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ON VECTOR VALUED MEASURE SPACES OF BOUNDED Φ-VARIATION CONTAINING COPIES OF ℓ_∞

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Abstract. Given a Young function Φ, we study the existence of copies of c_0 and ℓ_∞ in $\mathrm{cabv}_\Phi(\mu, X)$ and in $\mathrm{cabsv}_\Phi(\mu, X)$, the countably additive, μ-continuous, and X-valued measure spaces of bounded Φ-variation and bounded Φ-semivariation, respectively.

1. Introduction

The interest in Lebesgue's and Bochner's integration theory in Analysis has been a powerful incentive in the study of the Young functions and the Orlicz spaces. In fact the Orlicz theory of measurable functions and measures appears in literature as a natural attempt to generalize the classical theory of vector measures and integration which was restricted to the L_p spaces, and also because of the characterization of the uniformly integrable sets in $L_1(\mu)$ given by de la Vallée Poussin in 1915 [1] in terms of Orlicz spaces. Again the classical Banach sequence spaces, especially the non reflexive ones, play a central role in the study of the Banach spaces. In this way, we present some results related to the existence of copies of c_0 and ℓ_∞ in Orlicz spaces of vector valued measures. This problem has been studied

(a) in [2] for $\mathrm{cabv}(\mu, X)$, the space of the countably additive, μ-continuous and X-valued measures of bounded variation endowed with the topology of the variation norm,
(b) in [4] for $\mathrm{ba}(\Sigma, X)$, the space of bounded X-valued vector measures and for $\mathrm{ca}(\Sigma, X)$, the space of countably additive and X-valued vector measures, both equipped with the semivariation norm.

Clearly this paper is a natural continuation of the results of a) and b).

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The notation is standard, see [6] and [8] for details.

A Young function is a convex function $\Phi\colon \mathbb{R} \to \mathbb{R}^+$ such that $\Phi(-x) = \Phi(x)$, $\Phi(0) = 0$ and $\lim\limits_{x\to\infty} \Phi(x) = \infty$.

From now on, (Ω, Σ, μ) will denote an atomless abstract finite measure space, where Σ is a σ-algebra on which μ is a σ-additive and nonnegative measure. For every Banach space X, $L_\Phi(\mu, X)$ is the space of classes of μ-measurable and X-valued functions $f\colon \Omega \to X$, such that there is a real constant $H > 0$ such that $\int_\Omega \Phi(H\|f(x)\|)\,\mathrm{d}\mu < \infty$ (with the identification of functions that coincide a.e.), which is a Banach space with the norm

$$NV_\Phi(f) := \inf\Bigl\{K > 0\colon \int_\Omega \Phi(\|f(x)\|/K)\,\mathrm{d}\mu \leqslant 1\Bigr\}.$$

For every convex function Φ on A, we say that $y = ax + b$ is a support line of Φ if $\Phi(x) \geqslant ax + b$, $\forall x \in A$. The properties of the Young functions imply the existence of support lines with $a > 0$ and $b \leqslant 0$. This fact can be used to prove that $L_\Phi(\mu, X)$ is continuously embedded in $L_1(\mu, X)$. We denote by $\chi(\mu, X)$ the set of step functions of $L_1(\mu, X)$.

Let F be a countably additive, X-valued and μ-continuous measure on (Ω, Σ, X). The Φ-variation of F, denoted by $I_\Phi(F)$, is defined by

$$I_\Phi(F) := \sup_\pi \Bigl\{\sum_n \Phi\Bigl(\frac{\|F(A_n)\|}{\mu(A_n)}\Bigr)\mu(A_n)\Bigr\}$$

where the supremum is taken over all partitions $\pi = \{A_n\}$ of Ω in Σ. If $I_\Phi(F) < \infty$, F is said to be of bounded Φ-variation.

We denote by $\mathrm{cabv}_\Phi(\mu, X)$ the space of μ-continuous countably additive and X-valued measures F such that there is a $K > 0$ with $I_\Phi(F/K) \leqslant 1$, which is a Banach space with the norm

$$NV_\Phi(F) := \inf\{K > 0\colon I_\Phi(F/K) \leqslant 1\}.$$

The space $L_\Phi(\mu, X)$ is an isometric subspace of $\mathrm{cabv}_\Phi(\mu, X)$ by the map $G\colon L_\Phi(\mu, X) \to \mathrm{cabv}_\Phi(\mu, X)$ such that $G(f)(E) = \int_E f\,\mathrm{d}\mu$, $\forall f \in L_\Phi(\mu, X)$ and for every $E \in \Sigma$, see [8].

If $x' \in X'$ and F is an X-valued measure, we denote by $x'F$ the scalar measure such that $x'F(E) = \langle x', F(E)\rangle$. The Φ-semivariation of F is

$$IS_\Phi(F) := \sup\{I_\Phi(x'F)\colon x' \in X',\ \|x'\| \leqslant 1\}.$$

If $IS_\Phi(F) < \infty$, then F is said to be of bounded Φ-semivariation. We denote by $\mathrm{cabsv}_\Phi(\mu, X)$ the Banach space of countably additive and μ-continuous X-valued measures F such that there is a $K > 0$ with $IS_\Phi(F/K) \leqslant 1$, endowed with the norm

$$NS_\Phi(F) := \inf\{K > 0\colon\ IS_\Phi(F/K) \leqslant 1\}.$$

It is clear that $\mathrm{cabv}_\Phi(\mu, X) \subset \mathrm{cabsv}_\Phi(\mu, X)$, with $NS_\Phi(F) \leqslant NV_\Phi(F)$ for every $F \in \mathrm{cabv}_\Phi(\mu, X)$. We denote by J the canonical injection of $\mathrm{cabv}_\Phi(\mu, X)$ into $\mathrm{cabsv}_\Phi(\mu, X)$.

Finally, we need the following result of Rosenthal:

Lemma 1 ([7] Proposition 1.2 and Remark 1)**.** *Let $T\colon \ell_\infty \to X$ be a linear and continuous map such that $\{\|T(e_n)\|\}$ does not converge to zero, where (e_n) is the unit vector sequence in ℓ_∞. Then there is an infinite subset D of $\mathbb{N}$ such that $T\big|_{\ell_\infty(D)}$ is an isomorphism.*

3. Main results

Theorem 1. *Let $\{f_n\}$ be a $\sigma\big(L_1(\mu), \chi(\mu)\big)$-null sequence in $L_1(\mu)$ with the following properties:*

(1) *$\exists M > 0$ such that $\mu(\{\omega \in \Omega\colon\ |f_n(\omega)| > M\}) = 0$, $\forall n \in \mathbb{N}$.*
(2) *$\exists B > 0$ and $\exists S > 0$ such that $\forall n \in \mathbb{N}$, $\exists A_n \in \Sigma$ with $\mu(A_n) \geqslant S$ and $f_n(\omega) \geqslant B$, $\forall \omega \in A_n$.*

Let Φ be a Young function such that $0 < \Phi(x) < \infty$ if $0 < x < \infty$ and $\exists x_0 > 0$: $\Phi(x_0) \leqslant 1/\mu(\Omega)$, and let X be a Banach space containing a copy of c_0. Then $\mathrm{cabv}_\Phi(\mu, X)$ and $\mathrm{cabsv}_\Phi(\mu, X)$ contain the respective subspaces S and S' isomorphic to ℓ_∞. Moreover, $S \cap L_\Phi(\mu, X)$ contains a subspace isomorphic to c_0.

P r o o f. It is enough to prove the theorem if $X = c_0$. For every $n \in \mathbb{N}$, let Λ_n be the scalar measure

$$\Lambda_n(E) = \langle f_n, \chi_E \rangle$$

for every $E \in \Sigma$. We define $G\colon \ell_\infty \to \mathrm{cabv}_\Phi(\mu, X)$ such that $G((\xi_i)) = F_{(\xi_i)}$, where $F_{(\xi_i)}(E) = (\xi_i \Lambda_i(E))$. If $(\xi_i) \neq 0$, it is clear that $(\xi_i \Lambda_i(E)) \in c_0$ for every $E \in \Sigma$, and $F_{(\xi_i)}$ is a countably additive and μ-continuous c_0-valued measure. For every partition $\{E_n\}$ of Ω contained in Σ and for every $K > 0$, we have

$$\sum_{n\in\mathbb{N}} \Phi\Big(\frac{\|F_{(\xi_i)}(E_n)\|_{c_0}}{K\mu(E_n)}\Big)\mu(E_n) \leqslant \Phi(M\|(\xi_i)\|_{\ell_\infty}/K)\mu(\Omega) < \infty.$$

Consequently, $F_{(\xi_i)}$ has bounded Φ-variation and

$$NV_\Phi(F_{(\xi_i)}) \leqslant \inf\{K > 0\colon\ \Phi(M\|(\xi_i)\|_{\ell_\infty}/K)\mu(\Omega) \leqslant 1\}.$$

If we take $K_0 > 0$ such that $\forall\varepsilon > 0$, $\Phi(M\|(\xi_i)\|_{\ell_\infty}/K_0)\mu(\Omega) \leqslant 1$ then

$$\Phi\big(M\|(\xi_i)\|_{\ell_\infty}/(K_0-\varepsilon)\big)\mu(\Omega) > 1 \geqslant \Phi(x_0)\mu(\Omega),$$

$M\|(\xi_i)\|_{\ell_\infty}/(K_0-\varepsilon) \geqslant x_0$ and $NV_\Phi(F_{(\xi_i)}) - \varepsilon \leqslant K_0 - \varepsilon \leqslant M\|(\xi_i)\|_{\ell_\infty}/x_0$, which implies that $NV_\Phi(F_{(\xi_i)}) \leqslant M\|(\xi_i)\|_{\ell_\infty}/x_0$ and hence G and JG are continuous. If $(\xi_i) = 0$, the conclusion follows directly.

If (e_n) is the unit basis in c_0, then $F_{e_n} \in L_\Phi(\mu, X)$ with Radon-Nikodym derivative $(0,\dots,0,f_n(.),0,\dots)$ for every $n \in \mathbb{N}$. Moreover, fix n, take the partition $\{E_k\}_{k=1}^2\colon$ $E_1 = A_n$, $E_2 = \Omega\setminus A_n$. For every $x = (x_i) \in \ell_1$ we have $|\langle x, F_{e_n}(E_1)\rangle| = |x_n\Lambda_n(E_1)| \geqslant |x_n|B\mu(E_1)$, and then

$$\sum_{k=1}^{2}\Phi\Big(\frac{|\langle x, F_{e_n}(E_k)\rangle|}{K\mu(E_k)}\Big)\mu(E_k) \geqslant \Phi(|x_n|B/K)\mu(E_1) \geqslant \Phi(|x_n|B/K)S.$$

Hence $I_\Phi(xF_{e_n}/K) \geqslant \Phi(|x_n|B/K)S$, therefore

$$NV_\Phi(xF_{e_n}) \geqslant \inf\{K > 0\colon\ \Phi(|x_n|B/K)S \leqslant 1\}.$$

For every support line $y = ax + b$ of Φ with $a > 0$, $b \leqslant 0$, taking $x = e_n$, we obtain

$$\begin{aligned}NS_\Phi(F_{e_n}) &\geqslant \inf\{K > 0\colon\ \Phi(B/K)S \leqslant 1\}\\ &\geqslant \inf\{K > 0\colon\ (aB/K + b)S \leqslant 1\} = aBS/(1-bS) > 0.\end{aligned}$$

Hence

$$\inf\{NV_\Phi(F_{e_n}),\ n\in\mathbb{N}\} \geqslant \inf\{NS_\Phi(F_{e_n}),\ n\in\mathbb{N}\} \geqslant aBS/(1-bS) > 0.$$

Then we use Lemma 1 to conclude that there are infinite subsets DV and DS of $\mathbb{N}$ such that $G\big|\ell_\infty(DV)$ and $JG\big|\ell_\infty(DS)$ are isomorphisms. □

Remarks.

1) Every Rademacherlike sequence in Ω, i.e., every orthogonal sequence $\{r_n\}$ such that $\mu(\{\omega\in\Omega\colon\ r_n(\omega) = 1\}) = \mu(\{\omega\in\Omega\colon\ r_n(\omega) = -1\}) = 1/2$ verifies the required condition. If μ is the Lebesgue measure in $[0,1]$, we also can take $f_n(\omega) = \sin(n\pi\omega)$.

2) Every continuous Young function such that $0 < \Phi(x) < \infty$ if $0 < x < \infty$ verifies the hypothesis of Theorem 1 for all finite measure spaces (Ω,Σ,μ).

3) A Young function satisfies $\Phi \in \Delta_2$ if $\exists H > 0$: $\forall x > 0$, $\Phi(2x) \leqslant H\Phi(x)$. Many properties of the space $L_\Phi(\mu, X)$ and $\mathrm{cabv}_\Phi(\mu, X)$ with $\Phi(x) = \|x\|^p$, $1 \leqslant p < \infty$, are fulfilled for $\Phi \in \Delta_2$ and the corresponding proofs are also valid in this setting. This happens mainly because if $\Phi \in \Delta_2$, the simple functions are dense in $L_\Phi(\mu, X)$. Moreover, if (Ω, Σ, μ) is separable, then $L_\Phi(\mu)$ is separable, [6]. For example, if (Ω, Σ, μ) is separable and $\Phi \in \Delta_2$ then

a) $L_\Phi(\mu, X)$ contains a copy of ℓ_∞ if and only if X does, Mendoza [5];

b) if X contains a copy of c_0, then $L_\Phi(\mu, X)$ contains a complemented copy of c_0, see Emmanuelle [3]. If moreover X contains no copies of ℓ_∞, a consequence of Theorem 1 is that $L_\Phi(\mu, X)$ is an uncomplemented subspace of $\mathrm{cabv}_\Phi(\mu, X)$, see Drewnowski and Emmanuelle [2].

Theorem 2. *Let Φ be a continuous Young function such that $\Phi(x) = 0$ iff $x = 0$. Then for every separable finite measure space (Ω, Σ, μ), the space $\mathrm{cabv}_\Phi(\mu, X)$ (or $\mathrm{cabsv}_\Phi(\mu, X)$) contains a copy of c_0 iff it contains a copy of ℓ_∞.*

P r o o f. We only prove the theorem for $\mathrm{cabv}_\Phi(\mu, X)$ (the proof in the case of $\mathrm{cabsv}_\Phi(\mu, X)$ is analogous). By virtue of Theorem 1 and the above remarks, it is enough to prove the statement if X contains no copies of c_0 and $\mathrm{cabv}_\Phi(\mu, X)$ contains a copy of c_0. Let $J\colon\ c_0 \to \mathrm{cabv}_\Phi(\mu, X)$ be an isomorphism. First of all we will see that $\sum\limits_{i=1}^{\infty} \xi_i J(e_i)(E) \in X$ for every $(\xi_i) \in \ell_\infty$ and for every $E \in \Sigma$. We know that the formal series $\sum\limits_{i=1}^{\infty} J(e_i)$ is weakly unconditionally Cauchy in $\mathrm{cabv}_\Phi(\mu, X)$. For every $E \in \Sigma$, we consider the map $H_E\colon\ \mathrm{cabv}_\Phi(\mu, X) \to X$ such that $H_E(F) = F(E)$ for every $F \in \mathrm{cabv}_\Phi(\mu, X)$. If $y = ax + b$ is a support line of Φ with $a > 0$, $b \leqslant 0$, for every $K > 0$ we have

$$I_\Phi(F/K) \geqslant \Phi\Big(\frac{\|F(E)\|}{K\mu(E)}\Big)\mu(E) \geqslant a\|F(E)\|/K + b\mu(E).$$

Then

$$N_\Phi(F) \geqslant \inf\{K > 0\colon\ a\|F(E)\|/K + b\mu(E) \leqslant 1\} = \frac{a}{1 - b\mu(E)}\|F(E)\|$$

therefore H_E is both continuous and weakly continuous. Hence the series $\sum\limits_{i=1}^{\infty} J(e_i)(E)$ is a weakly unconditionally Cauchy series in X, and as X does not contain copies of c_0, by virtue of a classical result of Bessaga and Pelczynski, the series is unconditionally convergent in X, and then $\sum\limits_{i=1}^{\infty} \xi_i J(e_i)(E) \in X$. For every $u = (\xi_i) \in \ell_\infty$, we define the measure

$$F_u\colon\ \Sigma \to X\colon\ F_u(E) = \sum_{i=1}^{\infty} \xi_i J(e_i)(E)\ \forall E \in \Sigma.$$

Let $(F_{u_n})_{n\in\mathbb{N}}$ be a sequence in $\mathrm{cabv}_\Phi(\mu, X)$ with $F_{u_n}(E) = \sum\limits_{i=1}^{n} \xi_i J(e_i)(E)$, $\forall E \in \Sigma$. It is clear that $F_u(E) = \lim\limits_n F_{u_n}(E)\ \forall E \in \Sigma$, and then by the Vitali-Hahn-Saks theorem F_u is μ-continuous and countably additive. Moreover, $NV_\Phi(F_{u_n}) \leqslant \|J\|\|u\|$, $\forall n \in \mathbb{N}$. Given a partition $\mathcal{P}$ of Ω by elements of Σ and a $\varepsilon > 0$, there is $n_{\mathcal{P},\varepsilon} \in \mathbb{N}$ such that

$$\sum_{E\in\mathcal{P}} \Phi\Big(\frac{\|F_u(E)\|}{\|J\|\|u\|\mu(E)}\Big)\mu(E) \leqslant \sum_{E\in\mathcal{P}} \Phi\left(\frac{\sum\limits_{i=1}^{n_{\mathcal{P},\varepsilon}} \|F_{u_n}(E)\| + \varepsilon}{\|J\|\|u\|\mu(E)}\right)\mu(E).$$

Thus $I_\Phi\big(\frac{F_u}{\|J\|\|u\|}\big) \leqslant \sup\limits_{n\in\mathbb{N}} I_\Phi\big(\frac{F_{u_n}}{\|J\|\|u\|}\big) \leqslant 1$, and $F_u \in \mathrm{cabv}_\Phi(\mu, X)$ with $NV_\Phi(F_u) \leqslant \|J\|\|u\|$. This implies that $G\colon\ \ell_\infty \to \mathrm{cabv}_\Phi(\mu, X)$ such that $G(u) = F_u$ is a well defined, linear and continuous map. As $G\big|c_0 = J$ and $\inf\limits_{n\in\mathbb{N}} NV_\Phi G(e_n) > 0$, we can use Lemma 1 to conclude that $\mathrm{cabv}_\Phi(\mu, X)$ contains a subspace isomorphic to ℓ_∞.

□

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