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SUBFIELDS OF LATTICE-ORDERED FIELDS THAT MIMIC MAXIMAL TOTALLY ORDERED SUBFIELDS

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0. Introduction

A *lattice-ordered field* (or *ℓ-field*) is a field $(L, +, \cdot)$ with a compatible lattice order $\geqslant$. A totally ordered subfield of L is called an *o-subfield*, and for any subset $S \subseteq L$, $S^{\geqslant} = \{s \in S \mid s \geqslant 0\}$. It is well known that every ℓ-field L in which $1 > 0$ has a maximal o-subfield $M(L)$ (cf. [4], [7]). For archimedean ℓ-fields, Schwartz proved in [7] that even if $1 \not> 0$, there is a subfield with properties similar to those of the o-subfields $M(L)$. Specifically, he showed that if $(L, +, \cdot, \geqslant)$ is an archimedean ℓ-field, then L contains a subfield K with the following properties:

(1) K is the largest subfield of L which has a compatible total order $\succcurlyeq$ such that $K^{\succcurlyeq} L^{\geqslant} \subseteq L^{\geqslant}$;

(2) K is the largest subfield of L such that for all $0 < a \in L$, aK is a totally ordered subgroup of $(L, +, \cdot, \geqslant)$;

(3) $K^{\succcurlyeq} = \{x \in L \mid \text{ for all } a, b \in L, a \wedge b = 0 \Longrightarrow xa \wedge b = 0\}$.

The object of this paper is to investigate the existence of such a subfield without assuming that L is archimedean.

In section 1, we show that partially ordered fields may have distinct maximal o-subfields, which may even be nonisomorphic. In the remainder of the paper, we restrict ourselves to ℓ-fields $(L, +, \cdot, \geqslant)$. In section 2, we introduce a compatible partial order $\succcurlyeq$ on L to show that a subfield satisfies condition (1) if and only if it also satisfies condition (2). In section 3, we show that, with respect to $\succcurlyeq$,

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the positive cone of a subfield which satisfies condition (1) or (2) is contained in the set determined by condition (3). And then in section 4, we find several conditions, each of which is equivalent to this set being the positive cone of an o-subfield of $(L, +, \cdot, \succcurlyeq)$. In sections 5 and 6, we use the results of sections 3 and 4 to show that common constructions of ℓ-fields in which $1 \not> 0$ lead to ℓ-fields in which the set determined by condition (3) is always the positive cone of such an o-subfield.

We will use the following terminology without further comment; for terminology and notation left undefined, see [2]. If $(L, +, \cdot, \geqslant)$ is an ℓ-field, then an element $b \in L$ can be "basic" in two senses. It can be an element of a basis of L as a vector space over some subfield T or it can be a positive element for which the interval $[0, b]$ is totally ordered. We distinguish between these two situations by calling a positive element b for which $[0, b]$ is totally ordered *ℓ-basic* and by calling a basis of L as a vector space over some subfield a *v-basis*.

1. Totally ordered subfields of partially ordered fields

We noted in the introduction above that an ℓ-field $(L, +, \cdot, \geqslant)$ in which $1 > 0$ contains a unique maximal o-subfield $M(L)$. In this section, we give examples which show that this may not be true for partially ordered fields in general.

These examples are based on Hahn fields, whose general construction is the following. Let F be a field and let T be a totally ordered group. Form the product $\prod_T F$. For $f \in \prod_T F$, let $\operatorname{Supp}(f) = \{t \in T \mid f(t) \neq 0\}$ denote the *support* of f. Then form the subset

$$_X\prod_T F = \Big\{ f \in \prod_T F \mid \operatorname{Supp}(f) \text{ is inversely well-ordered} \Big\}.$$

It is well known ([2], [4], [5], [6]) that $_X\prod_T F$ is a field with respect to coordinatewise addition and to convolution as multiplication:

$$(f+g)_\delta = f_\delta + g_\delta; \quad (fg)_\delta = \sum_{\alpha+\gamma=\delta} f_\alpha g_\gamma.$$

If F is totally ordered, then $_X\prod_T F$ is also totally ordered with respect to the order

$$f > 0 \iff f_\mu > 0, \text{ where } \mu = \bigvee \operatorname{Supp}(f).$$

Example 1.1. Construct the Hahn field $H = {}_X\prod_{\mathbb{Q}(\sqrt{2})} \mathbb{Q}$. We will partially order H in an unusual way. Let

$$\begin{aligned}
L_1 &= \{f \in H \mid f_{a+b\sqrt{2}} = 0 \text{ if } b \neq 0\},\\
P(L_1) &= \Big\{f \in L_1 \mid f_\mu > 0 \text{ where } \mu = \bigvee \operatorname{Supp}(f)\Big\},\\
L_2 &= \{f \in H \mid f_{a+b\sqrt{2}} = 0 \text{ if } a \neq 0\},\\
P(L_2) &= \{f \in L_2 \mid f_\mu > 0 \text{ where } \mu = \bigvee \operatorname{Supp}(f)\}.
\end{aligned}$$

Then L_1, being a copy of the field ${}_X\prod_{\mathbb{Q}} \mathbb{Q}$, is a subfield of H, and $P(L_1)$ is the strictly positive cone of a compatible total order on L_1 (viz., the order defined in the preceding paragraph). Similarly L_2 is a subfield of H and $P(L_2)$ is the strictly positive cone of a compatible total order on L_2. Now let $P(H)$ denote the set of all finite sums of the form $\sum\limits_i f_i g_i$, where $f_i \in P(L_1)$ and $g_i \in P(L_2)$. We will show that $P(H)$ is the strictly positive cone of a compatible partial order on H, with respect to which $1 > 0$ and L_1 and L_2 are distinct maximal o-subfields.

We first show that $P(H)$ determines a compatible partial order on H. Suppose that $\sum\limits_i f_i g_i, \sum\limits_k a_k b_k \in P(H)$, and let

$$\begin{aligned}
\mu &= \bigvee \operatorname{Supp}\left(\sum_i f_i g_i + \sum_k a_k b_k\right),\\
\nu &= \bigvee \operatorname{Supp}\left(\sum_i f_i g_i \sum_k a_k b_k\right),\\
\varphi_i &= \bigvee \operatorname{Supp}(f_i), \quad \gamma_i = \bigvee \operatorname{Supp}(g_i),\\
\alpha_k &= \bigvee \operatorname{Supp}(a_k), \quad \beta_k = \bigvee \operatorname{Supp}(b_k),\\
\lambda_1 &= \bigvee \{\varphi_i + \gamma_i\}, \quad \lambda_2 = \bigvee \{\alpha_k + \beta_k\}.
\end{aligned}$$

Then $\mu = \lambda_1 \vee \lambda_2$ and $\nu = \lambda_1 + \lambda_2$, and

$$\begin{aligned}
\left(\sum_i f_i g_i + \sum_k a_k b_k\right)_\mu &= \left(\sum_i (f_i g_i)_\mu\right) + \left(\sum_k (a_k b_k)_\mu\right)\\
&= \left(\sum_i \sum_{\varphi_i + \gamma_i = \mu} (f_i)_{\varphi_i} (g_i)_{\gamma_i}\right) + \left(\sum_k \sum_{\alpha_k + \beta_k = \mu} (a_k)_{\alpha_k} (b_k)_{\beta_k}\right) > 0,
\end{aligned}$$

$$\begin{aligned}
\left(\sum_i f_i g_i \sum_k a_k b_k\right)_\nu &= \left(\sum_i (f_i g_i)\right)_{\lambda_1} \left(\sum_k (a_k b_k)\right)_{\lambda_2}\\
&= \left(\sum_i \sum_{\varphi_i + \gamma_i = \lambda_1} (f_i)_{\varphi_i} (g_i)_{\gamma_i}\right)\left(\sum_k \sum_{\alpha_k + \beta_k = \lambda_2} (a_k)_{\alpha_k} (b_k)_{\beta_k}\right) > 0.
\end{aligned}$$

It follows that $P(H)+P(H) \subseteq P(H)$ and $P(H)P(H) \subseteq P(H)$. Furthermore, $\lambda_1 = \bigvee \operatorname{Supp}(-\sum_i f_i g_i)$, and

$$\Big(-\sum_i f_i g_i\Big)_{\lambda_1} = -\Big(\sum_{\varphi_i+\gamma_i=\lambda_1} (f_i)_{\varphi_i}(g_i)_{\gamma_i}\Big) < 0$$

so that $-\sum_i f_i g_i \notin P(H)$. Thus it is also the case that $P(H) \cap (-P(H)) = \emptyset$, and we conclude from [2, p. 105] that $P(H)$ is the strictly positive cone of a compatible partial order $\geqslant$ on H.

Since $1 \in P(L_1)$ and $1 \in P(L_2)$, $1 \in P(H)$, and hence $(H, +, \cdot, \geqslant)$ is a partially ordered field in which $1 > 0$. As well, since $P(L_1) = P(H) \cap L_1$ and $P(L_2) = P(H) \cap L_2$, L_1 and L_2 are both o-subfields of H. We will show that both L_1 and L_2 are maximal o-subfields of H.

For $Z \in P(H)$, let

$$R(Z) = \{q \in \mathbb{Q} \mid q + s\sqrt{2} \in \operatorname{Supp}(Z) \text{ for some } s \in \mathbb{Q}\},$$
$$S(Z) = \{q \in \mathbb{Q} \mid r + q\sqrt{2} \in \operatorname{Supp}(Z) \text{ for some } r \in \mathbb{Q}\}.$$

We first show that

(a) for all $Z \in P(H)$, both $R(Z)$ and $S(Z)$ are inversely well-ordered.

Since $Z \in P(H)$, $Z = \sum_k a_k b_k$ for $a_k \in P(L_1)$ and $b_k \in P(L_2)$. So if $\alpha+\beta\sqrt{2} \in S(Z)$, then for some k, $\alpha+\beta\sqrt{2} \in S(a_k b_k)$, and hence for some k, $\alpha \in S(a_k)$ and $\beta \in S(b_k)$. It follows that $R(Z) \subseteq \bigcup_k R(a_k)$ and $S(Z) \subseteq \bigcup_k S(b_k)$, and thus since all the $R(a_k)$ and $S(b_k)$ are inversely well-ordered, both $R(Z)$ and $S(Z)$ are inversely well-ordered. This proves (a).

Now suppose that $X \in P(H)$, that $X \notin P(L_1)$, and, by way of contradiction, that $L_1(X)$ is totally ordered.

We will need the following notation. For $w \in \mathbb{Q}(\sqrt{2})$, define $E_w \in P(H)$ by letting

$$(E_w)_\lambda = \begin{cases} 1 & \text{if } \lambda = w \\ 0 & \text{if } \lambda \neq w; \end{cases}$$

and for $0 \neq Z \in H$ and $0 < i \in \mathbb{Z} >$, let

$$\mu_0(Z) = \bigvee \operatorname{Supp}(Z), \quad \mu_i(Z) = \bigvee (\operatorname{Supp}(Z) \setminus \{\mu_0(Z), \dots, \mu_{i-1}(Z)\}).$$

We begin by showing that

(b) there exist $Y \in L_1(X) \cap P(H)$ and $0 < \gamma, q \in \mathbb{Q}$ such that $\mu_0(Y) = \gamma\sqrt{2}$, $\mu_1(Y) < q < \mu_0(Y)$, and for all $\alpha \in R(Y)$ and all $0 < n \in \mathbb{Z}$, $\alpha \neq nq$.

Since $X \in P(H)$, $X = \sum_k a_k b_k$ for $a_k \in P(L_1)$ and $b_k \in P(L_2)$. If $b_k \in P(L_1)$, then $X - a_k b_k \in L_1(X)$, and hence we may assume that $b_k \notin P(L_1)$ for all k. Thus, if $\bigvee \mathrm{Supp}(X) = \alpha + \gamma\sqrt{2}$, then $\gamma \neq 0$. Suppose that $\gamma < 0$. Then since $L_1(X)$ is totally ordered, X^{-1} is also in $L_1(X) \cap P(H)$ and $\mu_0(X^{-1}) = (-\alpha) + (-\gamma)\sqrt{2}$, where $-\gamma > 0$. So since $L_1(X) = L_1(X^{-1})$, we may also assume that $\gamma > 0$. Furthermore, since $E_{-\alpha} \in P(L_1)$, $E_{-\alpha}X \in L_1(X)$, and hence, since $\mu_0(E_{-\alpha}X) = \gamma\sqrt{2}$, we have found $Y \in L_1(X)$ such that $\mu_0(Y) = \gamma\sqrt{2}$ for $\gamma > 0$. Now (a) implies both that $\bigvee R(Y)$ exists in $R(Y)$ and that for all $0 < n < N$,

$$r_n = \bigvee \left\{ r \in R(Y) \mid \frac{r}{n} < \mu_0(Y) \right\}$$

exists in $R(Y)$ and $\frac{r_n}{n} < \mu_0(Y)$. Then for some $0 < N \in \mathbb{Z}$, $\frac{\bigvee R(Y)}{N} < \mu_0(Y)$, and for all $n \geqslant N$, $\frac{r_n}{n} \leqslant \frac{\bigvee R(Y)}{N} = \frac{r_N}{N}$. Let $M = \frac{r_1}{1} \vee \ldots \vee \frac{r_N}{N} \vee \mu_1(Y)$. Then $M < \mu_0(Y)$ and hence, since $0 < \mu_0(Y)$, there exists $0 < q \in \mathbb{Q}$ such that $M < q < \mu_0(Y)$. If $q = n\alpha$ for $\alpha \in R(Y)$, then $\frac{\alpha}{n} = q < \mu_0(Y)$ and hence $q = \frac{\alpha}{n} \leqslant \frac{r_n}{n} \leqslant M$, a contradiction. So for all $\alpha \in R(Y)$ and all $0 < n \in \mathbb{Z}$, $\alpha \neq nq$, and since $\mu_1(Y) \leqslant M$, $\mu_1(Y) < q < \mu_0(Y)$. This proves (b).

So we may assume that $Y \in L_1(X) \cap P(H)$ and $\gamma, q \in \mathbb{Q}$ satisfy the conditions given in (b). Then $E_q \in L_1$, and hence $Z = E_q + Y$ is an element in $L_1(X) \cap P(H)$ such that $\mu_0(Z) = \mu_0(Y) = \gamma\sqrt{2}$ and $\mu_1(Z) = q$. Now we are assuming that $L_1(X)$ is totally ordered, and hence we must have $Z^{-1} \in P(H)$. We will show that $R(Z^{-1})$ is unbounded.

Certainly $\mu_0(Z^{-1}) = -\gamma\sqrt{2}$. Since $ZZ^{-1} = 1$ and $\mu_1(Z) = q$, we must have

$$\begin{aligned} q - \gamma\sqrt{2} &= \mu_1(Z) + \mu_0(Z^{-1}) \\ &= \mu_0(Z) + \mu_1(Z^{-1}) = \gamma\sqrt{2} + \mu_1(Z^{-1}), \end{aligned}$$

and hence $\mu_1(Z^{-1}) = q - 2\gamma\sqrt{2}$. But $\mu_1(Z) + \mu_1(Z^{-1}) \notin \mathrm{Supp}(1) = \mathrm{Supp}(ZZ^{-1})$ and hence either

$$\begin{aligned} 2q - 2\gamma\sqrt{2} &= \mu_1(Z) + \mu_1(Z^{-1}) \\ &= \mu_0(Z) + \mu_2(Z^{-1}) = \gamma\sqrt{2} + \mu_2(Z^{-1}) \end{aligned}$$

or for some $\alpha + \beta\sqrt{2} \in \mathrm{Supp}(Z)$,

$$\begin{aligned} 2q - 2\gamma\sqrt{2} &= \mu_1(Z) + \mu_1(Z^{-1}) \\ &= (\alpha + \beta\sqrt{2}) + \mu_0(Z^{-1}) = \alpha + (\beta - \gamma)\sqrt{2}). \end{aligned}$$

In the latter case, $\alpha \in R(Z)$ and $q = \frac{\alpha}{2}$, a contradiction. So we must have $\mu_2(Z^{-1}) = 2q - 3\gamma\sqrt{2}$. Similarly, either

$$\begin{aligned}3q - 3\gamma\sqrt{2} &= \mu_1(Z) + \mu_2(Z^{-1})\\ &= \mu_0(Z) + \mu_3(Z^{-1}) = \gamma\sqrt{2} + \mu_3(Z^{-1})\end{aligned}$$

or for some $\alpha + \beta\sqrt{2} \in \mathrm{Supp}(Z)$, one of the following two equations holds:

$$\begin{aligned}3q - 3\gamma\sqrt{2} &= \mu_1(Z) + \mu_2(Z^{-1})\\ &= (\alpha + \beta\sqrt{2}) + \mu_0(Z^{-1}) = \alpha + (\beta - \gamma)\sqrt{2}),\\ 3q - 3\gamma\sqrt{2} &= \mu_1(Z) + \mu_2(Z^{-1})\\ &= (\alpha + \beta\sqrt{2}) + \mu_1(Z^{-1}) = (\alpha + q) + (\beta - 2\gamma)\sqrt{2}).\end{aligned}$$

In the third case, $\alpha \in R(Z)$ and $q = \frac{\alpha}{2}$ and in the second case, $\alpha \in R(Z)$ and $q = \frac{\alpha}{3}$. Since both cases contradict our assumptions concerning q, $\mu_3(Z^{-1}) = 3q - 4\gamma\sqrt{2}$. Continuing in this fashion, we conclude that for any $0 < n \in \mathbb{Z}$, $\mu_n(Z^{-1}) = nq - (n+1)\gamma\sqrt{2}$, and hence that $R(Z^{-1}) \supseteq \{nq \mid 0 < n \in \mathbb{Z}\}$. But then Z^{-1} is an element of $P(H)$ for which $R(Z^{-1})$ has no maximum element. This contradicts (a) and therefore, $L_1(X)$ is not totally ordered.

It follows that L_1 is a maximal o-subfield of H. A similar argument shows that L_2 is also a maximal o-subfield and hence that H has two distinct maximal o-subfields. Note that L_1 and L_2, both being isomorphic to ${}_X\prod_{\mathbb{Q}} \mathbb{Q}$ as totally ordered fields, are also isomorphic to each other as totally ordered fields.

In Example 1.1, we constructed a partially ordered field with $1 > 0$ having distinct maximal o-subfields that are isomorphic. A similar construction may be used to find a partially ordered field with $1 > 0$ having distinct maximal o-subfields that are not isomorphic.

Example 1.2. Let $\mathbb{U}$ be a subgroup of $\mathbb{R}$ that is uncountable, divisible and order-dense and for which $\mathbb{U} \cap \mathbb{Q} = \{0\}$, and let $\mathbb{T} = \mathbb{Q} \oplus \mathbb{U}$. Then $\mathbb{T}$ is a totally ordered group with respect to the usual total order on $\mathbb{R}$, and we may construct the Hahn field $H = {}_X\prod_{\mathbb{T}} \mathbb{Q}$. We will define a partial order on H in similar to that defined on ${}_X\prod_{\mathbb{Q}(\sqrt{2})} \mathbb{Q}$ in Example 1.1. Let

$$\begin{aligned}&L_1 = \{f \in H \mid f_{q+u} = 0 \text{ if } u \neq 0\},\\ &P(L_1) = \Big\{f \in L_1 \mid f_\mu > 0 \text{ where } \mu = \bigvee \mathrm{Supp}(f)\Big\},\\ &L_2 = \{f \in H \mid f_{q+u} = 0 \text{ if } q \neq 0\},\\ &P(L_2) = \Big\{f \in L_2 \mid f_\mu > 0 \text{ where } \mu = \bigvee \mathrm{Supp}(f)\Big\}.\end{aligned}$$

Then L_1, being a copy of the field ${}_X\prod_{\mathbb{Q}} \mathbb{Q}$, is a subfield of H, and $P(L_1)$ is the strictly positive cone of the usual compatible total order on L_1. Similarly L_2, being a copy of the field ${}_X\prod_{\mathbb{U}} \mathbb{Q}$, is a subfield of H, and $P(L_2)$ is the strictly positive cone of the usual compatible total order on L_2. Now let $P(H)$ denote the set of all finite sums of the form $\sum_i f_i g_i$, where $f_i \in P(L_1)$ and $g_i \in P(L_2)$. An argument similar to that given in Example 1.1 shows that $P(H)$ is the strictly positive cone of a compatible partial order on H, with respect to which $1 > 0$ and L_1 and L_2 are distinct maximal o-subfields. Since $\mathbb{Q}$ is countable and $\mathbb{U}$ is uncountable, ${}_X\prod_{\mathbb{Q}} \mathbb{Q}$ is not isomorphic to ${}_X\prod_{\mathbb{U}} \mathbb{Q}$. So L_1 and L_2 are maximal o-subfields of H that are not isomorphic.

2. A partially ordered field associated with a lattice-ordered field

As noted in the introduction, if $(L, +, \cdot, \geqslant)$ is an archimedean ℓ-field, then, even if $1 \not> 0$, L nonetheless contains a subfield K with properties similar to those of $M(L)$, the specific properties (1), (2) and (3) being given in the introduction. In the sequel, we will investigate subfields with similar properties in ℓ-fields that are not assumed to be archimedean.

To this end, let $(L, +, \cdot, \geqslant)$ be an ℓ-field and let

$$\overline{P}(L) = \{x \in L \mid \text{ for all } a \geqslant 0,\ ax \geqslant 0\}.$$

It is easy to check that $\overline{P}(L) \cap (-\overline{P}(L)) = \{0\}$, that $\overline{P}(L) + \overline{P}(L) \subseteq \overline{P}(L)$, and that $\overline{P}(L)\overline{P}(L) \subseteq \overline{P}(L)$, and hence by [2, p. 105] that $\overline{P}(L)$ is the positive cone of a compatible partial order, $\succcurlyeq$, on L, i.e. that $(L, +, \cdot, \succcurlyeq)$ is a partially ordered field, where $L^{\succcurlyeq} = \overline{P}(L)$. In the archimedean case, property (1) implies that the subfield K is a maximal o-subfield of $(L, +, \cdot, \succcurlyeq)$. In the general case, since $1 \in \overline{P}(L)$, L contains a copy of the rational numbers which is given the usual total order by $\succcurlyeq$, and thus by Zorn's Lemma, $(L, +, \cdot, \succcurlyeq)$ contains at least one maximal o-subfield. So to address property (1) in general, we let $\mathscr{M}_1(L)$ denote the set of maximal o-subfields of $(L, +, \cdot, \succcurlyeq)$. Note that the examples of section 1 show that there exist partially ordered fields in which $1 > 0$ and which have more than one maximal o-subfield.

As noted above, L contains a copy Q of the rational numbers. By [2, p. 67], Q has the following property:

(c) for all $0 < a \in L, aQ$ is a totally ordered subgroup of $(L, +, \geqslant)$.

Thus by Zorn's Lemma, L contains at least one subfield T which is maximal with respect to property (c). So to address property (2), we let $\mathscr{M}_2(L)$ denote the set of all subfields of $(L, +, \cdot)$ which are maximal with respect to property (c).

Finally, to address property (3), we let

$$\begin{aligned} P_3(L) &= \{x \in L \mid \text{ for all } a, b \in L, a \wedge b = 0 \Longrightarrow ax \wedge b = 0\}, \\ M_3(L) &= \{x - y \mid x, y \in P_3(L)\}. \end{aligned}$$

(In the terminology of [1], $P_3(L)$ is the set of all x such that the function $a \mapsto ax$ defines a "polar preserving endomorphism" of L.) The work of Schwartz [7] shows that if $1 > 0$, then $M_3(L) = M(L)$ and that if L is archimedean, then $K = M_3(L)$ is a subfield of L satisfying properties (1), (2) and (3) given in the introduction and as well $\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\}$. In general, if $1 > 0$ and $x \in \overline{P}(L)$, then $x = 1x \geqslant 0$ and hence $\overline{P}(L) \subseteq L^{\geqslant}$. Since always $\overline{P}(L) \supseteq L^{\geqslant}$, in this case $\overline{P}(L) = L^{\geqslant}$ so that $\mathscr{M}_1(L) = \{M(L)\} = \{M_3(L)\}$. (A stronger result is Corollary 2.4 below.)

What can be said in general, if we assume neither that L is archimedean nor that $1 > 0$? We begin our answer to this question by showing that $\mathscr{M}_1(L) = \mathscr{M}_2(L)$ for all ℓ-fields L. Note that this leaves open the possibilities first that $\mathscr{M}_1(L)$ contains more than one element and second that the subfields that comprise $\mathscr{M}_1(L)$ may have no relation to $M_3(L)$. In section 3, we will show that any subfield in $\mathscr{M}_1(L) = \mathscr{M}_2(L)$ is contained in $M_3(L)$; in section 4, we will show that $\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\}$ if and only if $M_3(L)$ is totally ordered. In sections 5 and 6, we will apply this result to known constructions.

To prove that $\mathscr{M}_1(L) = \mathscr{M}_2(L)$, we need the following lemma.

Lemma 2.1. *Let $(L, +, \cdot, \geqslant)$ be an ℓ-field and let S be a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, aS is a totally ordered subgroup of $(L, +, \geqslant)$. If $s \in S$ is such that $0 < ps$ for some $0 < p \in L$, then $s \in \overline{P}(L)$.*

Proof. Note that by hypothesis, for all $0 < a \in L$ and all $0 \neq t \in S$, either $at > 0$ or $at < 0$. If $as^{-1} < 0$ for some $0 < a \in L$, then $pa = psas^{-1} < 0$, a contradiction. So $as^{-1} > 0$ for all $0 < a \in L$. But if $as < 0$ for some $0 < a \in L$, then $-as > 0$, and thus $-a^2 = (-as)as^{-1} > 0$, a contradiction. So $0 < as$ for all $0 < a \in L$, i.e., $s \in \overline{P}(L)$. □

Proposition 2.2. *Suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field. Then for any $S \subseteq L$, the following statements are equivalent:*

(i) *S is an o-subfield of $(L, +, \cdot, \succcurlyeq)$;*
(ii) *S is a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, aS is a totally ordered subgroup of $(L, +, \geqslant)$.*

Proof. Suppose that S is an o-subfield of $(L, +, \cdot, \succcurlyeq)$. Then S is a subfield of $(L, +, \cdot)$. Let $0 < a \in L$ and let $s, t \in S$. Without loss of generality, we may assume that $s \succcurlyeq t$. Then $s - t \in \overline{P}(L)$ and hence $0 \leqslant a(s-t) = as - at$, i.e., $as \leqslant at$. Since aS is obviously a subgroup of $(L, +)$, it follows that aS is a totally ordered subgroup of $(L, +, \succcurlyeq)$.

Conversely, suppose that S is a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, aS is a totally ordered subgroup of $(L, +, \geqslant)$. Pick $0 < p \in L$. If $0 \neq s \in S$ is such that $0 < ps$, then by Lemma 2.1, $s \in \overline{P}(L)$. Otherwise, $0 < -ps$ and by Lemma 2.1, $-s \in \overline{P}(L)$. It follows that S is an o-subfield of $(L, +, \cdot, \succcurlyeq)$. □

Corollary 2.3. *For any ℓ-field $(L, +, \cdot, \geqslant)$, $\mathscr{M}_1(L) = \mathscr{M}_2(L)$.*

P r o o f. By Proposition 2.2, the set $\mathscr{S}_1(L)$ of o-subfields of $(L, +, \cdot, \succcurlyeq)$ is the same as the set $\mathscr{S}_2(L)$ of subfields K of $(L, +, \cdot)$ such that for all $0 < a \in L$, aK is a totally ordered subgroup of $(L, +, \geqslant)$. Therefore, since $\mathscr{M}_1(L)$ is the set of maximal elements of $\mathscr{S}_1(L)$ and $\mathscr{M}_2(L)$ is the set of maximal elements of $\mathscr{S}_2(L)$, $\mathscr{M}_1(L)$ must equal $\mathscr{M}_2(L)$. □

Corollary 2.4. *For any ℓ-field $(L, +, \cdot, \geqslant)$ in which $1 > 0$,*

$$\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\} = \{M(L)\}.$$

P r o o f. Corollary 2.4 follows from Corollary 2.3 and the observation above that if $1 > 0$, then $\mathscr{M}_1(L) = \{M(L)\} = \{M_3(L)\}$. □

3. Totally ordered subfields and polar preserving multiplication

The object of this section is to show that every subfield in $\mathscr{M}_1(L)$ is contained in $M_3(L)$ and to derive some of the elementary properties of the elements of $M_3(L)$.

Lemma 3.1. *Let $(L, +, \cdot, \geqslant)$ be an ℓ-field and let S be a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, aS is a totally ordered subgroup of $(L, +, \geqslant)$. If $s \in S$ is such that $0 < ps < p$ for some $0 < p \in L$, then $0 < as < a$ for all $0 < a \in L$.*

P r o o f. Let $0 < a \in L$. By Lemma 2.1, $0 < as$. Since aS is totally ordered, $as < a$ or $as = a$ or $as > a$. If $as = a$, then $s = 1$ and hence $ps = p$, a contradiction. If $as > a$, then $pas > pa$, and since $a > 0$ and $p > ps$, $pa > psa$. But then $pas > pa > psa = pas$, a contradiction. We conclude that $as < a$. □

Lemma 3.2. *Let $(L, +, \cdot, \geqslant)$ be an ℓ-field and let S be a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, aS is a totally ordered subgroup of $(L, +, \geqslant)$. If $s \in S$ is such that $0 < ps$ for some $0 < p \in L$, then $s^{-1} \in \overline{P}(L)$.*

P r o o f. Let $0 < a \in L$, and note that by Lemma 2.1, $0 < as$. Since S is a subfield, $s^{-1} \in S$ and hence by hypothesis either $0 < as^{-1}$ or $0 > as^{-1}$. If $0 > as^{-1}$, then $a^2 = (as^{-1})(as) < 0(as) = 0$, a contradiction, and hence $0 < as^{-1}$. It follows that $s^{-1} \in \overline{P}(L)$. $\square$

Proposition 3.3. *Let $(L, +, \cdot, \geqslant)$ be an ℓ-field and let S be a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, aS is a totally ordered subgroup of $(L, +, \geqslant)$. If $s \in S$ is such that $0 < ps$ for some $0 < p \in L$, then $s \in P_3(L)$.*

P r o o f. Suppose that $a \wedge b = 0$ in L, and note that by hypothesis either $as \leqslant a$ or $as > a$. If $as \leqslant a$, then since $as > 0$ by Lemma 2.1, $0 \leqslant as \wedge b \leqslant a \wedge b = 0$, i.e., $as \wedge b = 0$. So suppose that $as > a$, and by way of contradiction that $as \wedge b > 0$ as well. Since S is a subfield, $s^{-1} \in S$ and $0 < as^{-1}$ by Lemma 3.2. Furthermore, by hypothesis, $as^{-1} > a$ or $as^{-1} = a$ or $as^{-1} < a$. If $as^{-1} > a$, then $a^2 = (as)(as^{-1}) > a(as^{-1}) > a^2$, a contradiction. If $as^{-1} = a$, then $a = as$, also a contradiction. So $as^{-1} < a$. But then by Lemma 3.1, $(as \wedge b)s^{-1} < as \wedge b \leqslant b$, and by Lemma 3.2, $0 < (as \wedge b)s^{-1} \leqslant (as)s^{-1} = a$. Then $0 < (as \wedge b)s^{-1} \leqslant a \wedge b$, a contradiction. We conclude that $as \wedge b = 0$, and hence that $s \in P_3(L)$. $\square$

Corollary 3.4. *If $(L, +, \cdot, \geqslant)$ is an ℓ-field and $T \in \mathscr{M}_2(L)$, then $T \subseteq M_3(L)$.*

P r o o f. By Corollary 2.3, $T \in \mathscr{M}_1(L)$. So T is an o-subfield of $(L, +, \cdot, \succcurlyeq)$, and thus $T = \{x - y \mid x, y \in T^{\succcurlyeq}\}$. But by Proposition 3.3, $T^{\succcurlyeq} \subseteq P_3(L)$. It follows that $T \subseteq M_3(L)$. $\square$

4. Properties of polar preserving multiplication

By Corollaries 2.3 and 3.4, if $(L, +, \cdot, \geqslant)$ is an ℓ-field and if $P_3(L)$ is the positive cone of an o-subfield of $(L, +, \cdot, \succcurlyeq)$, then $\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\}$ and hence $(L, +, \cdot, \geqslant)$ contains a unique subfield with properties (1), (2) and (3) given in the introduction. The object of this section is to find conditions under which $P_3(L)$ is the positive cone of such an o-subfield.

It is easy to see that $P_3(L)$ already possesses several of the properties which determine such a cone. Specifically, we have the following.

Proposition 4.1. *If $(L, +, \cdot, \geqslant)$ is an ℓ-field, then $M_3(L)$ is a partially ordered subring of $(L, +, \cdot, \succcurlyeq)$ whose positive cone is $P_3(L)$.*

P r o o f. It is easy to see that $M_3(L)$ is a subring of L, and since $P_3(L) \subseteq \overline{P}(L)$, $P_3(L) \cap (-P_3(L)) = \{0\}$. Suppose that $x, y \in P_3(L)$ and that $a \wedge b = 0$. Then

$ax \wedge b = 0$ and hence $axy \wedge b = 0$ so that $xy \in P_3(L)$. So $P_3(L)P_3(L) \subseteq P_3(L)$. Furthermore, $ax \wedge b = 0 = ay \wedge b$ in L, and thus (see [2, p. 70]), $(a(x+y)) \wedge b = (ax+ay) \wedge b = 0$ so that $x+y \in P_3(L)$. Then $P_3(L) + P_3(L) \subseteq P_3(L)$, and therefore $M_3(L)$ is a partially ordered subring of $(L, +, \cdot, \succcurlyeq)$ whose positive cone is $P_3(L)$. □

So to show that $M_3(L)$ is an o-subfield of $(L, +, \cdot, \succcurlyeq)$, it is only necessary to show that it is totally ordered in $(L, +, \cdot, \succcurlyeq)$ and that its nonzero elements possess multiplicative inverses that are also in $M_3(L)$. We will show that nonzero elements of $P_3(L)$ possess multiplicative inverses in $P_3(L)$, and hence the only condition that remains to be checked is that $M_3(L)$ is totally ordered.

Lemma 4.2. *Suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field and $e \in P_3(L)$. Then for all $a, b \in L$, $e(a \wedge b) = ea \wedge eb$ and $e(a \vee b) = ea \vee eb$.*

P r o o f. Since $e \in P_3(L)$, we have

$$\begin{aligned}
(a - (a \wedge b)) \wedge (b - (a \wedge b)) = (a \wedge b) - (a \wedge b) = 0 \\
\Longrightarrow (ea - e(a \wedge b)) \wedge (b - (a \wedge b)) = 0 \\
\Longrightarrow (ea - e(a \wedge b)) \wedge (eb - e(a \wedge b)) = 0 \\
\Longrightarrow (ea \wedge eb) - e(a \wedge b) = 0 \\
\Longrightarrow ea \wedge eb = e(a \wedge b).
\end{aligned}$$

As well,

$$e(a \vee b) = -e((-a) \wedge (-b)) = -((-ea) \wedge (-eb)) = ea \vee eb.$$

□

Proposition 4.3. *If $(L, +, \cdot, \geqslant)$ is an ℓ-field and $0 \neq e \in P_3(L)$, then $e^{-1} \in P_3(L)$.*

P r o o f. By Lemma 4.2, for all $a, b \in L$,

$$\begin{aligned}
e^{-1}(a \wedge b) &= e^{-1}((ee^{-1}a) \wedge (ee^{-1}b)) \\
&= e^{-1}e((e^{-1}a) \wedge (e^{-1}b)) = (e^{-1}a) \wedge (e^{-1}b).
\end{aligned}$$

So if $a \wedge b = 0$, then $a \wedge eb = 0$ because $e \in P_3(L)$ and hence

$$(e^{-1}a) \wedge b = (e^{-1}a) \wedge (e^{-1}eb) = e^{-1}(a \wedge eb) = 0.$$

It follows that $e^{-1} \in P_3(L)$. □

We are now in a position to use our previous work to determine when $M_3(L)$ is an o-subfield.

Theorem 4.4. *For any ℓ-field $(L, +, \cdot, \geqslant)$, the following statements are equivalent:*

(i) *$P_3(L)$ is a totally ordered subset of $(L, \succcurlyeq)$;*
(ii) *$M_3(L)$ is an o-subfield of $(L, +, \cdot, \succcurlyeq)$;*
(iii) *$M_3(L)$ is the unique maximal o-subfield of $(L, +, \cdot, \succcurlyeq)$;*
(iv) *$M_3(L)$ is a subfield of $(L, +, \cdot)$ such that for all $0 < a \in L$, $aM_3(L)$ is a totally ordered subgroup of $(L, +, \geqslant)$;*
(v) *$M_3(L)$ is the unique maximal subfield K of $(L, +, \cdot)$ such that for all $0 < a \in L$, aK is a totally ordered subgroup of $(L, +, \geqslant)$;*
(vi) *$\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\}$.*

P r o o f. It suffices to show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (iii) $\Longrightarrow$ (i), and of these implications, (iii) $\Longrightarrow$ (i) is obvious, (v) $\Longrightarrow$ (iii) follows immediately from Corollary 2.3, (vi) $\Longrightarrow$ (v) follows from the definition of $\mathscr{M}_2(L)$, and (ii) $\Longrightarrow$ (iv) follows immediately from Proposition 2.2. Thus it remains to show merely that (i) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (vi).

(i) $\Longrightarrow$ (ii): By Proposition 4.1, $M_3(L)$ is a partially ordered subring of $(L, +, \cdot, \succcurlyeq)$ whose positive cone is $P_3(L)$. Suppose $x - y \in M_3(L)$, where $x, y \in P_3(L)$. Since $P_3(L)$ is totally ordered in $(L, +, \succcurlyeq)$, x is comparable to y in $(L, +, \succcurlyeq)$ and hence $x - y$ is comparable to 0, i.e., $x - y \in P_3(L) \cup (-P_3(L))$. It follows that $M_3(L) = P_3(L) \cup (-P_3(L))$ and that $M_3(L)$ is a totally ordered subring of $(L, +, \cdot, \succcurlyeq)$. But by Proposition 4.3, if $0 \neq e \in P_3(L) \cup (-P_3(L))$, then $e^{-1} \in P_3(L) \cup (-P_3(L))$, and hence $M_3(L)$ is an o-subfield of $(L, +, \cdot, \succcurlyeq)$.

(iv) $\Longrightarrow$ (vi): Zorn's Lemma implies that there exists $T \in \mathscr{M}_2(L)$ such that $T \supseteq M_3(L)$; so by Corollary 3.4, $M_3(L) = T \in \mathscr{M}_2(L)$. If $S \in \mathscr{M}_2(L)$, then $S \subseteq M_3(L)$ by Corollary 3.4. But by Corollary 2.3, $\mathscr{M}_2(L) = \mathscr{M}_1(L)$, and hence $S, M_3(L) \in \mathscr{M}_1(L)$. Then S is a maximal o-subfield of $(L, +, \cdot, \succcurlyeq)$ and $M_3(L)$ is an o-subfield of $(L, +, \cdot, \succcurlyeq)$ containing S; so $S = M_3(L)$. We conclude that $\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\}$. $\square$

This section is concerned with a particular method of constructing ℓ-fields in which $1 \not> 0$. Specifically we will be concerned with ℓ-fields of the following sort.

Let L be an extension field of a totally ordered field $(T, +, \cdot, \geqslant)$; let B be a v-basis for L over T; suppose that for all $c, d \in B$, if $cd = \beta_1 b_1 + \ldots + \beta_n b_n$ for $b_1, \ldots, b_n \in B$, then $\beta_i \geqslant 0$ for all i; and let $\geqq_B$ be the binary relation:

$$\alpha_1 b_1 + \ldots + \alpha_n b_n \geqq_B 0 \iff \alpha_i \geqslant 0 \text{ for all } i.$$

In [9], Wilson observed that $(L, +, \cdot, \geqq_B)$ is an ℓ-field. We will refer to $\geqq_B$ as the *Wilson order* on L determined by B.

We first note that in many cases, ℓ-fields with Wilson orders have $1 \ngeqq_B 0$. To prove this and for use in the sequel, we begin with the following lemma.

Lemma 5.1. *Let $(L, +, \cdot, \geqq_B)$ be the ℓ-field described above.*

(i) *If $0 < \alpha, \gamma \in T$ and $a, c \in B$ are such that $\alpha a \geqq_B \gamma c$, then $a = c$.*

(ii) *If $v >_B 0$, then there exist $0 < \beta \in T$ and $b \in B$ such that $v \geqq_B \beta b$.*

(iii) *If $b \in B$ and $0 < \beta \in T$, then βb is an ℓ-basic element of $(L, +, \geqq_B)$.*

(iv) *If $0 <_B v \in L$ and $v = \beta_1 b_1 + \ldots + \beta_n b_n$, where $\beta_i \in T$ and the b_i are distinct elements of B, then the $\beta_i b_i$ are pairwise disjoint and $v = \beta_1 b_1 \vee \ldots \vee \beta_n b_n$.*

Proof. (i) If $a \neq c$, then since $\alpha a + (-\gamma)c \geqq_B 0$, the definition of $\geqq_B$ implies that $-\gamma \geqslant 0$, a contradiction.

(ii) We have $v = \beta_1 b_1 + \ldots + \beta_n b_n$, where $\beta_i \geqslant 0$ for all i, and since $v \neq 0$, we may assume that $\beta_1 > 0$. Then $v - \beta_1 b_1 = \beta_2 b_2 + \ldots + \beta_n b_n \geqq_B 0$ and hence $v \geqq_B \beta_1 b_1$.

(iii) Suppose by way of contradiction that βb is not ℓ-basic. Then there exist $v, w \in L$ such that $\beta b \geqq_B v >_B 0$, $\beta b \geqq_B w >_B 0$, and $v \wedge w = 0$. By (ii), there must exist $0 < \gamma \in T$, $c \in B$, $0 < \delta \in T$, and $d \in B$ such that $v \geqq_B \gamma c$ and $w \geqq_B \delta d$. Then $\beta b \geqq_B \gamma c$, $\beta b \geqq_B \delta d$, and $\gamma c \wedge \delta d = 0$. Now by (i), $c = b = d$, and since T is totally ordered, we may assume that $\gamma \geqslant \delta$. Then $\gamma b \geqq_B \delta b$ and hence $0 = \gamma b \wedge \delta b = \delta b$. Since neither b nor δ is 0, this is a contradiction. We conclude that βb is ℓ-basic.

(iv) Since $v >_B 0$, we may assume that each $\beta_i > 0$. So by (iii), each $\beta_i b_i$ is ℓ-basic. Since the b_i are distinct, this observation, together with (i), implies that the $\beta_i b_i$ are pairwise incomparable, and thus, since the $\beta_i b_i$ are ℓ-basic, that they are in fact pairwise disjoint. It follows from [2, p. 70] that $v = \beta_1 b_1 \vee \ldots \vee \beta_n b_n$. $\square$

Proposition 5.2. *For the ℓ-field $(L, +, \cdot, \geqq_B)$ described above, $1 \geqq_B 0$ if and only if $B \cap T \neq \emptyset$.*

Proof. Suppose first that $1 \geqq_B 0$. Then 1 may be written as a linear combination of elements of B, and hence by Lemma 5.1 (iv), 1 may be written as the disjoint join $1 = \omega_1 b_1 \vee \ldots \vee \omega_n b_n$ for $0 < \omega_i \in T$ and $b_i \in B$. But by [4, Proposition 1.4], 1 is ℓ-basic, and hence $n = 1$, i.e., $1 = \omega_1 b_1$. Then $b_1 = {\omega_1}^{-1} \in T$, and thus $B \cap T \neq \emptyset$.

Conversely suppose that $b \in B \cap T$. Then since $b \in B$, $b >_B 0$ and hence $b^2 >_B 0$. But if $b < 0$ in T, then $-b > 0$ in T and hence $-b^2 = (-b)b >_B 0$ by definition of $\geqq_B$. This is a contradiction and thus $b > 0$ in T. But then $b^{-1} > 0$ in T and hence $1 = (b^{-1})b \geqq_B 0$. □

So there are many ℓ-fields with Wilson orders $\geqq_B$ for which $1 \ngeqq_B 0$. For instance, the following v-bases of $\mathbb{Q}(\sqrt{3})$ determine Wilson orders on $\mathbb{Q}(\sqrt{3})$ for which $1 \ngeqq_B 0$:

$$B = \{1 - \sqrt{3},\ 3 - \sqrt{3}\}, \quad B = \{1 + 2\sqrt{3},\ 1 + \sqrt{3}\};$$

and the following v-bases of $\mathbb{Q}(\sqrt[3]{3})$ determine Wilson orders on $\mathbb{Q}(\sqrt[3]{3})$ for which $1 \ngeqq_B 0$:

$$B = \{1 + \sqrt[3]{3} + \sqrt[3]{9},\ 2 + 2\sqrt[3]{3} + \sqrt[3]{9},\ 3 + \sqrt[3]{3} + \sqrt[3]{9}\},$$
$$B = \{1 + \sqrt[3]{3} + \sqrt[3]{9},\ 3 + 2\sqrt[3]{3} + \sqrt[3]{9},\ 2 + \sqrt[3]{3} + \sqrt[3]{9}\}.$$

We want to show that, for any ℓ-field L whose order is a Wilson order, $P_3(L)$ is totally ordered. It then follows from Theorem 4.4 that L contains a unique maximal subfield, viz. $M_3(L)$, having the properties (1), (2) and (3) given in the introduction. In fact, in this case, as we will show, $M_3(L) = T$. We first prove two lemmas which are true for ℓ-fields in general.

Lemma 5.3. *Let $(L, +, \cdot, \geqslant)$ be any ℓ-field. If $e \in P_3(L)$ and b is an ℓ-basic element of L, then eb is also an ℓ-basic element of L and either $eb \geqslant b$ or $eb \leqslant b$.*

Proof. Suppose that $x, y \leqslant eb$ and let $z = x \wedge y$. Then $(x - z) \wedge (y - z) = 0$, and thus since b is ℓ-basic, either $b \wedge (x - z) = 0$ or $b \wedge (y - z) = 0$. But then since $e \in P_3(L)$, either $x - z = eb \wedge (x - z) = 0$ or $y - z = eb \wedge (y - z) = 0$. It follows that either $x \wedge y = z = x$ or $x \wedge y = z = y$, i.e., that either $x \leqslant y$ or $x \geqslant y$, and therefore, eb is ℓ-basic. If eb is not comparable to b, then, since both elements are ℓ-basic, $eb \wedge b = 0$. But by Proposition 4.3, $e^{-1} \in P_3(L)$ and hence by definition of $P_3(L)$, $b \wedge b = ((e^{-1})(eb)) \wedge b = 0$, a contradiction. We conclude that eb is comparable to b. □

Lemma 5.4. *Let $(L, +, \cdot, \geqslant)$ be any ℓ-field. If $e \in P_3(L)$, then either for all ℓ-basic elements b of L, $eb \geqslant b$ or for all ℓ-basic elements b of L, $eb \leqslant b$.*

P r o o f. Suppose that b and c are ℓ-basic elements of L. By Lemma 5.3, eb is comparable to b and ec is comparable to c. If $eb \geqslant b$, then $ebc \geqslant bc$. But if also $ec < c$, then $ecb < cb$, a contradiction. So $ec \geqslant c$ as well. Similarly, if $eb \leqslant b$, $ec \leqslant c$. $\square$

Proposition 5.5. *For the ℓ-field $(L, +, \cdot, \geqq_B)$ described above, $P_3(L)$ is a totally ordered subset of $(L, \succcurlyeq_B)$, and therefore*

(i) *$M_3(L)$ is the unique maximal o-subfield of $(L, +, \cdot, \succcurlyeq_B)$;*
(ii) *$M_3(L)$ is the unique maximal subfield K of $(L, +, \cdot)$ such that for all $0 < a \in L$, aK is a totally ordered subgroup of $(L, +, \geqq_B)$;*
(iii) *$\mathscr{M}_1(L) = \mathscr{M}_2(L) = \{M_3(L)\}$.*

P r o o f. Suppose that $d, e \in P_3(L)$ and let $r = de^{-1}$. Note that by Propositions 4.3 and 4.1, $r \in P_3(L)$. We will show that r is comparable to 1 in $(L, \succcurlyeq_B)$, and hence that d is comparable to e in $(L, \succcurlyeq_B)$. Note that by Lemma 5.4, either $rb \geqq_B b$ for all ℓ-basic b, or $rb \leqq_B b$ for all ℓ-basic b. Suppose first that $rb \geqq_B b$ for all ℓ-basic b, and suppose that $0 <_B a \in L$. Then by Lemma 5.1 (iv), $a = a_1 \vee \ldots \vee a_n$ for disjoint ℓ-basic elements $a_1, \ldots, a_n$, and by Lemma 4.2,

$$ar = a_1 r \vee \ldots \vee a_n r \geqslant a_1 \vee \ldots \vee a_n = a = a \cdot 1$$

so that $a(r-1) \geqq_B 0$. It follows that $r - 1 \in \overline{P}(L)$, i.e., that $r - 1 \succcurlyeq_B 0$. A similar argument shows that in the other case, when $rb \leqq_B b$ for all ℓ-basic b, $1 - r \succcurlyeq_B 0$, and hence that r is comparable to 1. We conclude that d is comparable to e and hence that $P_3(L)$ is totally ordered. That (i), (ii) and (iii) hold then follows from Theorem 4.4. $\square$

To conclude this section, we show that for any ℓ-field L with a Wilson order $\geqq_B$, $M_3(L) = T$.

Proposition 5.6. *For the ℓ-field $(L, +, \cdot, \geqq_B)$ described above, $T^{\geqslant} = P_3(L)$ and hence $M_3(L) = T$.*

P r o o f. Suppose $0 < \tau \in T$ and $0 <_B a \in L$. Then $a = \alpha_1 a_1 + \ldots + \alpha_n a_n$, where $0 < \alpha_i \in T$ and the a_i are distinct elements of B, and $\tau a = \tau\alpha_1 a_1 + \ldots + \tau\alpha_n a_n$. Since $0 < \tau\alpha_i \in T$ for all i, $\tau a >_B 0$. So $\tau \in P_3(L)$, and hence, since clearly $0 \in P_3(L)$, $T^{\geqslant} \subseteq P_3(L)$. Conversely suppose that $0 \neq e \in P_3(L)$ and let $b \in B$. Then $eb >_B 0$ and hence $eb = \beta_1 b_1 + \ldots + \beta_n b_n$, where $0 < \beta_i \in T$ and the b_i are distinct elements of B. By Lemma 5.3, eb is ℓ-basic, and by Lemma 5.1, the $\beta_i b_i$ are pairwise disjoint ℓ-basic elements. So $eb = \beta_1 b_1$. Since $(T, +, \cdot, \geqslant)$ is totally ordered, $\beta_1 - 1 \geqslant 0$ or $1 - \beta_1 \geqslant 0$. In the former case, $(\beta_1 - 1)b_1 \geqq_B 0$, and in the latter case,

$(\beta_1 - 1)b_1 \geqq_B 0$; so in both cases, $\beta_1 b_1$ is comparable to b_1. But by Lemma 5.3, eb is comparable to b. So since eb, $\beta_1 b_1$, b, and b_1 are all ℓ-basic, b is comparable to b_1, and hence by Lemma 5.1, $b_1 = b$. But then $e = \beta_1 b_1 b^{-1} = \beta_1 \in T^{\geqslant}$. Thus, since $0 \in T^{\geqslant}$, $T^{\geqslant} \supseteq P_3(L)$, so that in fact $T^{\geqslant} = P_3(L)$. Then

$$M_3(L) = \{x - y \mid x, y \in P_3(L)\} = \{x - y \mid x, y \in T^{\geqslant}\} = T.$$

□

6. Changing the multiplication

In section 5, we noted that Wilson orders can be used to construct ℓ-fields with $1 \not> 0$. In this section, we consider another way of constructing such ℓ-fields, viz., by changing the definition of the multiplication. We will show that in this case as well, when $1 > 0$ in the original ℓ-field, $M_3(L)$ is a subfield, and hence that such ℓ-fields also have a unique maximal subfield satisfying conditions (1), (2) and (3).

Specifically suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field and that $u \in L$ is such that $L^{\geqslant} L^{\geqslant} \subseteq uL^{\geqslant}$. Then, as observed in [3], $(L, +, \otimes_u, \geqslant)$ is an ℓ-field, where $\otimes_u$ is defined by letting $x \otimes_u y = xyu^{-1}$. It is easy to see that the multiplicative identity of $(L, +, \otimes_u, \geqslant)$ is u and hence we have the following

Proposition 6.1. *Suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field and that $u \in L$ is such that $L^{\geqslant} L^{\geqslant} \subseteq uL^{\geqslant}$, and form the ℓ-field $(L, +, \otimes_u, \geqslant)$. Then the multiplicative identity of $(L, +, \otimes_u)$ is positive if and only if $u > 0$.*

We first show that changing the multiplication translates one set $P_3(L)$ into another.

Proposition 6.2. *Suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field and that $u \in L$ is such that $L^{\geqslant} L^{\geqslant} \subseteq uL^{\geqslant}$, and form the ℓ-field $(L, +, \otimes_u, \geqslant)$. Then $P_3(L, +, \otimes_u, \geqslant) = uP_3(L, +, \cdot, \geqslant)$.*

P r o o f. Suppose that $e \in P_3(L, +, \cdot, \geqslant)$ and that $a \wedge b = 0$ in $(L, \geqslant)$. Then

$$(ue \otimes_u a) \wedge b = (ueau^{-1}) \wedge b = ea \wedge b = 0,$$

and hence $ue \in P_3(L, +, \otimes_u, \geqslant)$. So $P_3(L, +, \otimes_u, \geqslant) \supseteq uP_3(L, +, \cdot, \geqslant)$. Conversely suppose that $z \in P_3(L, +, \otimes_u, \geqslant)$ and again that $a \wedge b = 0$ in $(L, \geqslant)$. Then

$$((u^{-1}z)(a)) \wedge b = (z \otimes_u a) \wedge b = 0,$$

and hence $u^{-1}z \in P_3(L, +, \cdot, \geqslant)$. So $P_3(L, +, \otimes_u, \geqslant) \subseteq uP_3(L, +, \cdot, \geqslant)$. □

Note that if $1 > 0$ in $(L, +, \cdot, \geqslant)$, then by the remarks at the beginning of section 2, Proposition 6.2 says that $P_3(L, +, \otimes_u, \geqslant)$ is a translation of the positive cone of the totally ordered field $M(L, +, \cdot, \geqslant)$.

We next show that if $(L, +, \otimes_u, \geqslant)$ is derived from an ℓ-field which has a unique maximal subfield with properties (1), (2) and (3), then $(L, +, \otimes_u, \geqslant)$ also contains such a subfield, and hence in particular, that if $1 > 0$ in $(L, +, \cdot, \geqslant)$, then $(L, +, \otimes_u, \geqslant)$ always contains such a subfield.

Proposition 6.3. *Suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field and that $u \in L$ is such that $L^{\geqslant} L^{\geqslant} \subseteq uL^{\geqslant}$, and form the ℓ-field $(L, +, \otimes_u, \geqslant)$. If $P_3(L, +, \cdot, \geqslant)$ is totally ordered with respect to the partial order determined by $\overline{P}(L, +, \cdot, \geqslant)$, then $P_3(L, +, \otimes_u, \geqslant)$ is totally ordered with respect to the partial order determined by $\overline{P}(L, +, \otimes_u, \geqslant)$. And therefore*

(i) *$M_3(L, +, \otimes_u, \geqslant)$ is the unique maximal subfield of the field $(L, +, \otimes_u)$ that is totally ordered with respect to the order determined by $\overline{P}(L, +, \otimes_u, \geqslant)$;*
(ii) *$M_3(L, +, \otimes_u, \geqslant)$ is the unique maximal subfield K of the field $(L, +, \otimes_u)$ such that for all $0 < a \in L$, $a \otimes_u K$ is a totally ordered subgroup of $(L, +, \geqslant)$;*
(iii) *$\mathscr{M}_1(L, +, \otimes_u, \geqslant) = \mathscr{M}_2(L, +, \otimes_u, \geqslant) = \{M_3(L, +, \otimes_u, \geqslant)\}$.*

Proof. Let $s, t \in P_3(L, +, \otimes_u, \geqslant)$. We must show that s is comparable to t with respect to the partial order determined by $\overline{P}(L, +, \otimes_u, \geqslant)$. By Proposition 6.2, $u^{-1}s, u^{-1}t \in P_3(L, +, \cdot, \geqslant)$, and by Theorem 4.4, $M_3(L, +, \cdot, \geqslant)$ is an o-subfield of $(L, +, \cdot)$ with respect to the partial order on L determined by $\overline{P}(L, +, \cdot, \geqslant)$. And therefore

$$M_3(L, +, \cdot, \geqslant) = P_3(L, +, \cdot, \geqslant) \cup (-P_3(L, +, \cdot, \geqslant)).$$

So either $u^{-1}s - u^{-1}t$ or $u^{-1}t - u^{-1}s$ is in $P_3(L, +, \cdot, \geqslant)$, and thus either $s - t$ or $t - s$ is in $uP_3(L, +, \cdot, \geqslant) = P_3(L, +, \otimes_u, \geqslant)$. So since $P_3(L, +, \otimes_u, \geqslant) \subseteq \overline{P}(L, +, \otimes_u, \geqslant)$, either $s - t$ or $t - s$ is in $\overline{P}(L, +, \otimes_u, \geqslant)$. It follows that s is comparable to t with respect to the partial order on L determined by $\overline{P}(L, +, \otimes_u, \geqslant)$. That (i), (ii) and (i) hold then follows from Theorem 4.4. □

Corollary 6.4. *Suppose that $(L, +, \cdot, \geqslant)$ is an ℓ-field in which $1 > 0$ and that $u \in L$ is such that $L^{\geqslant} L^{\geqslant} \subseteq uL^{\geqslant}$ and form the ℓ-field $(L, +, \otimes_u, \geqslant)$. Then $P_3(L, +, \otimes_u, \geqslant)$ is totally ordered with respect to the order on L determined by $\overline{P}(L, +, \otimes_u, \geqslant)$, and hence $M_3(L, +, \otimes_u, \geqslant)$ is a subfield of $(L, +, \otimes_u)$ having properties* (i) *and* (ii) *of Proposition 6.3. As well, $M_3(L, +, \otimes_u, \geqslant) = uM(L, +, \cdot, \geqslant)$.*

P r o o f. As noted in the introduction and at the beginning of section 2, if $1 > 0$ in $(L, +, \cdot, \geqslant)$, then

$$\begin{aligned}
&P_3(L, +, \cdot, \geqslant) = M(L, +, \cdot, \geqslant)^{\geqslant},\\
&M(L, +, \cdot, \geqslant) \text{ is an } o\text{-subfield of } (L, +, \cdot, \geqslant), \text{ and}\\
&\overline{P}(L, +, \cdot, \geqslant) = (L, +, \cdot, \geqslant)^{\geqslant}.
\end{aligned}$$

So $P_3(L, +, \cdot, \geqslant)$ is totally ordered with respect to the order on L determined by $\overline{P}(L, +, \cdot, \geqslant)$ and hence by Proposition 6.3, $P_3(L, +, \otimes_u, \geqslant)$ is totally ordered with respect to the partial order on L with positive cone $\overline{P}(L, +, \otimes_u, \geqslant)$ and $M_3(L, +, \otimes_u, \geqslant)$ is a subfield of $(L, +, \otimes_u)$ having properties (i) and (ii) of Proposition 6.3. As well, since $P_3(L, +, \cdot, \geqslant) = M(L, +, \cdot, \geqslant)^{\geqslant}$, $M(L, +, \cdot, \geqslant) = M_3(L, +, \cdot, \geqslant)$, and thus by Proposition 6.2,

$$M_3(L, +, \otimes_u, \geqslant) = uM_3(L, +, \cdot, \geqslant) = uM(L, +, \cdot, \geqslant).$$

$\square$

We note finally that the technique described above may be used to construct an ℓ-field which does not have a Wilson order.

Example 6.5. Consider the field ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$ with pointwise order; this is an ℓ-field in which $1 > 0$ and for which $M({}_X\prod_{\mathbb{Z}}\mathbb{Q}) = \{f \mid f_n = 0 \text{ if } n \neq 0\}$ (cf. [5]). If the order on ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$ were a Wilson order determined by a v-basis B and a subfield T, then according to Proposition 5.6, T would have to be $M({}_X\prod_{\mathbb{Z}}\mathbb{Q})$, and according to Lemma 5.1 (iii), each element of B would have to ℓ-basic. However, it is easy to see that the subspace of ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$ over $M({}_X\prod_{\mathbb{Z}}\mathbb{Q})$ generated by the ℓ-basic elements consists of the vectors with finite support and hence cannot be all of ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$. So we conclude that the pointwise order on ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$ is not a Wilson order. Nonetheless, the ℓ-field ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$ still has a unique maximal subfield satisfying conditions (1), (2) and (3) given in the introduction. And by Corollary 6.4, so does any ℓ-field that is created by changing the multiplication on ${}_X\prod_{\mathbb{Z}}\mathbb{Q}$ by the method described in this section.

7. Questions

All known examples of ℓ-fields have orders derived from Wilson orders or from orders that are constructed by changing the multiplication on an ℓ-field in which $1 > 0$. In general, the answer to the following question is unknown.

Question 7.1. Is there an ℓ-field whose order is not a Wilson order and which cannot be constructed from an ℓ-field in which $1 > 0$ by a change of multiplication?

As well, all ℓ-fields with Wilson orders have a v-basis of ℓ-basic elements, and all known ℓ-fields with $1 > 0$ have the property that every positive element exceeds an ℓ-basic element. Thus the following question also remains unanswered.

Question 7.2. Does there exist an ℓ-field with a positive element which does not exceed an ℓ-basic element?

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