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VARIATIONAL MEASURES IN THE THEORY OF THE INTEGRATION IN $\mathbb{R}^m$

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Abstract. We study properties of variational measures associated with certain conditionally convergent integrals in $\mathbb{R}^m$. In particular we give a full descriptive characterization of these integrals.

Keywords: variational measures and derivates of set functions, Riemann generalized integrals

MSC 2000: 28A15, 26A39, 26A45

1. Introduction

It is known (see for example [15]) that on the real line $\mathbb{R}$ a continuous function F of bounded variation is absolutely continuous if and only if the Lebesgue-Stieltjes measure generated by F is absolutely continuous with respect to the Lebesgue measure μ. Then on $\mathbb{R}$ the class of all Lebesgue primitives coincides with the class of all functions whose Lebesgue-Stieltjes measure is absolutely continuous with respect to μ.

An extension of this result to the Henstock-Kurzweil integral in $\mathbb{R}$ was proved in [4], by using a variational measure associated to this integral.

In the present paper we consider certain conditionally convergent integrals in $\mathbb{R}^m$ and we associate to each of them a variational measure constructed by means of the derivation base used to define the integral. We study the properties of these measures (see Theorems 1, 2, 3 and 7) and we apply them to characterize the primitives of the integrals constructed by means of Kurzweil's and Kempisty's bases; we also characterize the primitives of Mawhin's integral (see Theorems 4, 5, 6). Moreover in

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Theorem 5, in Theorem 7 and in Corollary 4 we improve the descriptive characterization of the conditionally convergent integrals considered by Kurzweil and Jarník in [6] and in [7].

2. Preliminaries

We recall some definitions and notations. Our ambient space is $\mathbb{R}^m$, where m is a fixed positive integer. In $\mathbb{R}^m$ we shall use the norm $\|x\| = \max\limits_{1\leqslant i\leqslant m} |x_i|$, where $x = (x_1, x_2, \ldots, x_m)$. Then the δ-neighbourhood of x, denoted by $U(x,\delta)$, is an open cube centered at x with side equal to 2δ. For a set $E \subset \mathbb{R}^m$ we denote by E^0, ∂E, and $|E|$ the interior, the boundary and the m-dimensional Lebesgue measure of E, respectively. Moreover for $x \in \mathbb{R}^m$, $d(x,E)$ denotes the distance of x from the set E. A set E with $|E| = 0$ is called *negligible*. The words "almost everywhere" (shortly a.e.) and "absolutely continuous" are always refered to m-dimensional Lebesgue measure. An *interval* is always a nondegenerate compact subinterval of $\mathbb{R}^m$. Throughout this paper Δ denotes a fixed interval and $\mathcal{I}$ the family of all subintervals of Δ. For $I \in \mathcal{I}$, $I = [a_1,b_1] \times [a_2,b_2] \times \ldots \times [a_m,b_m]$ with $a_i < b_i$, $i = 1, 2, \ldots, m$, we put $d(I) = \max\limits_i (b_i - a_i)$ and $r(I) = \min\limits_i (b_i - a_i)/d(I)$. The numbers $d(I)$ and $r(I)$ are called the *diameter* and the *regularity* of I respectively. Let $\alpha \in (0,1)$; if $r(I) > \alpha$ we say that the interval I is *α-regular*.

A *derivation base* (or simply a *base*) on Δ is, by definition, a nonempty subset $\mathcal{B}$ of $\mathcal{I} \times \Delta$. Given a base $\mathcal{B}$, an interval I is called a *$\mathcal{B}$-interval* if $(I,x) \in \mathcal{B}$, for some $x \in \Delta$. For a set $E \subset \Delta$ we write

$$\mathcal{B}(E) = \{(I,x) \in \mathcal{B}\colon\ I \subset E\} \text{ and } \mathcal{B}[E] = \{(I,x) \in \mathcal{B}\colon\ x \in E\}. \tag{1}$$

If $\delta(x)$ is a positive function defined on Δ we denote

$$\mathcal{B}_\delta = \{(I,x) \in \mathcal{B}\colon\ I \subset U(x,\delta(x))\}. \tag{2}$$

Note that $\mathcal{B}_\delta$ is also a base on Δ. So the meaning of $\mathcal{B}_\delta(E)$ and $\mathcal{B}_\delta[E]$ is clear from (1) and (2).

All functions in this paper are real valued. A function F defined on $\mathcal{I}$ is said to be *additive* if $F(I \cup J) = F(I) + F(J)$, for each nonoverlapping intervals I and J in $\mathcal{I}$, with $I \cup J \in \mathcal{I}$.

Let $\mathcal{B}$ be a base such that for any $\delta > 0$ and for any $x \in \Delta$ the set $\mathcal{B}_\delta[\{x\}]$ is nonempty. The *lower derivate* of F at a point $x \in \Delta$ with respect to $\mathcal{B}$ is defined as

$$\underline{D}_{\mathcal{B}}F(x) = \sup\nolimits_\delta \inf \left\{ \frac{F(I)}{|I|}\colon\ (I,x) \in \mathcal{B}_\delta[\{x\}] \right\}.$$

The *upper derivate* of F at a point $x \in \Delta$ with respect to $\mathcal{B}$ is defined as

$$\overline{D}_{\mathcal{B}}F(x) = \inf_{\delta}\sup\left\{\frac{F(I)}{|I|}:\ (I,x) \in \mathcal{B}_{\delta}[\{x\}]\right\}.$$

If $\underline{D}_{\mathcal{B}}F(x) = \overline{D}_{\mathcal{B}}F(x) \neq \infty$ we say that F is $\mathcal{B}$*-derivable* at x and we set $D_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x) = \overline{D}_{\mathcal{B}}F(x)$; $D_{\mathcal{B}}F(x)$ is called the $\mathcal{B}$-derivate of F at x.

A $\mathcal{B}$*-partition* is a finite collection $\pi = \{(I_1, x_1), \ldots, (I_p, x_p)\}$ where I_i are nonoverlapping $\mathcal{B}$-intervals and $x_i \in \Delta$, for $i = 1, \ldots, p$.

Given $E \subset \Delta$, a positive function δ on E is said to be a *gauge* on E. We say that a partition π is

- *anchored in* E if $\{x_1, \ldots, x_p\} \subset E$,
- *a partition in* E if $\bigcup_{i=1}^{p} I_i \subset E$,
- *a partition of* E if $\bigcup_{i=1}^{p} I_i = E$,
- δ*-fine* if $I_i \subset U(x_i, \delta(x_i))$, for $i = 1, \ldots, p$.

We say that a base $\mathcal{B}$ is:

- a *fine base* on a set $E \subset \Delta$, if for any $x \in E$ and for any $\delta > 0$ the set $\mathcal{B}_{\delta}[\{x\}]$ is nonempty;
- a *Perron base* if, for any couple (I, x) of $\mathcal{B}$, $x \in I$.

We say that a base $\mathcal{B}$ fulfils:

property i): if, for any $I \in \mathcal{I}$ and for any gauge δ on Δ, there exists a δ-fine $\mathcal{B}$-partition of I (*partitioning property*);

property ii): if for any $\mathcal{B}^* \subset \mathcal{B}$, fine on a set $E \subset \Delta$, there exists a disjoint sequence $B_1, B_2, \ldots$ of sets from $\mathcal{B}^*$ such that $|E \setminus \left(\bigcup_i B_i\right)| = 0$ (*Vitali property*);

property iii): if for each $\mathcal{B}$-interval J and for each $x \in J$ we have $(J, x) \in \mathcal{B}$;

property iv): if, for each additive function F on $\mathcal{I}$, $D_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x) \neq \infty$ ($D_{\mathcal{B}}F(x) = \overline{D}_{\mathcal{B}}F(x) \neq \infty$) holds at almost all points x at which $\underline{D}_{\mathcal{B}}F(x) > -\infty$, ($\overline{D}_{\mathcal{B}}F(x) < +\infty$) (*Ward property*).

3. Variational measure associated with a function Henstock integrable with respect to a given base $\mathcal{B}$.

Let $\mathcal{B}$ be a base on Δ satisfying the partitioning property.

Definition 1. A function f on Δ is called Henstock integrable with respect to $\mathcal{B}$ (br. $\mathcal{B}H$-integrable) whenever there exists an additive function F on $\mathcal{I}$ satisfying the following condition:

given $\varepsilon > 0$, we can find a gauge $\delta(x)$ so that

$$(3) \qquad \sum_{(x,I)\in\pi} |f(x)|I| - F(I)| < \varepsilon$$

for any δ-fine $\mathcal{B}$-partition π in Δ.

The equivalence of Definition 1 and the definition given in [10] follows immediately from Henstock's Lemma (see [10], Theorem 1.6.1). The function F is uniquely determined by f and we call it the *indefinite $\mathcal{B}$-integral of f* in Δ.

In order to study the differentiation properties of an interval function F and to investigate whether F is an indefinite $\mathcal{B}H$-integral, we introduce the notion of variation of F associated with the $\mathcal{B}H$-integral. It is the multidimensional extension of the notion of the fine variational measure given in [16] in the case of an interval function defined on the real line.

Let F be an additive function on $\mathcal{I}$ and let $E \subset \Delta$ be an arbitrary set. For a fixed gauge δ on E we set

$$(4) \qquad V_{\mathcal{B}}(F,\delta,E) = \sup_\pi \sum_i |F(I_i)|\,,$$

where $\pi = \{(I_1,x_1),\ldots,(I_p,x_p)\}$ is a δ-fine $\mathcal{B}$-partition anchored in E.

The *$\mathcal{B}$-variation of* F on E is defined as follows

$$(5) \qquad V_{\mathcal{B}}F(E) = \inf V_{\mathcal{B}}(F,\delta,E),$$

where the "inf" is taken over all gauges δ on E.

By an argument similar to that used in [16, Theorem 3.7], it is easy to see that the set function $V_{\mathcal{B}}F(\cdot)$ is a Borel metric outer measure in Δ, called the *$\mathcal{B}H$-variational measure generated by F*.

In case the base $\mathcal{B}$ is the family $\mathcal{L} = \mathcal{I} \times \Delta$, the corresponding integral is the *McShane integral* that it is known to be equivalent to the Lebesgue integral (see [9] and [11]). Note that if $(I,x) \in \mathcal{L}$, the point x does not need belong to I. The following proposition is, in a certain sense, a multidimensional extension of a consequence of De La Vallée Poussin's Decomposition Theorem (see [15, p. 128]). A similar result was obtained in [5] by using a variation defined by means of Perron figures partitions. We recall that an additive function F on $\mathcal{I}$ is said to be *absolutely continuous* (br. *AC*) if given $\varepsilon > 0$, there is $\eta > 0$ such that

$$(6) \qquad \sum_{I\in P} |F(I)| < \varepsilon$$

for any collection P of nonoverlapping intervals in $\mathcal{I}$ with $\sum_{I\in P} |I| < \eta$.

Proposition 1. *An additive function F on $\mathcal{I}$ is AC if and only if $E \to V_{\mathcal{L}}F(E)$ is absolutely continuous.*

P r o o f. Let F be AC and let N be a negligible subset of Δ. Fix $\varepsilon > 0$ and choose $\eta > 0$ so that (6) is fulfilled. Take an open set O with $O \supset N$ and $|O| < \eta$. For each $x \in N$ define $\delta(x) < d(x, \partial O)$. Therefore $\sum\limits_i |F(I_i)| < \varepsilon$ for each δ-fine $\mathcal{L}$-partition $\{(I_i, x_i)\colon\ i = 1, \ldots, p\}$ anchored in N. Then $V_{\mathcal{L}}F(N) \leqslant V_{\mathcal{L}}(F, \delta, N) \leqslant \varepsilon$ and, by the arbitrariness of ε, $V_{\mathcal{L}}F(N) = 0$.

Conversely, suppose that the variational measure $V_{\mathcal{L}}F$ is absolutely continuous. As $V_{\mathcal{L}}F(\{x\}) = 0$ for each $x \in \Delta$, we can find a positive constant $\gamma(x)$ on Δ such that $\sum\limits_i |F(B_i)| < 1$ for each γ-fine $\mathcal{L}$-partition $\{(B_i, x)\}$ anchored on $\{x\}$. Since Δ is compact, there exist $y_1, \ldots, y_n$ on Δ such that $\bigcup\limits_{i=1}^{n} U(y_i, \gamma(y_i)) \supset \Delta$. Now let δ be a gauge such that for each $x \in \Delta$ there exists an index $i = 1, \ldots, n$ such that $U(x, \delta(x)) \subset U(y_i, \gamma(y_i))$.

For each δ-fine $\mathcal{L}$-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ we obtain

$$\sum_{j=1}^{p} |F(A_j)| = \sum_{i=1}^{n} \sum_{(A_j, y_i) \in \pi_i} |F(A_j)| < n,$$

where π_i is the γ-fine $\mathcal{L}$-partition $\{(A_j, y_i)\colon\ A_j \subset U(y_i, \gamma(y_i))\}$. Therefore $V_{\mathcal{L}}F(\Delta) < +\infty$. Now fix $\varepsilon > 0$. By [14, Theorem 6.11] there exists $\eta > 0$ such that $V_{\mathcal{L}}F(E) < \varepsilon$ for each Borelian set $E \subset \Delta$ with $|E| < \eta$. Let $\{I_i\colon\ i = 1, \ldots, p\}$ be a collection of nonoverlapping intervals in $\mathcal{I}$ with $\sum\limits_{i=1}^{p} |I_i| < \eta$. Then there exists a gauge $\sigma(x)$ on $\bigcup\limits_{i=1}^{p} I_i$ such that $V_{\mathcal{L}}\Big(F, \sigma, \bigcup\limits_{i=1}^{p} I_i\Big) < \varepsilon$. Since the base $\mathcal{L}$ satisfies the partitioning property (see for example [11, Cousin's Lemma]), for each index i there exists a σ-fine $\mathcal{L}$-partition $\{(C_j^i, z_j^i)\}$ of I_i, $i = 1, \ldots, p$. Find one; then by the additivity of F, it follows that

$$\sum_i |F(I_i)| \leqslant \sum_i \sum_j |F(C_j^i)| \leqslant V_{\mathcal{L}}\Big(F, \sigma, \bigcup_i I_i\Big) < \varepsilon.$$

Thus the function F is AC. $\square$

Corollary 1. *An additive function F on $\mathcal{I}$ is the indefinite Lebesgue integral of a given function f on Δ if and only if the measure $V_{\mathcal{L}}F$ is absolutely continuous.*

Now we use the $\mathcal{B}$-variational measures introduced above to find similar characterizations for the $\mathcal{B}H$-integrals.

Proposition 2. *Let f be $\mathcal{B}H$-integrable on Δ and let F be its indefinite $\mathcal{B}H$-integral. Then the measure $V_{\mathcal{B}}F$ is absolutely continuous.*

P r o o f. Let N be a negligible set in Δ. For each $n = 1, 2, \ldots$, put $N_n = \{x \in N \colon n - 1 < |f(x)| \leqslant n\}$.

Choose an $\varepsilon > 0$ and find an open set O_n such that $N_n \subset O_n$ and $|O_n| < \varepsilon 2^{-n} n^{-1}$. Let δ_0 be a gauge on Δ such that

$$\sum_{i=1}^{p} |f(x_i)|I_i| - F(I_i)| < \varepsilon,$$

for each δ_0-fine $\mathcal{B}$-partition $\{(I_i, x_i) \colon i = 1, \ldots, p\}$ in Δ.

Define a gauge δ on N by setting $\delta(x) = \min(\delta_0(x), d(x, \partial O_n))$ if $x \in N_n$ and choose an arbitrary δ-fine $\mathcal{B}$-partition $\{(J_i, y_i) \colon i = 1, \ldots, s\}$ anchored in N. Then we have

$$\begin{aligned}\sum_{i=1}^{s} |F(J_i)| &\leqslant \sum_{i=1}^{s} \bigl|F(J_i) - f(y_i)|J_i|\bigr| + \sum_{i=1}^{s} |f(y_i)||J_i| \\ &< \varepsilon + \sum_{n} \sum_{y_i \in N_n} |f(y_i)||J_i| < \varepsilon + \varepsilon \sum_{n=1}^{\infty} 2^{-n} = 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, we infer $V_{\mathcal{B}}F(N) = 0$. □

Proposition 3. *Let F be an additive function on $\mathcal{I}$. If $V_{\mathcal{B}}F$ is absolutely continuous and if F is a.e. $\mathcal{B}$-derivable on Δ, then F is the indefinite $\mathcal{B}H$-integral of $D_{\mathcal{B}}F(x)$.*

P r o o f. Let us denote by N the negligible set of all $x \in \Delta$ at which F is not $\mathcal{B}$-derivable and define $f(x) = D_{\mathcal{B}}F(x)$ if $x \in \Delta \setminus N$, $f(x) = 0$ if $x \in N$.

We are proving that F is the indefinite $\mathcal{B}H$-integral of f. Fix $\varepsilon > 0$ and find a gauge δ_1 on N so that $\sum\limits_{i=1}^{p} |F(I_i)| < \varepsilon$ for each δ_1-fine $\mathcal{B}$-partition $\{(I_i, x_i) \colon i = 1, \ldots, p\}$ anchored in N. For each $x \in \Delta \setminus N$ there is a positive number $\delta_2(x)$ such that

$$|f(x)|J| - F(J)| < \frac{\varepsilon |J|}{|\Delta|},$$

for each $\mathcal{B}$-interval J with $(J, x) \in \mathcal{B}$ and $J \subset U(x, \delta_2(x))$. Now define a gauge δ on Δ in the following way: $\delta(x) = \delta_1(x)$ if $x \in N$, $\delta(x) = \delta_2(x)$ if $x \in \Delta \setminus N$.

Then, for each δ-fine $\mathcal{B}$-partition $\{(I_i, x_i) \colon i = 1, \ldots, p\}$ we have

$$\sum_{i=1}^{p} |f(x_i)|I_i| - F(I_i)| < \sum_{x_i \in N} |F(I_i)| + \frac{\varepsilon}{|\Delta|} \sum_{x_i \notin N} |I_i| < 2\varepsilon.$$

Thus f is $\mathcal{B}H$-integrable and F is its indefinite $\mathcal{B}H$-integral. □

The following result was proved in [4, Theorems 1 and 5] in the case of a variational measure associated with the Henstock-Kurzweil integral in $\mathbb{R}$ (see also [12, Proposition 11]).

Theorem 1. *Let F be an additive function on $\mathcal{I}$ and assume that $\mathcal{B}$ is a Perron base possessing property* iii). *If $V_{\mathcal{B}}F$ is absolutely continuous, then it is σ-finite.*

P r o o f. Suppose that $V_{\mathcal{B}}F$ is not σ-finite. Let U be the union of all open sets O such that $V_{\mathcal{B}}F(\Delta \cap O) < +\infty$. By the Lindelöf Theorem, the measure $V_{\mathcal{B}}F$ is σ-finite on $\Delta \cap U$. Define $P = \Delta \setminus U$. The set P is compact, nonempty and

$$V_{\mathcal{B}}F(P \cap O)) = +\infty, \tag{7}$$

for each open set O with $P \cap O \neq \emptyset$. Moreover, as $V_{\mathcal{B}}F$ is absolutely continuous, $V_{\mathcal{B}}F(\{x\}) = 0$ for each $x \in \Delta$. Thus P is perfect.

Let us denote by $\hat{P}$ the set of all points $z \in P$ possessing the following property: there exists at least one $\mathcal{I}$-interval J such that $z \in \partial J$ and $J^0 \cap P = \emptyset$. For each $z \in \hat{P}$ select an $\mathcal{I}$-interval $J(z)$ with the above property. Put $A = \bigcup\limits_{z \in \hat{P}} (J(z))^0$. A is an open set disjoint with P and $\hat{P} \subset \partial A$. Since ∂A is negligible, also $\hat{P}$ is negligible. Therefore, as $V_{\mathcal{B}}F$ is absolutely continuous, from (7) we obtain

$$V_{\mathcal{B}}F(O \cap (P \setminus \hat{P})) = +\infty$$

for each open set O with $P \cap O \neq \emptyset$.

Now we will construct a set $N \subset P$ with $|N| = 0$ and $V_{\mathcal{B}}(F, N) \geqslant 1$, thus obtaining a contradiction to the assumption of absolute continuity of $V_{\mathcal{B}}F$.

Let I_0 be an interval whose interior meets P. Choose a gauge δ on $(I_0)^0$ such that $\delta(x) < d(x, \partial I_0)$ for each $x \in (I_0)^0$. As $V_{\mathcal{B}}F((I_0)^0 \cap (P \setminus \hat{P})) = +\infty$, we can find a δ-fine $\mathcal{B}$-partition $\pi = \{(I_i^{(1)}, y_i^{(1)}) : i = 1, \ldots, p\}$ anchored on $(I_0)^0 \cap (P \setminus \hat{P})$ such that

$$\sum_i |F(I_i^{(1)})| > 1 \text{ and } \sum_i |I_i^{(1)}| < 2^1.$$

Since $\mathcal{B}$ is a Perron base and $y_i^{(1)} \in P \setminus \hat{P}$, we have $(I_i^{(1)})^0 \cap P \neq \emptyset$. Moreover, as P is perfect we can assume $p > 1$.

We proceed by induction. If $\{I_i^{(k-1)}\}$, $k \geqslant 2$, is a finite collection of nonoverlapping $\mathcal{B}$-intervals with $(I_i^{(k-1)})^0 \cap P \neq \emptyset$, we can construct a new finite collection of nonoverlapping $\mathcal{B}$-intervals $\{I_i^{(k)}\}$, such that:

1) $P \cap (I_i^{(k)})^0 \neq \emptyset$;
2) each $I_i^{(k)}$ is contained in the interior of some $I_j^{(k-1)}$;

3) each $I_i^{(k-1)}$ contains at least two intervals $I_j^{(k)}$;

4) $\sum\limits_i |I_i^{(k)}| < k^{-1}$;

5) $\sum\limits_{i\colon I_i^{(k)} \subset I_j^{(k-1)}} |F(I_i^{(k)})| > 1$ for each j.

Put $N = \bigcap\limits_k \bigcup\limits_i I_i^{(k)}$. By conditions 1) – 4) it follows that the set N is a perfect subset of Δ and $|N| = 0$.

Choose a gauge $\delta(x)$ on N and, let

$$N_n = \{x \in N\colon\ \delta(x) > n^{-1}\} \quad \text{for } n = 1, 2, \ldots.$$

By Baire Category Theorem, there exists an index n such that N_n is dense in $N \cap J$, for some open interval J. There exist l and i such that $I_i^{(l)} \subset J$, $|I_i^{(l)}| < n^{-1}$. By property iii) the family $\{(I_j^{(l+1)}, z_j)\}$ with $z_j \in I_j^{(l+1)} \cap N_n$, $I_j^{(l+1)} \subset I_i^{(l)}$ is a δ-fine $\mathcal{B}$-partition anchored in N. Thus, by condition 5) we infer:

$$V_{\mathcal{B}}(F, \delta, N) \geqslant \sum_{j\colon I_j^{(l+1)} \subset I_i^{(l)}} |F(I_j^{(l+1)})| > 1.$$

Therefore $\mathrm{V}_F(N) \geqslant 1$, giving the desired contradiction. □

4. Essential variation and derivates

Let $E \subset \Delta$. An *essential gauge on* E is a non-negative function δ defined on E and positive a.e. The definition of δ-fine partition in the case of an essential gauge instead of a gauge coincides with the definition given in Section 2. For the δ-variation defined by this kind of partitions we keep the notation given by formula (4). If in the definition given by formula (5) "inf" is taken over all essential gauges on E we denote by $V_{\mathcal{B}}^{\text{ess}}F$ the resulting *essential variational measure generated by* F. The inequality

$$V_{\mathcal{B}}^{\text{ess}}F(E) \leqslant V_{\mathcal{B}}F(E) \quad \text{for each } E \subset \Delta \tag{8}$$

follows from the definition.

A more general version of the next theorem was proved in [1] in the case of an abstract measure space with finite measure and a derivation base satisfying the Vitali property.

Theorem 2. *Let $\mathcal{B}$ be a fine base on Δ possessing property* ii) *and let F be an additive function on $\mathcal{I}$. Then for any measurable set $E \subset \Delta$, we have*

$$V_{\mathcal{B}}^{\mathrm{ess}} F(E) = \int_E \overline{D}_{\mathcal{B}} |F|(x) \, \mathrm{d}x, \tag{9}$$

where the integral is the usual Lebesgue integral in $\mathbb{R}^m$.

Remark 1. If $\mathcal{B}$ is only a fine base, instead of the equality (9) we have (see [1])

$$V_{\mathcal{B}}^{\mathrm{ess}} F(E) \leqslant \int_E \overline{D}_{\mathcal{B}} |F|(x) \, \mathrm{d}x.$$

Theorem 3. *Let $\mathcal{B}$ be a fine base on Δ and let F be an additive function on $\mathcal{I}$ such that $V_{\mathcal{B}} F$ is absolutely continuous. Then*

$$V_{\mathcal{B}}^{\mathrm{ess}} F(E) = V_{\mathcal{B}} F(E)$$

for any $E \subset \Delta$.

Proof. By (8) it is enough to prove $V_{\mathcal{B}}^{\mathrm{ess}} F(E) \geqslant V_{\mathcal{B}} F(E)$. Suppose, by contradiction, $V_{\mathcal{B}}^{\mathrm{ess}} F(E) < V_{\mathcal{B}} F(E)$. Then there exists an essential gauge δ_1 and a positive number η such that $V_{\mathcal{B}}^{\mathrm{ess}}(F, \delta_1, E) + \eta < V_{\mathcal{B}} F(E)$. Put $N_{\delta_1} = \{x \in \Delta \colon \delta_1(x) = 0\}$. Since $V_{\mathcal{B}} F$ is absolutely continuous, then there exists a gauge δ_2 on N_{δ_1} such that $V_{\mathcal{B}}(F, \delta_2, N_{\delta_1}) < \eta$. Define:

$$\delta(x) = \begin{cases} \delta_1(x), & \text{if } x \in E \setminus N_{\delta_1}, \\ \delta_2(x), & \text{if } x \in N_{\delta_1}. \end{cases}$$

Thus we have

$$\begin{aligned} V_{\mathcal{B}}(F, \delta, E) &\leqslant V_{\mathcal{B}}^{\mathrm{ess}}(F, \delta_1, E) + V_{\mathcal{B}}(F, \delta_2, N_{\delta_1}) \\ &< V_{\mathcal{B}}^{\mathrm{ess}}(F, \delta_1, E) + \eta < V_{\mathcal{B}} F(E), \end{aligned}$$

which gives the required contradiction. □

Corollary 2. *Let $\mathcal{B}$ be a fine base on Δ with properties* ii) *and* iv) *and let F be an additive function on $\mathcal{I}$ such that $V_{\mathcal{B}} F$ is absolutely continuous. Then F is a.e. $\mathcal{B}$-derivable on Δ if and only if $V_{\mathcal{B}} F$ is σ-finite.*

Proof. It follows immediately from Theorems 2 and 3. □

Corollary 3. *Let $\mathcal{B}$ be a fine Perron base possessing properties* ii)–iv), *and let F be an additive function on $\mathcal{I}$ such that $V_{\mathcal{B}}F$ is absolutely continuous. Then F is a.e. $\mathcal{B}$-derivable on Δ and $|D_{\mathcal{B}}F|$ is the Radon-Nikodym derivate of $V_{\mathcal{B}}F$ with respect to Lebesgue measure.*

Proof. It follows immediately from Theorem 1 and Corollary 2. □

5. Applications

We apply the previous results to some non absolutely convergent integrals.

Definition 2. (see [10]) A derivation base is called the *Kurzweil base* and is denoted by $\mathcal{B}_1$, if $(I, x) \in \mathcal{B}_1$ for each interval $I \in \mathcal{I}$ and for each $x \in I$.

Definition 3. (see [10]) Let $\alpha \in (0, 1)$. A derivation base is called the *Kempisty α-base* and is denoted by $\mathcal{B}_2^\alpha$, if $(I, x) \in \mathcal{B}_2^\alpha$ for each interval $I \in \mathcal{I}$, with $r(I) > \alpha$, and for each $x \in I$.

The bases $\mathcal{B}_1$ and $\mathcal{B}_2^\alpha$ are fine Perron bases possessing properties i) and iii).

The $\mathcal{B}_1 H$-integral is known as the classical Henstock integral. Generally the indefinite $\mathcal{B}_1 H$-integral is not $\mathcal{B}_1$-differentiable (see [10]). Moreover, as the $\mathcal{B}_1$-base does not satisfy Vitali property, Theorem 2 fails. By Propositions 2 and 3 we deduce the following partial descriptive characterization of the $\mathcal{B}_1$-integral

Theorem 4. *Let F be an additive function on $\mathcal{I}$ a.e. $\mathcal{B}_1$-derivable on Δ. Then F is the indefinite $\mathcal{B}_1 H$-integral of $D_{\mathcal{B}_1}F$ if and only if $V_{\mathcal{B}_1}F$ is absolutely continuous.*

For each $\alpha \in (0, 1)$, the $\mathcal{B}_2^\alpha$ base possesses properties ii) and iv) (see [15]). Then, by Propositions 2 and 3, by Theorem 1 and by Corollary 2 we infer the following full descriptive characterization of the $\mathcal{B}_2^\alpha H$-integral.

Theorem 5. *Let $\alpha \in (0, 1)$ and let F be an additive function on $\mathcal{I}$. Then $D_{\mathcal{B}_2^\alpha}F$ exists a.e. on Δ and F is its indefinite $\mathcal{B}_2^\alpha H$-integral if and only if $V_{\mathcal{B}_2^\alpha}F$ is absolutely continuous.*

Remark 2. The previous Theorem improves the descriptive characterization given by Kurzweil and Jarník (see [7, Theorem 3]) in which the a.e. existence of $D_{\mathcal{B}_2^\alpha}F$ is required as an additional condition. In [7] the statement "F is α-variationally normal on I" is used instead of "$V_{\mathcal{B}_2^\alpha}F$ is absolutely continuous on I".

Now we consider the integral introduced in [8] by Mawhin (see also [15]).

Definition 4. Let $f\colon \Delta \to \mathbb{R}$. We say that f is integrable in the sense of Mawhin (shortly M-integrable) on Δ whenever there is an additive function F on $\mathcal{I}$ satisfying the following condition:

for every $\varepsilon > 0$ and every $\alpha \in (0,1)$ we can find a gauge $\delta(x)$ such that

$$\sum_{(x,I)\in\pi} |f(x)|I| - F(I)| < \varepsilon$$

for any δ-fine $\mathcal{B}_2^\alpha$-partition π in Δ.

The function F is uniquely determined and is called the *indefinite M-integral of f* in Δ. The equivalence of the previous definition and the original definition of Mawhin follows immediately from [10, Theorem 1.6.1]. If f is M-integrable then it is $\mathcal{B}_2^\alpha H$-integrable for each $\alpha \in (0,1)$ and both the indefinite integrals are equal. Concerning the above definition Kurzweil and Jarník proved in [7, Theorem 1 and Theorem 2] that, while the value of α is irrelevant in the definition of $\mathcal{B}_2^\alpha$-derivate (i.e. if an additive function F is $\mathcal{B}_2^\alpha$-derivable at a point x, then, for any $\beta \in (0,1)$, it is $\mathcal{B}_2^\beta$-derivable at x), the value of α is essential in the definition of the $\mathcal{B}_2^\alpha H$-integral.

Let F be an additive function on $\mathcal{I}$ and let $E \subset A$ be an arbitrary set. We define the *$\mathcal{M}$-variation of* F on E as follows:

$$V_{\mathcal{M}}F(E) = \sup_{\alpha\in(0,1)} V_{\mathcal{B}_2^\alpha}F(E).$$

As each $V_{\mathcal{B}_2^\alpha}F$ is an outer metric measure, it is easy to show that also $V_{\mathcal{M}}F$ is an outer metric measure in Δ. The notion of derivate associated with the M-integral is the classical notion of ordinary derivate (see [15]). According to [7, Theorem 1], we say that the function F is *derivable in the ordinary sense* at $x \in \Delta$ if $D_{\mathcal{B}_2^\alpha}F(x)$ exists for some $\alpha \in (0,1)$. Then we put $F'(x) = D_{\mathcal{B}_2^\alpha}F(x)$ and call it the ordinary derivate of F at x.

Theorem 6. *Let F be an additive function on $\mathcal{I}$. Then F' exists a.e. on Δ and F is its indefinite M-integral if and only if $V_{\mathcal{M}}F$ is absolutely continuous.*

Proof. Suppose $V_{\mathcal{M}}F$ to be absolutely continuous. Since for any $\alpha \in (0,1)$ $V_{\mathcal{B}_2^\alpha}F \leqslant V_{\mathcal{M}}F$, then also $V_{\mathcal{B}_2^\alpha}F$ is absolutely continuous. Hence, by Theorem 5, $D_{\mathcal{B}_2^\alpha}F$ exists a.e. on Δ. Thus F' exists a.e. on Δ and F is its indefinite M-integral.

Conversely, assume F is the indefinite-$\mathcal{M}$-integral of F'. Therefore F is the indefinite $\mathcal{B}_2^\alpha$-integral of F' for any $\alpha \in (0,1)$. Thus, by Theorem 5, we obtain the absolute continuity of each variational measure $V_{\mathcal{B}_2^\alpha}$, $\alpha \in (0,1)$. Then also $V_{\mathcal{M}}F$ is absolutely continuous. $\square$

6. The ϱ-integral of Jarník-Kurzweil

We recall the definition of the ϱ-integral introduced in [6].

Let ϱ be a fixed real function $\varrho\colon \Delta \times (0, +\infty) \to [0, 1]$ satisfying the following conditions:

ϱ_1) $\varrho(x,t) < 1$ for $x \in \Delta$, $t > 0$;

ϱ_2) $\limsup_{t \to 0^+} \varrho(x,t) < 1$ for $x \in \Delta$;

ϱ_3) $\liminf_{t \to 0^+} \varrho(x,t) > 0$ for $x \in \Delta$.

A *ϱ-partition* is a $\mathcal{B}_1$-partition $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ with the property $r(I_i) > \varrho(x_i, d(I_i))$ for $i = 1, \ldots, p$.

Definition 5. A function f on Δ is called ϱ-integrable on Δ whenever there is an additive function F on $\mathcal{I}$ satisfying the following condition:

given $\varepsilon > 0$, we can find a gauge $\delta(x)$ such that

$$\sum_{(x,I) \in \pi} |f(x)|I| - F(I)| < \varepsilon$$

for any δ-fine ϱ-partition π in Δ.

Therefore the ϱ-integral is the $\mathcal{B}H$-integral with respect to the base $\mathcal{B}_\varrho = \{(I, x) : x \in I \in \mathcal{I} \text{ and } r(I) > \varrho(x, d(I))\}$. We note that by [6, Lemma 1.8] the above definition is equivalent to that given in [6].

For simplicity in the following we write respectively $V_\varrho(F, \delta, E)$ and $V_\varrho F(E)$, instead of $V_{\mathcal{B}_\varrho}(F, \varrho, E)$ and $V_{\mathcal{B}_\varrho} F(E)$, for the $\mathcal{B}_\varrho$-variation of a function F on a set E.

We observe that, without any other hypotheses on ϱ, the base $\mathcal{B}_\varrho$ generally does not verify property iii). Moreover, even the Vitali property is not satisfied in the standard way. So we cannot apply Theorem 1 and Theorem 2 to $\mathcal{B}_\varrho F$.

Nonetheless, by using the previous conditions concerning ϱ, it is not difficult to prove a $\mathcal{B}_\varrho$-version of Theorem 2 (see [3, Theorem 1]).

In [6], on the hypotheses $\varrho_1) - \varrho_3)$, the indefinite ϱ-integrals are characterized as additive interval functions which are derivable in the ordinary sense a.e. in Δ and whose associated ϱ-variations are absolutely continuous on the set of all points of nonderivability.

It is easy to prove, by using condition ϱ_3), that if an additive function F is derivable in the ordinary sense a each point of a set E, then $V_\varrho F$ is absolutely continuous on E. Therefore the next theorem improves the descriptive characterization of the ϱ-integral given in [6, Theorem 3.2]. For simplicity, we will write $\overline{D}_\alpha F$ instead of $\overline{D}_{\mathcal{B}_2^\alpha} F$.

Theorem 7. *Let F be an additive function on $\mathcal{I}$. If $V_\varrho F$ is absolutely continuous, then F' exists a.e. on Δ.*

For the proof we need the following Lemma, proved in [2, Lemma 1].

Lemma 8. *Let $P \subset \Delta$ be a measurable set with $0 < |P|$ and let $\mathcal{B}$ be a fine Perron base on Δ possessing the Vitali property. Then, for every $0 < \tau < 1$ and every $0 < \lambda < \sigma < 1$, there are finitely many disjoint $\mathcal{B}$-intervals $C_1, \dots, C_p$ such that:*

(a) $|P \cap C_j| > \tau |C_j|$ *for* $j = 1, \dots, p$;

(b) $\lambda |P| < \left| \bigcup_{j=1}^{p} C_j \right| < \sigma |P|$.

P r o o f of Theorem 7. Assume, by contradiction, that F' does not exists a.e. on Δ. Let α_k, $k = 0, 1, \dots$, be an increasing sequence of positive numbers converging to 1. For each $k = 0, 1, \dots$ put $A_k = \{x \in \Delta \colon \overline{D}_{\alpha_k} F(x) = +\infty\}$ and $B = \bigcap_k A_k$. The set B is measurable and, by [15, Theorem 11.15 of Chapter IV] and [7, Theorem 1], it results $|B| > 0$.

For $k = 0, 1, \dots$ choose $\eta_k > 1$ such that

$$\prod_{k=0}^{\infty} \eta_k^3 < 2. \tag{10}$$

Choose also a decreasing sequence d_k, $k = 0, 1, \dots$, of real numbers converging to 0.

By induction we can construct a sequence of $\mathbf{\Sigma}_k \subset \mathbb{N}^{k+1}$ of indices and a family $\{J_\sigma\}$, $\sigma \in \mathbf{\Sigma}_k$, of disjoint $\mathcal{I}$-intervals such that

1_k) $\mathbf{\Sigma}_k$ is finite for each $k \in \mathbb{N}$;

2_k) each interval J_σ, $\sigma \in \mathbf{\Sigma}_k$ is α_k-regular;

3_k) for each $\tau \in \mathbf{\Sigma}_{k+1}$ there are $\sigma \in \mathbf{\Sigma}_k$ and $j = 1, \dots, p_\sigma$ such that $\tau = (\sigma, j)$ and $J_\tau \subset J_\sigma$;

4_k) $d(J_\sigma) < d_k$ and $|J_{\sigma,j}| < 2^{-2}\eta_k^2 |B \cap J_\sigma|$ for $\sigma \in \mathbf{\Sigma}_k$ and $k \in \mathbb{N}$;

5_k) $|J_\sigma \cap B| > \eta_k^{-2} |J_\sigma|$ for each $\sigma \in \mathbf{\Sigma}_k$ and $k = 1, 2, \dots$;

6_k) $2^{-1}\eta_k^2 |B \cap J_\sigma| < \left| \bigcup_j J_{\sigma,j} \right| < 2^{-1}\eta_k^3 |B \cap J_\sigma|$ for $\sigma \in \mathbf{\Sigma}_k$ and $k \in \mathbb{N}$;

7_k) $F(J_\sigma) > 2^{2k} |J_\sigma|$ for $\sigma \in \mathbf{\Sigma}_k$ and $k = 1, 2, \dots$

For $k = 0$ choose an α_0-regular $\mathcal{I}$-interval J_0 with $d(J_0) < d_0$ and $0 < |J_0 \cap B|$. Define $\mathbf{\Sigma}_0 = \{0\}$. For each $x \in B \cap J_0$ there exists a sequence $\{C_n(x)\}$ of α_1-regular $\mathcal{I}$-intervals such that $d(C_k(x)) \to 0$ as $k \to \infty$ and $F(C_k(x)) > 2^2 |C_k(x)|$.

The base $\mathcal{B}_2^{\alpha_1}$ has the Vitali property. So we apply Lemma 8 to $J_0 \cap B$ and find a finite family $\{J_{0,1}, \dots, J_{0,p_0}\}$ of disjoint α_1-regular $\mathcal{I}$-subintervals of J_0 such that, if $\mathbf{\Sigma}_1 \equiv \{(0,1), \dots, (0, p_0)\}$, conditions 5_1), 6_0) and 7_1) and the second part of 4_0) hold true. Moreover, 1_0), 2_1) 3_0) and the first part of 4_0) obviously hold.

Assume $\mathbf{\Sigma}_k$ and J_σ, $\sigma \in \mathbf{\Sigma}_k$, have been defined. By condition 5_k) we have $|J_\sigma \cap B| > 0$. So we can apply Lemma 8 to each set $J_\sigma \cap B$ and find disjoint α_{k+1}-regular intervals $J_{\sigma,j} \subset J_\sigma$, $j = 1, \ldots, p_\sigma$, such that $d(J_{\sigma,j}) < d_{k+1}$ and conditions $5_{k+1}), 6_k), 7_{k+1})$ and the second part of 4_k) hold true. Define $\mathbf{\Sigma}_{k+1}$ as the set of all indices $(\sigma,j) \in \mathbb{N}^{k+2}$ with $\sigma \in \mathbf{\Sigma}_k$ and $j = 1, \ldots, p_\sigma$.

Clearly $N = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in \mathbf{\Sigma}_k} J_\sigma$ is a compact set. Since by 4_k) and 6_k) we deduce $p_\sigma > 1$ for each $\sigma \in \mathbf{\Sigma}_k$, the set N is also perfect. Moreover, by 6_k) and by (10) we have

$$\begin{aligned}\Big|\bigcup_{\sigma\in\mathbf{\Sigma}_{k+1}} J_\sigma\Big| &= \Big|\bigcup_{\sigma\in\mathbf{\Sigma}_k}\bigcup_j J_{\sigma,j}\Big| \\ &< 2^{-1}\,\eta_k^3\Big|\bigcup_{\sigma\in\mathbf{\Sigma}_k} J_\sigma\Big| \\ &< 2^{-k-1}\,\eta_k^3\ldots\eta_0^3\,|J_0\cap B| < 2^{-k}|J_0\cap B|.\end{aligned}$$

Thus $|N| = 0$. We will prove $V_\varrho F(N) = +\infty$, which gives a contradiction to the assumption of absolute continuity of $V_\varrho F$.

By condition ϱ_2), for each $x \in N$ choose $0 < \nu(x) < 1$ and $0 < t_0(x)$ such that

$$\varrho(x,t) < \nu(x) \quad \text{for each } 0 < t < t_0(x). \tag{11}$$

Take any gauge $\delta(x)$ defined on N and for n, s, q positive integers, put

$$N_{n,s,q} = \left\{x \in N\colon\ \delta(x) > \frac{1}{n},\ t_0(x) > \frac{1}{s} \text{ and } \nu(x) < 1 - \frac{1}{q}\right\}.$$

Obviously $N = \bigcup_n\bigcup_s\bigcup_q N_{n,s,q}$. By the Baire Category Theorem, $N_{n,s,q}$ is dense in some portion of N, defined by an interval J, for some n, s, q. We may select $k \in \mathbb{N}$ and $\sigma \in \mathbf{\Sigma}_k$ such that $J_\sigma \subset J$, $d_k < \min(\frac{1}{n}, \frac{1}{s})$ and $\alpha_k > 1 - \frac{1}{q}$.

For $l \geqslant 1$ define $\mathbf{\Sigma}^*_{k+l} = \{\sigma' \in \mathbf{\Sigma}_{k+l}\colon \sigma' = (\sigma, \sigma'')\}$. For each $\sigma' \in \mathbf{\Sigma}^*_{k+l}$ let $x_{\sigma'} \in J_{\sigma'} \cap N_{n,s,q}$. By (11), by 2_{k+l}) and by the definition of $N_{n,s,q}$ we infer

$$\varrho(x_{\sigma'}, d(J_{\sigma'})) < \nu(x_{\sigma'}) < 1 - \frac{1}{q} < \alpha_k < \alpha_{k+l} < r(J_{\sigma'}).$$

Therefore $\{(J_{\sigma'}, x_{\sigma'})\colon\ \sigma' \in \mathbf{\Sigma}^*_{k+l}\}$ is a δ-fine ϱ-partition anchored in N.

Then by applying subsequently $7_k), 6_k) and 5_k)$ we obtain

$$
\begin{aligned}
V_\varrho(F,\delta,N) \geqslant & \sum_{\sigma' \in \mathbf{\Sigma}^*_{k+l}} F(J_{\sigma'}) \\
& > 2^{2(k+l)} \Big| \bigcup_{\sigma' \in \mathbf{\Sigma}^*_{k+l}} J_{\sigma'} \Big| \\
& = 2^{2(k+l)} \sum_{\sigma, j_1 \in \mathbf{\Sigma}^*_{k+1}} \cdots \sum_{\sigma, j_1, \dots, j_{l-1} \in \mathbf{\Sigma}^*_{k+l-1}} \Big| \bigcup_{j_l} J_{\sigma, j_1, \dots, j_l} \Big| \\
& > 2^{2(k+l)} 2^{-1} \eta^2_{k+l-1} \sum_{\sigma, j_1 \in \mathbf{\Sigma}^*_{k+1}} \cdots \sum_{\sigma, j_1, \dots, j_{l-1} \in \mathbf{\Sigma}^*_{k+l-1}} |B \cap J_{\sigma, j_1, \dots, j_{l-1}}| \\
& > 2^{2(k+l)} 2^{-1} \sum_{\sigma, j_1 \in \mathbf{\Sigma}^*_{k+1}} \cdots \sum_{\sigma, j_1, \dots, j_{l-1} \in \mathbf{\Sigma}^*_{k+l-1}} |J_{\sigma, j_1, \dots, j_{l-1}}| > \dots \\
& > 2^{2(k+l)} 2^{-l} |J_\sigma| = 2^k 2^l |J_\sigma|.
\end{aligned}
$$

Since k and J_σ are fixed and l is arbitrary we conclude $V_\varrho(F,\delta,N) = +\infty$. Hence $V_\varrho F(N) = +\infty$. □

Corollary 4. *Let F be an additive function on $\mathcal{I}$. Then F' exists a.e. on Δ and F is its indefinite ϱ-integral if and only if $V_\varrho F$ is absolutely continuous.*

P r o o f. The proof follows immediately from [6, Theorem 3.2], and from Theorem 7. □

Note that Corollary 4 holds also in the case of the strong ϱ-integral introduced in [6], if we use the corresponding variation.

All the integrals considered here are defined by means of interval partitions associated with a gauge δ. But if the integrals are defined by means of figures partitions associated with a non negative function δ, which is null on a set of σ-finite $(m-1)$-dimensional Hausdorff measures, then it is necessary to use different ideas. The related results have been obtained independently and by different methods in [2] and in [5].

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