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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 1, 139–141

Persistent URL: `http://dml.cz/dmlcz/127633`

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THE DIMENSION FUNCTION OF HOLOMORPHIC SPACES OF A REAL SUBMANIFOLD OF AN ALMOST COMPLEX MANIFOLD

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(Received December 2, 1997)

Abstract. Let M be a real submanifold of an almost complex manifold $(\overline{M}, \overline{J})$ and let $H_x = T_xM \cap \overline{J}(T_xM)$ be the maximal holomorphic subspace, for each $x \in M$. We prove that $c\colon M \to \mathbb{N}$, $c(x) = \dim_{\mathbb{R}} H_x$ is upper-semicontinuous.

Keywords: holomorphic space, submanifold, almost complex

MSC 2000: primary 53C40, secondary 53C15

Let $(\overline{M}, \overline{J})$ be an almost complex manifold and let M be a real submanifold of $\overline{M}$. Then one can consider, $\forall x \in M$, the maximal holomorphic subspaces $H_x = T_xM \cap \overline{J}(T_xM)$ and define the dimension function $c\colon M \to \mathbb{N}$, $c(x) = \dim_{\mathbb{R}} H_x$. Submanifolds verifying that the function c is constant are called *generic submanifolds* (cf. [C]). As is well known, this condition is equivalent to the following: the maximal holomorphic subspaces H_x define a distribution over M. Many results have been found for generic submanifolds and particular cases, such as almost complex, totally real, Cauchy-Riemann, etc.

However, no global properties of this function have been proved. The purpose of this note is to prove that the function c is upper-semicontinuous, i.e., the sets $A_n = \{x \in M / c(x) < n\}$ are open subsets of M.

One can easily prove the following properties, for all $x \in M$: (a) $c(x)$ is an even number and (b) $2\dim M - \dim \overline{M} \leqslant c(x) \leqslant \dim M$. In particular, if M is a real hypersurface of $\overline{M}$, then c is the constant function $c(x) = \dim M - 2$, $\forall x \in M$.

On the other hand, observe that $(M, \overline{J}\big|_M)$ is a complex submanifold of $(\overline{M}, \overline{J})$ iff $c(x) = \dim M$, $\forall x \in M$. (See [K-N, prop. IX.2.3].)

The work of the first author is partially supported by Spanish Grant DGICYT PB95-0124.

Let M be a real submanifold of an almost complex manifold $(\overline{M}, \overline{J})$. We shall use the following notation: $G_r(T_xM)$ (resp. $G_r(T_x\overline{M})$) is the Grassmann manifold of r-planes of T_xM (resp. of $T_x\overline{M}$) and $\pi\colon\ G_r(M) \to M$ and $\overline{\pi}\colon\ \overline{G}_r(M) \to M$ denote the Grassmann bundles over M with fibres $\pi^{-1}(x) = G_r(T_xM)$ and $\overline{\pi}^{-1}(x) = G_r(T_x\overline{M})$, $\forall x \in M$. Observe that the almost complex structure $\overline{J}$ on $\overline{M}$ induces a continuous fibred automorphism $\overline{J}\colon\ \overline{G}_r(M) \to \overline{G}_r(M)$ which is involutive. i.e., $\overline{J} \circ \overline{J} = \mathrm{id}$.

We can prove the following

Lemma. *Let M be a real submanifold of an almost complex manifold $(\overline{M}, \overline{J})$ and let $x \in M$ and $r \in \mathbb{N}$ be such that $c(x) < r \leqslant \dim M$. Then $G_r(T_xM) \cap \overline{J}(G_r(T_xM)) = \emptyset$.*

Proof. As $r > c(x)$, then $\overline{J}(W_r) \neq W_r$, $\forall W_r \in G_r(T_xM)$. Let us assume that there exists $W_r \in G_r(T_xM) \cap \overline{J}(G_r(T_xM))$. Then $W_r = \overline{J}(W_r')$ with $W_r' \in G_r(T_xM)$, and one can easily prove that $W_r + W_r'$ is a $\overline{J}$-invariant subspace of T_xM with $\dim(W_r + W_r') \geqslant r > c(x)$, which is not possible, thus proving the lemma. □

Then we have:

Theorem. *If M is a real submanifold of an almost complex manifold $(\overline{M}, \overline{J})$, then the function $c\colon\ M \to \mathbb{N}$, $c(x) = \dim_{\mathbb{R}} H_x$ is upper-semicontinuous.*

Proof. We shall prove that the sets $A_n = \{x \in M / c(x) < n\}$ are open subsets of M showing that $\forall x \in M$, there exists a neighborhood U of x in M such that $\forall y \in U$, $c(y) \leqslant c(x)$.

If $c(x) = \dim M$, the result is trivial. Let us assume that $c(x) < \dim M$. We shall prove that for every $r \in \mathbb{N}$ such that $c(x) < r \leqslant \dim M$, there exists a neighborhood U_r of x in M, verifying $c(y) < r$, $\forall y \in U_r$. Then the solution will be the neighborhood $U = \bigcap\{U_r, c(x) < r \leqslant \dim M\}$.

Let us consider $x \in M$ and $r \in \mathbb{N}$ such that $c(x) < r \leqslant \dim M$. Then, by the lemma, one has $G_r(T_xM) \cap \overline{J}(G_r(T_xM)) = \emptyset$. As $G_r(T_xM)$ is compact and $\overline{G}_r(M)$ is a Hausdorff space, there exist two neighborhoods $\overline{V}$ of $G_r(T_xM)$ and $\overline{W}$ of $\overline{J}(G_r(T_xM))$ such that $\overline{V} \cap \overline{W} = \emptyset$. Then $\overline{N} = \overline{V} \cap \overline{J}(\overline{W})$ is a neighborhood of $G_r(T_xM)$ verifying $\overline{N} \cap \overline{J}(\overline{N}) = \emptyset$.

Let us consider $N = \overline{N} \cap G_r(M)$. Then N is a neighborhood of $G_r(T_xM)$ in $G_r(M)$. One can consider that N is included in $\pi^{-1}(U')$, where U' is a trivialization neighborhood of x, and then there exists a diffeomorphism

$$\varphi\colon\ \pi^{-1}(U') \to U' \times G_r(\mathbb{R}^{\dim M})$$

given by $\varphi(W_r) = (\pi(W_r), \alpha(W_r))$.

For each $W_r \in G_r(T_xM)$ there exists a basic neighborhood $\varphi^{-1}(A^{W_r} \times B^{W_r})$, where A^{W_r} (resp. B^{W_r}) is a neighborhood of $\pi(W_r)$ (resp. $\alpha(W_r)$) and $\varphi^{-1}(A^{W_r} \times B^{W_r}) \subset N$, thus defining an open covering of $G_r(T_xM)$. Taking into account that $G_r(T_xM)$ is compact, one obtains a finite family $\{\varphi^{-1}(A^{W_r^i} \times B^{W_r^i}), 1 \leqslant i \leqslant k\}$ covering $G_r(T_xM)$.

Finally, let $U_r = \bigcap\{A^{W_r^i}, 1 \leqslant i \leqslant k\}$. Then $\pi^{-1}(U_r) \subset N$ and $\pi^{-1}(U_r) \cap \overline{J}(\pi^{-1}(U_r)) = \emptyset$, thus proving that $c(y) < r$, $\forall y \in U_r$. $\square$

Remarks. (1) Let M be a real compact submanifold of the complex euclidean space $\mathbb{C}^m$. As is well known (cf. [W, p. 11]), M cannot be immersed into $\mathbb{C}^m$ as a holomorphic submanifold. Therefore, there exists a point $x \in M$ such that $c(x) < \dim M$. The above theorem shows that $A_{\dim M}$ is a non-empty open subset of M.

(2) The theorem is also true if one replaces the almost complex manifold $(\overline{M}, \overline{J})$ by an almost product manifold $(\overline{M}, \overline{J})$, i.e., if $\overline{J} \circ \overline{J} = \mathrm{id}$.

References

[C] *B. Y. Chen*: Geometry of Submanifolds and its Applications. Sci. Univ. Tokyo, 1981.

[K-N] *S. Kobayashi and K. Nomizu*: Foundations of Differential Geometry, II. Interscience, New York, 1969.

[W] *R. O. Wells, Jr.*: Differential Analysis on Complex Manifolds. Springer, New York, 1980.

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