Sangwon Park

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THE GENERAL STRUCTURE OF INVERSE POLYNOMIAL MODULES

Sangwon Park, Pusan

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Abstract. In this paper we compute injective, projective and flat dimensions of inverse polynomial modules as $R[x]$-modules. We also generalize Hom and Ext functors of inverse polynomial modules to any submonoid but we show Tor functor of inverse polynomial modules can be generalized only for a symmetric submonoid.

Keywords: module, inverse polynomial, homological dimensions, Hom, Ext, Tor

MSC 2000: 16E30, 13C11, 16D80

1. Introduction

Northcott in [4] considered the module $K[x^{-1}]$ of inverse polynomials over the polynomial ring $K[x]$ (with K a field). The idea for this module came from Macaulay's work in [1]. McKerrow elaborated on Norchcott's work (in [3]) and considered the module $M[x^{-1}]$ over $R[x]$ (with R a ring and M a left R-module). An easy modification of argument in Northcott [4] shows that if R is a left noetherian ring and E is an injective left $R[x]$-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. In [5] and [6] we considered the behaviors of these so-call Macaulay-Northcott modules when we apply the torsion and extension functors to them. In this paper we will consider various homological dimensions of these modules. We also generalize Hom, Ext and Tor functors of Macaulay-Northcott modules to any submonoid. An interesting result is the fact that $\operatorname{Tor}_i^{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong \operatorname{Tor}_{i-1}^R(M,N)[x^{-S}]$ for a symmetric submonoid S of N (the set of natural numbers).

Definition 1.1. Let R be a ring and M be a left R-module, then $M[x^{-1}]$ is a left $R[x]$-module such that

$$x(m_0 + m_1x^{-1} + \ldots + m_nx^{-n}) = m_1 + m_2x^{-1} + \ldots + m_nx^{-n+1}$$

and such that

$$r(m_0 + m_1x^{-1} + \ldots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \ldots + rm_nx^{-n}$$

where $r \in R$. We call $M[x^{-1}]$ a Macaulay-Northcott module.

Definition 1.2. Let $\mathcal{C}$ be the category of left R-module and $\mathcal{D}$ be the category of left $R[x]$-module. Let $f\colon {}_RM \to {}_RN$ be a R-linear map, then $T\colon \mathcal{C} \to \mathcal{D}$ defined by $T(M) = M[x^{-1}]$ and $T(f) = f$ (where $f(m_0 + m_1x^{-1} + \ldots + m_nx^{-n+1}) = f(m_0) + f(m_1)x^{-1} + \ldots + f(m_n)x^{-1})$ is an additive and exact functor between $\mathcal{C}$ and $\mathcal{D}$. We call T the Macaulay-Northcott functor.

2. The general structure of inverse polynomial modules

Lemma 2.1. *Let M be an essential extension of N as a left R-module, then $M[x^{-1}]$ is an essential extension of $N[x^{-1}]$ as a left $R[x]$-module.*

Proof. Let $m_0 + m_1x^{-1} + \ldots + m_ix^{-i} \in M[x^{-1}]$ w.l.o.g. (without loss of generality) let $m_i \neq 0$, then there is $r_i \in R$, $r_i \neq 0$ such that $m_ir_i \in N$, $m_ir_i \neq 0$. So $r_ix^i(m_0 + m_1x^{-1} + \ldots + m_ix^{-i}) = r_im_i \in N[x^{-1}]$. Hence, $M[x^{-1}]$ is an essential extension of $N[x^{-1}]$. $\square$

Remark 2.2. Let R be a left noetherian ring. If E is an injective envelope of M then $E[x^{-1}]$ is an injective envelope of $M[x^{-1}]$.

Lemma 2.3. *If $0 \to M \to E^0 \to E^1 \to \ldots$ is a minimal injective resolution of M as a left R-module then*

$$0 \to M[x^{-1}] \to E^0[x^{-1}] \to E^1[x^{-1}] \to \ldots$$

is a minimal injective resolution of $M[x^{-1}]$ as a left $R[x]$-module.

Proof. This follows from the lemma above and from the exactness of the Macaulay-Northcott functor. $\square$

Definiton 2.4 ([7]). Let $0 \to B \xrightarrow{\varepsilon} E^0 \xrightarrow{d^0} E^1 \to \ldots$ be an injective resolution; denote $\operatorname{im}\varepsilon$ by L^0 and, for $n \geqslant 1$, denote $\operatorname{im} d^{n-1}$ by L^n. For $n \geqslant 0$, L^n is the $n-$th cosyzygy.

We will need the following results.

Lemma 2.5 (Theorem 9.8, [7]). *The followings are equivalent for a left R-module N:*

(1) $\operatorname{id}(N) \leqslant n$;

(2) $\operatorname{Ext}^k(M, N) = 0$ *for all module M and all $k \geqslant n+1$;*

(3) $\operatorname{Ext}^{n+1}(M, N) = 0$ *for all modules M;*

(4) *every injective resolution of a left R-module N has an injective $(n-1)$st cosyzygy.*

Theorem 2.6. *Let R be a left Noetherian ring. Then*

$$\operatorname{id}_{R[x]}(M[x^{-1}]) = \operatorname{id}_R(M).$$

P r o o f. Suppose $\operatorname{id}_R(M) = n$ and

$$0 \to M \to E^0 \to E^1 \to \ldots \to E^n \to 0$$

is an injective resolution of M. Then

$$0 \to M[x^{-1}] \to E^0[x^{-1}] \to E^1[x^{-1}] \to \ldots \to E^n[x^{-1}] \to 0$$

is an injective resolution of $M[x^{-1}]$. Let

$$K^n = \ker(E^{n-2} \to E^{n-1})$$

then K^n is an injective R-module but K^i is not an injective R-module for $i < n$. Thus $K^n[x^{-1}]$ is an injective $R[x]$-module and $K^i[x^{-1}]$ is not an injective $R[x]$-module for $i < n$. Therefore, $\operatorname{id}_{R[x]}(M[x^{-1}]) = n$. Suppose $\operatorname{id}_R(M) = \infty$ and

$$0 \to M \to E^0 \to E^1 \to \ldots \to E^n \to \ldots$$

is an injective resolution of M. Then

$$M[x^{-1}] \to E^0[x^{-1}] \to E^1[x^{-1}] \to \ldots \to E^n[x^{-1}] \to \ldots$$

is an injective resolution of $M[x^{-1}]$. Let

$$K^n = \ker(E^{n-2} \to E^{n-1})$$

then K^i is not an injective R-module for all i. Thus $K^i[x^{-1}]$ is not an injective $R[x]$-module for all i. Therefore, $\operatorname{id}_{R[x]}(M[x^{-1}]) = \infty$. Similarly if $\operatorname{id}_{R[x]} M[x^{-1}] = n$, then $\operatorname{id}_R(M) = n$ and if $\operatorname{id}_{R[x]} M[x^{-1}] = \infty$, then $\operatorname{id}_R(M) = \infty$. Hence, $\operatorname{id}_{R[x]}(M[x^{-1}]) = \operatorname{id}_R(M)$. $\square$

Lemma 2.7. *There is a short exact sequence of $R[x]$-modules*

$$0 \to M[x] \to M[x, x^{-1}] \to M[x^{-1}] \to 0.$$

Proof. Let M be a R-module, then $M[x] \subset M[x, x^{-1}]$ is a submodule. Let $\varphi\colon \frac{M[x,x^{-1}]}{M[x]} \to M[x^{-1}]$ be defined by

$$\begin{aligned}\varphi\big((a_0 + a_1x + \ldots + a_nx^n + b_1x^{-1} + \ldots + b_mx^{-m}) + M[x]\big)\\ = b_1 + b_2x^{-1} + \ldots + b_mx^{-m+1}.\end{aligned}$$

Then φ is an isomorphism. Hence,

$$0 \to M[x] \to M[x, x^{-1}] \to M[x^{-1}] \to 0$$

is a short exact sequence of $R[x]$-modules. □

Theorem 2.8. *Let R be a left noetherian ring. Then*

$$\begin{aligned} n \leqslant \operatorname{pd}_{R[x]}(M[x^{-1}]) &\leqslant n+1 \quad \textit{if} \quad \operatorname{pd}_R(M) = n,\\ \operatorname{pd}_{R[x]}(M[x^{-1}]) &= \infty \qquad\quad \textit{if} \quad \operatorname{pd}_R(M) = \infty.\end{aligned}$$

Proof. Suppose $\operatorname{pd}_R(M) = n$ then $\operatorname{pd}_{R[x]}(M[x]) = n$. Consider the following exact sequence

$$0 \to M[x] \to M[x, x^{-1}] \to M[x^{-1}] \to 0.$$

Then we have the long exact sequence with Ext

$$\begin{aligned} 0 \to\ &\operatorname{Hom}(M[x^{-1}], N) \to \operatorname{Hom}(M[x, x^{-1}], N) \to \operatorname{Hom}(M[x], N)\\ \to\ &\operatorname{Ext}^1(M[x^{-1}], N) \to \ldots \to \operatorname{Ext}^{n+1}(M[x^{-1}], N)\\ \to\ &\operatorname{Ext}^{n+1}(M[x, x^{-1}], N) \to \operatorname{Ext}^{n+1}(M[x], N)\\ \to\ &\operatorname{Ext}^{n+2}(M[x^{-1}], N) \to \operatorname{Ext}^{n+2}(M[x, x^{-1}], N) \to \ldots\end{aligned}$$

for a left R-module N. So $\operatorname{Ext}^{n+1}(M[x], N) = 0$. Since $\operatorname{pd}_R(S^{-1}M) \leqslant \operatorname{pd}_R(M) + 1$ and $M[x, x^{-1}] \cong S^{-1}M[x]$, where $S = \{1, s, s^2, \ldots\}$ for $s \in R$,

$$\operatorname{pd}_{R[x]}(M[x, x^{-1}]) \leqslant \operatorname{pd}_{R[x]}(M[x]) + 1.$$

So $\operatorname{Ext}^{n+2}(M[x, x^{-1}], N) = 0$. Thus $\operatorname{Ext}^{n+2}(M[x^{-1}], N) = 0$. Therefore,

$$\operatorname{pd}_{R[x]}(M[x^{-1}]) \leqslant n + 1.$$

Since $\operatorname{Ext}^n(M[x^{-1}], N[x^{-1}]) \cong \operatorname{Ext}^n(M,N)[[x]]$ (Theorem 1.2, [6]), and since $\operatorname{pd}_R(M) = n$, $\operatorname{Ext}^n(M,N)[[x]] \neq 0$, $\operatorname{Ext}^n(M[x^{-1}], N[x^{-1}]) \neq 0$. Therefore,

$$\operatorname{pd}_{R[x]}(M[x^{-1}]) \geqslant n.$$

Hence, we have

$$n \leqslant \operatorname{pd}_{R[x]}(M[x^{-1}]) \leqslant n+1.$$

Suppose $\operatorname{pd}_R(M) = \infty$. Let $n \in \mathbb{Z}^+$ then $\operatorname{Ext}^{n+1}(M,N) \neq 0$ for some N. But $\operatorname{Ext}^{n+1}(M[x^{-1}], N[x^{-1}]) \cong \operatorname{Ext}^{n+1}(M,N)[[x]]$. Therefore,

$$\operatorname{Ext}^{n+1}(M[x^{-1}], N[x^{-1}]) \neq 0.$$

Hence, we have $\operatorname{pd}_{R[x]}(M[x^{-1}]) = \infty$. $\square$

Theorem 2.9. *Let R be a commutative and noetherian and M be a finitely generated left R-module. Then*

$$\operatorname{pd}_{R[x]}(M[x^{-1}]) = n+1 \quad \textit{if} \quad \operatorname{pd}_R(M) = n.$$

P r o o f. Suppose $\operatorname{pd}_R(M) = n$ then $\operatorname{Tor}_n^R(M,N) \neq 0$ for some finitely generated left R-module N by (II) p. 130 [2]. By Theorem 2.1 [6],

$$\operatorname{Tor}_{n+1}^{R[x]}(M[x^{-1}], N[x^{-1}]) \cong \operatorname{Tor}_n^R(M,N)[x^{-1}].$$

So $\operatorname{Tor}_{n+1}^{R[x]}(M[x^{-1}], N[x^{-1}]) \neq 0$. Therefore, $\operatorname{pd}_{R[x]}(M[x^{-1}]) > n$. Hence, by the previous theorem we conclude that $\operatorname{pd}_{R[x]}(M[x^{-1}]) = n+1$. $\square$

Theorem 2.10. *If $\operatorname{fd}_R(M) = n$ then $\operatorname{fd}_{R[x]}(M[x^{-1}]) = n+1$ and if $\operatorname{fd}_R(M) = \infty$ then $\operatorname{fd}_{R[x]}(M[x^{-1}]) = \infty$.*

P r o o f. Suppose $\operatorname{fd}_R(M) = n$ then $\operatorname{fd}_{R[x]}(M[x]) = n$. Consider the following exact sequence $0 \to M[x] \to M[x,x^{-1}] \to M[x^{-1}] \to 0$. Then we have a long exact sequence with Tor

$$\begin{gathered} \ldots \to \operatorname{Tor}_{n+2}(N, M[x^{-1}]) \to \operatorname{Tor}_{n+1}(N, M[x]) \\ \to \operatorname{Tor}_{n+1}(N, M[x,x^{-1}]) \to \operatorname{Tor}_{n+1}(N, M[x^{-1}]) \to \ldots \to \operatorname{Tor}_1(N, M[x]) \\ \to N \otimes M[x] \to N \otimes M[x,x^{-1}] \to N \otimes M[x^{-1}] \to 0. \end{gathered}$$

Then $\operatorname{Tor}_{n+1}(N, M[x]) = 0$. Since $M[x,x^{-1}] \cong S^{-1}M[x]$ where $S = \{1, s, s^2, \ldots\}$, $s \in R$ and

$$\operatorname{fd}_{R[x]}(M[x,x^{-1}]) \leqslant \operatorname{fd}_{R[x]}(M[x]) + 1 = n+1,$$

$\mathrm{Tor}_{n+2}(N, M[x, x^{-1}]) = 0$. Therefore, $\mathrm{Tor}_{n+2}(N, M[x^{-1}]) = 0$. Hence,

$$\mathrm{fd}_{R[x]}(M[x^{-1}]) \leqslant n+1.$$

Since $\mathrm{fd}_R(M) = n$, $\mathrm{Tor}_n(N, M) \neq 0$ for some N. Since

$$\mathrm{Tor}_{n+1}(N[x^{-1}], M[x^{-1}]) \cong \mathrm{Tor}_n(N, M)[x^{-1}],$$

$\mathrm{Tor}_{n+1}(N[x^{-1}], M[x^{-1}]) \neq 0$. Thus $\mathrm{fd}_{R[x]}(M[x^{-1}]) \geqslant n+1$. Therefore, we have $\mathrm{fd}_{R[x]}(M[x^{-1}]) = n+1$. Suppose $\mathrm{fd}_R(M) = \infty$. Let $n \in \mathbb{Z}^+$ then $\mathrm{Tor}_{n+1}(N, M) \neq 0$ for some N. Since

$$\mathrm{Tor}_{n+1}(N[x], M[x^{-1}]) \cong \mathrm{Tor}_{n+1}(N, M)[x^{-1}],$$

$\mathrm{Tor}_{n+1}(N[x], M[x^{-1}]) \neq 0$. Hence, we have $\mathrm{fd}_{R[x]}(M[x^{-1}]) = \infty$. $\square$

Let $S \subset \mathbb{N}$ ($\mathbb{N}$ the natural numbers) be a submonoid where we assume that for some n_0, all $n \geqslant n_0$ are in $\mathbb{N}$. If S is a symmetric, then it has a conductor $c \in S$, i.e. c is such that the function $n \to -n + c - 1$ from $\mathbb{Z}$ to $\mathbb{Z}$ maps S bijectively to its complement in $\mathbb{Z}$.

If S is a submonoid of N (N is the set of all natural numbers), then $R[x^S]$ is defined to be the ring of all sums $\sum\limits_{i \in S} r_i x^i$ with $r_i \in R$. If M is a left R-module, then we let $M[x^{-S}]$ consist of all $\sum\limits_{i \in S} m_i x^{-i}$. Then $M[x^{-S}]$ is naturally an $R[x^S]$-module (where $x^j \cdot x^{-i} = x^{-(i-j)}$ if $i - j \in S$ and $x^j \cdot x^{-i} = 0$ if $i - j \notin S$ for $i, j \in S$).

Theorem 2.11. *Let M be a left R-module and S be a symmetric submonoid of N (the set of natural numbers). Then there is a short exact sequence*

$$0 \to M[x^S] \to M[x, x^{-1}] \to M[x^{-S}] \to 0$$

as $R[x^S]$-modules.

Proof. Let M be a R-module, then $M[x^S] \subset M[x, x^{-1}]$ is a submodule as $R[x^S]$-modules. Let $\sum\limits_{n \in \mathbb{Z}} a_n x^n = \ldots + a_{-1}x^{-1} + a_0 + a_1 x^1 + a_2 x^2 + \ldots$ for an element of $M[x, x^{-1}]$. Let $\varphi\colon\ M[x, x^{-1}]/M[x^S] \to M[x^{-S}]$ be defined by

$$\varphi\Bigl(\sum_{n \in \mathbb{Z}} a_n x^n + M[x^S]\Bigr) = \sum_{n \in S} a_{-n+c-1} x^{-n}$$

where c is the conductor of S. Then φ is an isomorphism. Hence,

$$0 \to M[x^S] \to M[x, x^{-1}] \to M[x^{-S}] \to 0$$

is a short exact sequence as $R[x^S]$-module. $\square$

We note that for a nonsymmetric submonid S, the above theorem does not hold.

Remark 2.12. We easily see that the natural isomorphism

$$\mathrm{Hom}_{R[x]}(M[x^{-1}], N[x^{-1}] \cong \mathrm{Hom}_R(M, N)[[x]]$$

(Theorem 1.1, [6]) can be extended to the case

$$\mathrm{Hom}_{R[x^S]}(M[x^{-S}], N[x^{-S}] \cong \mathrm{Hom}_R(M, N)[[x^S]]$$

for any submonid $S \subset N$, and also for a left noetherian ring R the isomorphism

$$\mathrm{Ext}^i_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong \mathrm{Ext}^i_R(M, N)[[x]]$$

(Theorem 1.2, [6]) can be extended to the case

$$\mathrm{Ext}^i_{R[x^S]}(M[x^{-S}], N[x^{-S}]) \cong \mathrm{Ext}^i_R(M, N)[[x^S]]$$

for any submonid $S \subset N$. But by the above theorem 2.8, the isomorphism

$$\mathrm{Tor}^{R[x]}_i(M[x^{-1}], N[x^{-1}]) \cong \mathrm{Tor}^R_{i-1}(M, N)[x^{-1}]$$

(Theorem 2.1, [6]) can be extended to the case

$$\mathrm{Tor}^{R[x^S]}_i(M[x^{-S}], N[x^{-S}]) \cong \mathrm{Tor}^R_{i-1}(M, N)[x^{-S}]$$

only for a symmetric submonid S.

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Author's address: Department of Mathematics, Dong-A University, Pusan, Korea 604-714, e-mail: `swpark@mail.donga.ac.kr`.