \tag{5}$$

Proposition 2.1. *Let f be a continuous function such that $f' \in L^p(a,b)$ for some $1 \leqslant p \leqslant \infty$. Then for any $x \in [a,b]$,*

$$\left|\frac{1}{b-a}\int_a^b f - f(x)\right| \leqslant \frac{1}{b-a}\, C(p,a,b,x)\, \|f'\|_p$$

where $C(p,a,b,x)$ is given by (5).

For $p=\infty$ ($p'=1$) we get Ostrowski's inequality ([4], p. 468; see also [1]), and for $p<\infty$ the inequality corresponds to the case $n=1$ of Theorem 1 in [4], p. 471. (See the general case of this theorem below.) Formula (3) is the same we get writing the right-hand side as

$$\int_a^b \big(f(t)-f(x)\big)\,\mathrm{d}t$$

and applying the fundamental theorem of Calculus to the difference $f(t)-f(x)$.

To see that the constants in Proposition 2.1 are sharp it is enough to observe that when $1<p\leqslant\infty$ the equality holds for any function f such that $f' = |\Phi|^{p'-1}\operatorname{sgn}\Phi$. When $p=1$ we choose as f' the characteristic function of $(x-\varepsilon,x)$ (or $(x,x+\varepsilon)$) and make $\varepsilon\to 0$ to get the bound $(x-a)/(b-a)$ ($(b-x)/(b-a)$, respectively). The inequality is also sharp when restricted to functions in $\mathcal{C}^1$, although the choices of f' do not give $\mathcal{C}^1$ functions (unless $x=a$ or $x=b$), but they can be approximated by a sequence of $\mathcal{C}^1$ functions whose derivatives converge (weakly) to f'.

If we apply the inequality to f' instead of f we obtain

$$\left|\frac{f(b)-f(a)}{b-a} - f'(x)\right| \leqslant \frac{1}{b-a}\, C(p,a,b,x)\, \|f''\|_p$$

where the constant $C(p,a,b,x)$ is as above. This inequality for $p=\infty$, $b=1$, $a=0$ appears in [4], p. 9.

2.2. Let now $\Phi_c(t) = \Phi(t)+c$ where Φ is as in (2) and c is a constant. Then (3) still holds with Φ_c instead of Φ for any function f such that $f(a)=f(b)$. We have the following proposition.

Proposition 2.2. *Let f be as in Proposition* 2.1 *and $f(a)=f(b)$. Then for any $x\in[a,b]$,*

$$\left|\frac{1}{b-a}\int_a^b f - f(x)\right| \leqslant \frac{(b-a)^{1/p'}}{2(p'+1)^{1/p'}}\|f'\|_p.$$

For $p = 1$ the constant on the right-hand side is replaced by its limit when $p' \to \infty$, namely, $1/2$.

It is enough to compute the infimum of $\|\Phi_c\|_{p'}$. This infimum coincides with the value of the constant $C(p, a, b, (a+b)/2)$ of (5). The result of Proposition 2.2 can also be explained in the following way: f can be extended by periodicity as a continuous function using (a, b) as the basic period, and the left-hand side of the inequality remains unchanged for the extended f if $[a, b]$ is replaced by any other interval of length $b - a$; for a fixed x, the interval centered at x gives the best bound among the values in (5) and due to the invariance of the constant under translations, this bound is $C(p, a, b, (a + b)/2)$.

The case $p = \infty$ in Proposition 2.2 gives the value $(b - a)/4$. It appears in [4], p. 9, applied to f' instead of f.

2.3. When $x = (a + b)/2$ another kind of improvement of Proposition 2.1 is possible. Due to the fact that Φ has integral zero the left-hand side of (3) does not change when f' is replaced with $f' + c$ where c is a constant.

Proposition 2.3. *Let f be a continuous function such that $f' \in L^p(a, b)$ for some $1 \leqslant p \leqslant \infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)^{1/p'}}{2(p'+1)^{1/p'}} \inf_{c\in\mathbb{R}} \|f' + c\|_p.$$

Although the values of c which minimize the L^p norm depend on f', we can describe the minimum at least for two precise values of p:

$$\inf_{c\in\mathbb{R}} \|f' + c\|_\infty = \frac{1}{2}\big(\sup_{t\in(a,b)} f'(t) - \inf_{t\in(a,b)} f'(t)\big),$$

$$\inf_{c\in\mathbb{R}} \|f' + c\|_2^2 = \|f'\|_2^2 - \frac{\big(f(b) - f(a)\big)^2}{b-a}.$$

2.4. Let Φ be given by

$$\Phi(t) = \begin{cases} (t-a)^m & \text{if } t \in (a, x), \\ (t-b)^m & \text{if } t \in (x, b). \end{cases} \tag{6}$$

Proposition 2.4. *Let f be a $\mathcal{C}^{m-1}$ function in $[a, b]$ such that $f^{(m)} \in L^p(a, b)$ for some $1 \leqslant p \leqslant \infty$ and $f^{(k)}(x) = 0$, $k = 1, \ldots, m-1$. Then for any $x \in [a, b]$,*

$$\left|\frac{1}{b-a}\int_a^b f - f(x)\right| \leqslant \frac{1}{m!(b-a)} \left(\frac{(x-a)^{mp'+1} + (b-x)^{mp'+1}}{mp'+1}\right)^{1/p'} \|f^{(m)}\|_p$$

if $1 < p \leqslant \infty$. *For* $p = 1$ *the expression in the parentheses has to be replaced by* $\max(x-a, b-x)^m$.

This result for the case $p = \infty$ appears in [1].

2.5. Define now Φ as follows:

$$m!\,\Phi(t) = \begin{cases} (t-a)(t-x)^{m-1} & \text{if } t \in (a,x), \\ (t-b)(t-x)^{m-1} & \text{if } t \in (x,b). \end{cases} \tag{7}$$

The relevant values needed for substitution in (1) are

$$\begin{gathered} m!\Phi^{(k)}(a) = k(m-1)\ldots(m-k+1)(a-x)^{m-k}, \\ m!\Phi^{(k)}(b) = k(m-1)\ldots(m-k+1)(b-x)^{m-k} \quad \text{for } k = 1, \ldots, m-1, \\ \Phi^{(k)}(x) = 0 \quad \text{for } k = 0, \ldots, m-2, \\ m!\Phi^{(m-1)}(x-) = (m-1)!(x-a), \quad \Phi^{(m-1)}(x+) = (m-1)!(x-b). \end{gathered}$$

Proposition 2.5. *Let* f *be a* $\mathcal{C}^{m-1}$ *function such that* $f^{(m)}$ *is in* $L^p(a,b)$. *Then for any* $x \in [a,b]$,

$$\left| \frac{1}{b-a} \int_a^b f - \frac{1}{m}\left[f(x) + \sum_{k=1}^{m-1} F_k(x) \right] \right| \leqslant \frac{1}{b-a} K(p,a,b,m,x)\, \|f^{(m)}\|_p$$

where $F_k(x)$ *is defined as*

$$\begin{aligned} F_k(x) &= \frac{m-k}{k!}[(x-b)^k f^{(k-1)}(b) - (x-a)^k f^{(k-1)}(a)], \\ K(p,a,b,m,x)^{p'} &= \frac{1}{(m!)^{p'}}[(x-a)^{mp'+1} + (b-x)^{mp'+1}] B\big((m-1)p'+1, p'+1\big) \end{aligned}$$

if $p > 1$, *and*

$$K(1,a,b,m,x) = \frac{(m-1)^{m-1}}{m!m^m} \max\big((x-a)^m, (b-x)^m\big).$$

Here $B(\alpha,\beta)$ stands for the beta function defined as $\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$. This result is due to A. M. Fink [3] and also appears as Theorem 1 of [4], p. 471. Fink uses a representation similar to ours in his proof. Sharpness of the constants is discussed as in Subsection 2.1.

2.6. Set $\Phi(t) = t - \dfrac{a+b}{2}$. Then we have the following proposition.

Proposition 2.6. *Let $f \in \mathcal{C}[a,b]$ be such that $f' \in L^p(a,b)$. Then*

$$\left|\frac{1}{b-a}\int_a^b f - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{(b-a)^{1/p'}}{2(p'+1)^{1/p'}}\|f'\|_p.$$

For $p=1$ the constant is 1/2 (which coincides with the limit when $p' \to \infty$ in the above inequality).

This result is in [4], p. 473. In this case, equality is possible with functions such that f' is continuous when $1 < p < \infty$, because we can choose f such that $f' = |\Phi|^{p'-1}\operatorname{sgn}\Phi$. It is possible to put $\inf\limits_{c\in\mathbb{R}}\|f'+c\|_p$ on the right-hand side as in Proposition 2.3.

2.7. Define now

$$\Phi_c(t) = \frac{(t-a)^2+(t-b)^2-c}{4} \tag{8}$$

where c is an arbitrary constant. Then $\Phi_c'(t) = (2t-a-b)/2$ and $\Phi_c''(t) = 1$. If $f'(a) = f'(b)$ we get

$$\frac{1}{b-a}\int_a^b f - \frac{f(a)+f(b)}{2} = \frac{1}{b-a}\int \Phi_c f''$$

and the following proposition is valid.

Proposition 2.7. *Let $f \in \mathcal{C}^1[a,b]$ be such that $f'(a) = f'(b)$ and $f'' \in L^p(a,b)$. Then*

$$\left|\frac{1}{b-a}\int_a^b f - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{1}{b-a}\inf_{c>0}\|\Phi_c\|_{p'}\|f''\|_p.$$

Particular values for the bound in this proposition for $a=0$, $b=1$ are: (i) if $p=\infty$, $c=5/32$ and $\|\Phi_{5/32}\|_1 = 1/8$; (ii) if $p=1$, $c=3/16$ and $\|\Phi_{3/16}\|_\infty = 1/16$; (iii) if $p=2$, $c=1/6$ and $\|\Phi_{2/3}\|_2 = 1/3\sqrt{5}$. For a general interval (a,b) the values of these norms must be multiplied by $(b-a)^{2+1/p'}$.

The results in the preceding section are quadrature formulae when they are understood as the approximation of the integral by a linear combination of values of the function and its derivatives at some points. In this section we deal with some of the classical quadrature formulae of numerical integration (see [2], for instance).

3.1. Take $\Phi(t) = \frac{1}{2}(t-a)(t-b)$ to produce

$$\int_a^b f - \frac{b-a}{2}\big(f(a)+f(b)\big) = \int_a^b \Phi f'' \tag{9}$$

from which we can write

$$\left|\int_a^b f - \frac{b-a}{2}\big(f(a)+f(b)\big)\right| \leqslant \|\Phi\|_{p'}\|f''\|_p.$$

Proposition 3.1. *Let f be a $\mathcal{C}^1$ function in $[a,b]$ such that $f'' \in L^p(a,b)$. Then*

$$\left|\int_a^b f - \frac{b-a}{2}\big(f(a)+f(b)\big)\right| \leqslant \frac{(b-a)^{2+1/p'}}{2} B(p'+1,p'+1)^{1/p'}\|f''\|_p$$

if $1 < p \leqslant \infty$. For $p = 1$ the bound is $(b-a)^2\|f''\|_1/8$.

The usual *trapezoid rule* uses as a bound the case $p = \infty$, that is, $(b-a)^3\|f''\|_\infty/12$. Sometimes, the difference between the integral and its approximation is given as $-(b-a)^3 f''(\xi)/12$ for some $\xi \in (a,b)$ assuming that f is $\mathcal{C}^2$. This result is obtained from the right-hand side of (9) by using the mean value theorem for integrals which leads to $f''(\xi)\int_a^b \Phi$.

3.2. If we choose

$$\Phi(t) = \begin{cases} (t-a+h)^3(3t-3a-h) & \text{for } t \in (a-h,a), \\ (t-a-h)^3(3t-3a+h) & \text{for } t \in (a,a+h), \end{cases} \tag{10}$$

we obtain the following proposition.

Proposition 3.2. *Let $f \in \mathcal{C}^3[a-h,a+h]$ be such that $f^{(4)} \in L^p(a-h,a+h)$. Then*

$$\left|\int_{a-h}^{a+h} f - \frac{h}{3}[f(a-h)+4f(a)+f(a+h)]\right| \leqslant \frac{1}{72}\|\Phi\|_{p'}\|f^{(4)}\|_p,$$

where Φ is given by (10). In particular, for $p = \infty$ we get

$$\left|\int_{a-h}^{a+h} f - \frac{h}{3}[f(a-h)+4f(a)+f(a+h)]\right| \leqslant \frac{h^5}{90}\|f^{(4)}\|_\infty.$$

The case $p=\infty$ corresponds to the well-known *Simpson's rule* and using the mean value theorem for integrals we can write for $f\in\mathcal{C}^4$

$$\int_{a-h}^{a+h} f-\frac{h}{3}[f(a-h)+4f(a)+f(a+h)]=-\frac{h^5}{90}f^{(4)}(\xi)$$

for some $\xi\in(a-h,a+h)$.

3.3. The definition

$$\Phi(t)=\begin{cases}(t-a)^2 & \text{for } t\in\left(a,\frac{a+b}{2}\right),\\ (t-b)^2 & \text{for } t\in\left(\frac{a+b}{2},b\right)\end{cases}$$

leads to the formula

$$\int_a^b f-(b-a)f\Big(\frac{a+b}{2}\Big)=\frac{1}{2}\int_a^b \Phi f''=\frac{(b-a)^3}{24}f''(\xi)$$

for some $\xi\in(a,b)$. This is the *middle-point rule.*

A bound in terms of $\|f''\|_p$ is obtained as in the other cases. This situation should be compared with Ostrowski's inequality in Subsection 2.1 where the bound was given in terms of $\|f'\|_p$. The use of f'' instead of f' is only possible when $x=(b+a)/2$; to see this, apply the inequality to $f(t)=t$.

4. Comparison of averages

In this section we will choose Φ so that it will allow to compare averages over different intervals.

4.1. Define now

$$\Phi(t)=\begin{cases}(x-b)(t-a) & \text{if } t\in(a,x),\\ (x-a)(t-b) & \text{if } t\in(x,b).\end{cases}$$

Integration by parts gives

$$\int_a^b \Phi(t)f'(t)\,\mathrm{d}t=(x-a)\int_a^b f-(b-a)\int_a^x f. \tag{11}$$

Proposition 4.1. *Let f be a continuous function in $[a,b]$ such that $f'\in L^p(a,b)$. Then for any $x\in[a,b]$,*

$$\left|\frac{1}{b-a}\int_a^b f-\frac{1}{x-a}\int_a^x f\right|\leqslant C(p,a,b,x)\|f'\|_p, \tag{12}$$

where

$$C(p,a,b,x)=\begin{cases}\dfrac{b-x}{(b-a)^{1/p}}\dfrac{1}{(1+p')^{1/p'}} & \text{if } 1<p\leqslant\infty,\\ \dfrac{b-x}{b-a} & \text{if } p=1.\end{cases}$$

If, moreover, f has zero integral over (a,b) we can write, for instance,

$$\left|\int_a^x f\right|\leqslant\frac{(x-a)(b-x)}{2}\|f'\|_\infty,$$

which is the precise form of Mahajani's inequality in [4], p. 474.

If we also assume that $f(a)=f(b)$, all the functions $\Phi_c(t)=\Phi(t)+c$ would provide the same right-hand side in (11) and the constant $C(p,a,b,x)$ can be taken as $\inf_c\|\Phi_c\|_{p'}$. For instance, if $p=1$, the new constant becomes half of the former one. This case is also in [4], p. 474.

4.2. Taking $a\leqslant c<d\leqslant b$, the averages over the intervals $[c,d]$ and $[a,b]$ can be compared using

$$\Phi(t)=\begin{cases}(t-a)(d-c) & \text{if } t\in(a,c),\\ [(d-c)-(b-a)]t+bc-ad & \text{if } t\in(c,d),\\ (t-b)(d-c) & \text{if } t\in(d,b).\end{cases}$$

Proposition 4.2. *Let f be a continuous function in $[a,b]$ such that $f'\in L^p(a,b)$. Then for any $c,d\in[a,b]$, $c<d$,*

$$\left|\frac{1}{d-c}\int_c^d f-\frac{1}{b-a}\int_a^b f\right|\leqslant K(a,b,c,d,p)\|f'\|_p$$

where

$$K(a,b,c,d,p)=\left(\frac{b-a}{(p'+1)L}\right)^{1/p'}[(c-a)^{p'+1}+(b-d)^{p'+1}]^{1/p'}\quad \text{if } p>1$$

and

$$K(a,b,c,d,1)=\frac{\max(c-a,b-d)}{b-a}.$$

(Here $L=(b-a)-(d-c)$.)

Limit cases of this inequality are Ostrowski's inequality ($c=d=x$) and the generalization of Majahani's inequality given in Proposition 4.1 ($a=c$). It is easy to check that the smallest value of the constant among the intervals $[c, d]$ of fixed length contained in $[a, b]$ is given by the interval centred at the middle point of $[a, b]$. Moreover, when this holds, the right-hand side of the inequality can be improved as in Proposition 2.3.

5. Weighted integrals

Given a weight function w in (a, b), we can obtain formulae for $\int_a^b fw$. This requires to insert w in the definition of Φ, which no longer will be a polynomial on each subinterval as in the previous sections. Although it is not necessary for most of the results, we will assume that w is nonnegative as is customary with densities.

Some interesting examples are obtained for $w(s) = s^{\alpha-1}$ $(\alpha > 0)$ or for the exponential $w(s) = \exp(-s^2)$ (in this case the interval (a, b) can be taken to be $(-\infty, \infty)$).

5.1. Take for instance,

$$\Phi(t) = \begin{cases} \displaystyle\int_a^t w(s)\,\mathrm{d}s & \text{if } t \in (a, x), \\ \displaystyle -\int_t^b w(s)\,\mathrm{d}s & \text{if } t \in (x, b). \end{cases} \tag{13}$$

We get

$$\int_a^b \Phi f' = \left(\int_a^b w\right) f(x) - \int_a^b wf$$

and the result is the following.

Proposition 5.1. *Let Φ be as in* (13) *and let f be a continuous function such that $f' \in L^p(a, b)$ for some $1 \leqslant p \leqslant \infty$. Then for any $x \in [a, b]$,*

$$\left|\int_a^b wf - \left(\int_a^b w\right) f(x)\right| \leqslant \|\Phi\|_{p'} \|f'\|_p.$$

Particular values of the norm are

$$\|\Phi\|_1 = \int_a^b |x - s| w(s)\,\mathrm{d}s, \qquad \|\Phi\|_\infty = \max\left(\int_a^x w, \int_x^b w\right).$$

As Φ depends on x, the same is true for its L^p-norm. The smallest value of this norm is obtained at the point x_0 where $\int_a^{x_0} w = \int_{x_0}^b w$.

We also have a result similar to Proposition 2.2.

Proposition 5.2. *Let f be a function satisfying the hypotheses of Proposition* 5.1 *and such that $f(a)=f(b)$. Then for any $x \in [a,b]$,*

$$\left|\int_a^b wf - \left(\int_a^b w\right) f(x)\right| \leqslant \inf_{c\in\mathbb{R}} \|\Phi + c\|_{p'} \|f'\|_p.$$

An improvement as in Proposition 2.3 is only possible when the point x is the centroid of the mass distribution over (a,b) with the density w, that is, when x satisfies

$$x \int_a^b w(s)\,\mathrm{d}s = \int_a^b sw(s)\,\mathrm{d}s. \tag{14}$$

5.2. Set now

$$\Phi(t) = \frac{1}{2}\left(\int_a^t w(s)\,\mathrm{d}s - \int_t^b w(s)\,\mathrm{d}s\right). \tag{15}$$

Proposition 5.3. *Let Φ be as in* (15) *and let f be a continuous function such that $f' \in L^p(a,b)$ for some $1 \leqslant p \leqslant \infty$. Then*

$$\left|\int_a^b fw - \left(\int_a^b w\right) \frac{f(a)+f(b)}{2}\right| \leqslant \|\Phi\|_{p'} \|f'\|_p.$$

Relevant values of the norm are

$$\|\Phi\|_\infty = \frac{1}{2}\int_a^b w, \qquad \|\Phi\|_1 = \int_a^{x_0} (s-a)w(s)\,\mathrm{d}s + \int_{x_0}^b (b-s)w(s)\,\mathrm{d}s$$

where x_0 is the point satisfying $\int_a^{x_0} w = \int_{x_0}^b w$.

5.3. Define

$$\Phi_c(t) = \frac{1}{2}\int_a^b |t-s| w(s)\,\mathrm{d}s + c. \tag{16}$$

Thus we can obtain a result similar to Proposition 2.7 if we assume an extra condition on w.

Proposition 5.4. *Assume that the centroid of the mass distribution of the density w (defined in* (14)*) is the point $(a+b)/2$ and let Φ_c be as in* (16)*. If f and f' are continuous, $f'(a) = f'(b)$ and $f'' \in L^p(a,b)$ for some $1 \leqslant p \leqslant \infty$, then*

$$\left|\int_a^b fw - \left(\int_a^b w\right) \frac{f(a)+f(b)}{2}\right| \leqslant \inf_c \|\Phi_c\|_{p'} \|f''\|_p.$$

5.4. It is also possible to obtain results related to those in Section 4 for weighted integrals. For simplicity we only include here a particular case which will be used in the next section. We will take $w(t) = t^{n-1}$ and compare the weighted averages of f in $(0,R)$ and in $(0,r)$ for $0 < r < R$. To this end we choose

$$\Phi(t) = \begin{cases} t^n(R^n - r^n) & \text{if } t \in (0,r), \\ r^n(R^n - t^n) & \text{if } t \in (r,R). \end{cases} \tag{17}$$

Proposition 5.5. *Let Φ be as in* (17) *and let f be a continuous function in $[0,R]$ such that $f' \in L^p(0,R)$ for some $1 \leqslant p \leqslant \infty$. Then, for $0 < r < R$,*

$$\left| \frac{n}{R^n} \int_0^R fw - \frac{n}{r^n} \int_0^r fw \right| \leqslant \frac{\|\Phi\|_{p'} \|f'\|_p}{R^n r^n}.$$

In particular, we have the bounds

$$\frac{n}{n+1}(R-r)\|f'\|_\infty \quad \text{and} \quad \left[1 - \left(\frac{r}{R}\right)^n\right] \|f'\|_1.$$

6. Some n-dimensional results

Using spherical coordinates and some results of the preceding section corresponding to the weight $w(t) = t^{n-1}$ we obtain in this section two n-dimensional results. S_r^{n-1} will denote the sphere of radius r centred at the origin and $\mathrm{d}\sigma_r$ its Lebegue measure (in both cases we drop the subindex for $r = 1$); B_r will be the ball of radius r centred at the origin. The symbol $⨍_D$ will stand for the mean value of the integral over the set D and $\frac{\partial f}{\partial \nu}$ for the radial derivative of f.

6.1. As in Subsection 5.1 and for the weight $w(t) = t^{n-1}$ we define

$$\Phi(t) = \begin{cases} \dfrac{t^n}{n} & \text{if } 0 < t < r, \\[2ex] \dfrac{t^n - R^n}{n} & \text{if } r < t < R. \end{cases} \tag{18}$$

Given a function f in B_R, for each $u \in S^{n-1}$ we can define $g_u(t) = f(tu)$ for $0 < t < R$. Then

$$\int_0^R \Phi(t) g_u'(t)\,\mathrm{d}t = \frac{R^n}{n} g_u(r) - \int_0^R g_u(t) t^{n-1}\,\mathrm{d}t. \tag{19}$$

Using the fact that $g'_u(t) = \frac{\partial f}{\partial \nu}(tu)$ and integrating over the unit sphere we get

$$\frac{R^n}{n}\int_{S^{n-1}} f(ru)\,\mathrm{d}\sigma(u) - \int_{S^{n-1}}\int_0^R f(tu)t^{n-1}\,\mathrm{d}t\,\mathrm{d}\sigma(u) = \int_0^R \Phi(t)\Big(\int_{S^{n-1}} \frac{\partial f}{\partial \nu}(tu)\,\mathrm{d}\sigma(u)\Big)\,\mathrm{d}t.$$

Proposition 6.1. *Let Φ be as in* (18) *and let f be a continuous function on B_R whose radial derivative is also continuous. Then, for $0 < r < R$,*

$$\left|\fint_{S_r^{n-1}} f - \fint_{B_R} f\right| \leqslant \frac{n}{R^n}\|\Phi\|_{p'}\left[\int_0^R \left|\fint_{S^{n-1}} \frac{\partial f}{\partial \nu}(tu)\,\mathrm{d}\sigma(u)\right|^p \mathrm{d}t\right]^{1/p}.$$

The result is deduced immediately from (19) taking into account that in $\mathbb{R}^n$ the measure of the unit sphere is n times the measure of the unit ball.

6.2. Our second result will compare the averages of a function over two balls centred at the origin and will be based on Proposition 5.5. If Φ is defined as in (17) and g_u is as before, then

$$\frac{n}{R^n}\int_0^R g_u(t)t^{n-1}\,\mathrm{d}t - \frac{n}{r^n}\int_0^r g_u(t)t^{n-1}\,\mathrm{d}t = \frac{1}{R^n r^n}\int_0^R \Phi(t)g'_u(t)\,\mathrm{d}t.$$

Integrating again over the unit sphere and applying Hölder's inequality we get the following proposition.

Proposition 6.2. *Let Φ be as in* (17) *and let f be a continuous function on B_R whose radial derivative is also continuous. Then, for $0 < r < R$,*

$$\left|\fint_{B_r} f - \fint_{B_R} f\right| \leqslant \frac{1}{R^n r^n}\|\Phi\|_{p'}\left[\int_0^R \left|\fint_{S^{n-1}} \frac{\partial f}{\partial \nu}(tu)\,\mathrm{d}\sigma(u)\right|^p \mathrm{d}t\right]^{1/p}.$$

As a particular case we have the following.

Corollary 6.3. *Let f be a continuous function on B_R whose radial derivative is also continuous. If $\fint_{B_R} f = 0$, then for $0 < r < R$,*

$$\left|\fint_{B_r} f\right| \leqslant \frac{n}{n+1}\,(R-r)\left\|\frac{\partial f}{\partial \nu}\right\|_{\infty}.$$

This is an n-dimensional version of Mahajani's inequality mentioned in [4], p. 474.

6.3. If f is assumed to be of class $\mathcal{C}^2$, Green's theorem reads

$$\int_{S_t^{n-1}} \frac{\partial f}{\partial \nu}(v)\,\mathrm{d}\sigma_t(v) = \int_{B_t} \Delta f$$

so that we have

$$\int_0^R \Phi(t)\left(\fint_{S^{n-1}} \frac{\partial f}{\partial \nu}(tu)\,\mathrm{d}\sigma(u)\right)\mathrm{d}t = \int_0^R \Phi(t)\frac{t}{n}\left(\fint_{B_t} \Delta f\right)\mathrm{d}t.$$

Propositions similar to 6.1 and 6.2 based on the right-hand side of this equality can be established. We leave the details to the reader.

All results in this section are sharp because when applied to a radial function they become exactly the same as their one-dimensional counterparts with respect to the weight $w(t) = t^{n-1}$.

6.4. Taking the function Φ as in (18) with $r = 0$ the following equality is obtained:

$$\fint_{B_R} f - f(0) = \int_0^R \left(1 - \frac{t^n}{R^n}\right)\frac{t}{n}\left(\fint_{B_t} \Delta f\right)\mathrm{d}t.$$

It provides an easy proof (whose details are left to the reader) of the equivalence of harmonicity and the mean value property for $\mathcal{C}^2$ functions.

Proposition 6.4. *If f is a $\mathcal{C}^2$ function in an open set A, the following conditions are equivalent:*

(i) *f is harmonic in A;*

(ii) *f has the mean value property on A (that is, for every $x \in A$ the mean value of f on every small ball centered at x is $f(x)$).*

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