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A POTENTIAL THEORETIC INEQUALITY

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Dedicated to Filippo Chiarenza

Abstract. In this paper is proved a weighted inequality for Riesz potential similar to the classical one by D. Adams. Here the gain of integrability is not always algebraic, as in the classical case, but depends on the growth properties of a certain function measuring some local potential of the weight.

Keywords: Adams trace inequality, Stummel class, Morrey Spaces

MSC 2000: 35B45, 35B65

0. Introduction

In his paper [1] D. Adams proved the following inequality

$$\left(\int_{\mathbb{R}^n} |u(x)|^q V(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant C(p,\lambda,n)\|V\|^{\frac{1}{q}}_{L^{1,\lambda}(\mathbb{R}^n)}\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{0.1}$$
$$\forall u \in C_0^\infty(\mathbb{R}^n), \quad q = p\frac{\lambda}{n-p}, \quad 1<p<n.$$

Here V is a non negative function in the Morrey space $L^{1,\lambda}(\mathbb{R}^n)$, $\lambda > n-p$, where

$$L^{1,\lambda}(\mathbb{R}^n) = \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)\colon \ \|f\|_{L^{1,\lambda}(\mathbb{R}^n)} \equiv \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \frac{1}{r^\lambda}\int_{|x-y|<r} |f(y)|\,\mathrm{d}y < +\infty \Big\}, \quad 0<\lambda<n. \tag{0.2}$$

Our purpose in this note is to establish an imbedding similar to (0.1), assuming more general hypotheses on the function V. We need to introduce the S_p class, $1 < p < n$,

$$S_p = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \colon \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-p}} \,\mathrm{d}y = \eta(r) \searrow 0 \text{ for } r \searrow 0 \right\}.$$

Sometimes we will call $\eta(r)$ the Stummel modulus of f. We note that, for $p = 2$, S_p is the famous Stummel-Kato class (see e.g. [3]). It is an easy task to see that $L^{1,\lambda}$ is contained in S_p if $\lambda > n - p$, while, in the case $\eta(r) \sim r^\alpha$ the belonging of f to S_p is equivalent to $f \in L^{1,n-p+\alpha}$ (see Lemma 1.1). For this reason, if $\eta(r)$ behaves like a power, one may then see the *algebraic* improvement of integrability q/p of u in Adams' inequality (0.1) either in terms of the belonging of V to $L^{1,\lambda}$ or in terms of the properties of the local Riesz potential of V given by the function $\eta(r)$.

In this paper we study the case of $\eta(r)$ not being, in general, a power. We show (Proposition 2.1) that if $V \in S_p$ has some kind of Dini continuity, i.e. its Stummel modulus η has a small power satisfying an integral condition at zero, then still there is an improvement (generally not algebraic) in the integrability of u.

This result is achieved by noting that, under conditions on η similar to the above mentioned, it is possible to get control of a "better" local potential than merely that of order p.

This is expressed by the belonging of V to $S_{p,\varphi}$ (see, in the following, Definition 1.1). In turn this improvement in the potential of V which may be controlled implies (Theorem 2.1) a "better" integrability of some other potential of an arbitrary function f in C_0^∞ with respect to the weight V.

This, we feel, is the heart of our proof. From this easily one deduces the *non algebraic* form of Adams' imbedding we discussed above.

About the technique we observe that it is fully elementary somehow following Hedberg's proof of Sobolev inequality [2] and especially some ideas in Welland [4].

We would like to thank Professor M. Frasca for many conversations on the subject of this note.

1. Function spaces and preliminary results

We begin establishing the relation between the space S_p and $L^{1,\lambda}$.

Lemma 1.1. *If V belongs to $L^{1,\lambda}(\mathbb{R}^n)$, $n - p < \lambda < n$, then V belongs to S_p, and*

$$\int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-p}} \,\mathrm{d}y \leqslant C(n,p,\lambda) r^{\lambda-n+p} \|V\|_{L^{1,\lambda}(\mathbb{R}^n)}.$$

Conversely, if V belongs to S_p and $\eta(r) \sim r^\alpha$ then V belongs to $L^{1,n-p+\alpha}(\mathbb{R}^n)$.

Proof. About the first part, we have

$$\begin{aligned}
\int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-p}}\,\mathrm{d}y &= \sum_{k=0}^{+\infty} \int_{\frac{r}{2^{k+1}}\leqslant|x-y|<\frac{r}{2^k}} \frac{|V(y)|}{|x-y|^{n-p}}\,\mathrm{d}y \\
&\leqslant \sum_{k=0}^{+\infty}\Big(\frac{2^{k+1}}{r}\Big)^{n-p}\int_{|x-y|<\frac{r}{2^k}} |V(y)|\,\mathrm{d}y \\
&\leqslant \|V\|_{L^{1,\lambda}(\mathbb{R}^n)} 2^{n-p}\sum_{k=0}^{+\infty}\Big(\frac{r}{2^k}\Big)^{\lambda}\Big(\frac{2^k}{r}\Big)^{n-p} \leqslant C(n,\lambda)\|V\|_{L^{1,\lambda}(\mathbb{R}^n)} r^{\lambda-(n-p)}.
\end{aligned}$$

The second part is obvious, indeed

$$\int_{|x-y|<r} |V(y)|\,\mathrm{d}y \leqslant r^{n-p}\int_{|x-y|<r}\frac{|V(y)|}{|x-y|^{n-p}}\,\mathrm{d}y \leqslant Cr^{n-p+\alpha}.$$

□

Lemma 1.2. *Let $V \in S_p$. Then there exists a positive constant $C_d = C_d\,(n)$ such that*

$$\eta(r) \leqslant C_d\eta\Big(\frac{r}{2}\Big), \quad r>0.$$

Proof. Let $x_0 \in \mathbb{R}^n$, $r>0$ and $B(x_0,r)$ the ball centered in x_0 with radius r. Then there exist $m = m(n) \in \mathbb{N}$, $x_1,\dots,x_{m(n)} \in B(x_0,r)$ such that

$$B(x_0,r) \subseteq \bigcup_{j=1}^{m} B\Big(x_j,\frac{r}{2}\Big).$$

We have

$$\int_{|x_0-y|<r}\frac{|V(y)|}{|x_0-y|^{n-p}}\,\mathrm{d}y \leqslant \sum_{j=1}^{m}\int_{|x_j-y|<\frac{r}{2}}\frac{|V(y)|}{|x_0-y|^{n-p}}\,\mathrm{d}y = \sum_{j=1}^{m} I_j$$

and

$$\begin{aligned}
I_j &= \int_{|x_0-y|\geqslant|x_j-y|,|x_j-y|<\frac{r}{2}}\frac{|V(y)|}{|x_0-y|^{n-p}}\,\mathrm{d}y \\
&\quad+\int_{|x_0-y|<|x_j-y|<\frac{r}{2}}\frac{|V(y)|}{|x_0-y|^{n-p}}\,dy = A_j + B_j.
\end{aligned}$$

Since

$$A_j \leqslant \int_{|x_j-y|<\frac{r}{2}} \frac{\bigl|V(y)\bigr|}{|x_j-y|^{n-p}}\,dy \leqslant \eta\Bigl(\frac{r}{2}\Bigr)$$

$$B_j \leqslant \int_{|x_0-y|<\frac{r}{2}} \frac{\bigl|V(y)\bigr|}{|x_0-y|^{n-p}}\,dy \leqslant \eta\Bigl(\frac{r}{2}\Bigr)$$

we obtain the conclusion. □

We need one more definition.

Definition 1.1. Let $1 < p < n$, $\varphi\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ be a non decreasing continuous function such that $\lim\limits_{t\to 0}\varphi(t) = 0$. We say that $V\colon\ \mathbb{R}^n \to \mathbb{R}$ belongs to the class $S_{p,\varphi}$ iff there exists a non decreasing function $\xi\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ with $\lim\limits_{r\to 0}\xi(r) = 0$ such that

$$\sup_{x\in\mathbb{R}^n}\int_{|x-y|<r} \frac{\bigl|V(y)\bigr|}{|x-y|^{n-p}\varphi\bigl(|x-y|\bigr)}\,\mathrm{d}y \leqslant \xi(r). \tag{1.1}$$

The following Lemma gives a sufficient condition in order that a function $V \in S_p$ belongs to an appropriate $S_{p,\varphi}$ class.

Lemma 1.3. *Let $V \in S_p$ such that $\exists\vartheta \in\ ]0,1[\ \colon \int_0^1 t^{-1}\eta^{1-\vartheta}(t)\,\mathrm{d}t < +\infty$, where $\eta(t)$ is the Stummel modulus of V.*

Then $V \in S_{p,\eta^\vartheta}$ and

$$\int_{|x-y|<r} \frac{\bigl|V(y)\bigr|}{|x-y|^{n-p}\eta^{\vartheta}\bigl(|x-y|\bigr)}\,\mathrm{d}y \leqslant \mu(r), \tag{1.2}$$

where

$$\mu(r) = \frac{2}{C}\int_0^r t^{-1}\eta^{1-\vartheta}(t)\,\mathrm{d}t.$$

P r o o f. Using Lemma 2.1 we have

$$\begin{aligned}\int_{|x-y|<r} \frac{\bigl|V(y)\bigr|}{|x-y|^{n-p}\eta^{\vartheta}\bigl(|x-y|\bigr)}\,\mathrm{d}y &= \sum_{k=0}^{+\infty}\int_{\frac{r}{2^{k+1}}\leqslant|x-y|<\frac{r}{2^k}} \frac{\bigl|V(y)\bigr|}{|x-y|^{n-p}\eta^{\vartheta}\bigl(|x-y|\bigr)}\,\mathrm{d}y\\ &\leqslant \sum_{k=0}^{+\infty}\eta\Bigl(\frac{r}{2^k}\Bigr)\Bigl[\eta\Bigl(\frac{r}{2^{k+1}}\Bigr)\Bigr]^{-\vartheta} \leqslant C^{-\vartheta}\sum_{k=0}^{+\infty}\Bigl[\eta\Bigl(\frac{r}{2^k}\Bigr)\Bigr]^{1-\vartheta}.\end{aligned}$$

The last series converges observing that

$$\int_0^r t^{-1}\eta^{1-\vartheta}(t)\,\mathrm{d}t = \sum_{k=0}^{+\infty}\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} t^{-1}\eta^{1-\vartheta}(t)\,\mathrm{d}t$$
$$\geqslant \sum_{k=0}^{+\infty}\Big[\eta\Big(\frac{r}{2^{k+1}}\Big)\Big]^{1-\vartheta}\frac{2^k}{r}\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}}\mathrm{d}t \geqslant \frac{1}{2}C^{1-\vartheta}\sum_{k=0}^{+\infty}\Big[\eta\Big(\frac{r}{2^k}\Big)\Big]^{1-\vartheta}.$$

□

2. Main result

Let f and h be measureble functions such that $f \in L^1_{loc}(\mathbb{R}^n)$ and $h \geqslant 0$, we set

$$I_p(f)(x) = \int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-p}}\,\mathrm{d}y,$$
$$I_{p,h}(f)(x) = \int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-p}h(|x-y|)}\,\mathrm{d}y.$$

Theorem 2.1. *Let $V \in S_{p,\varphi}$ with $\varphi(t)$ and $\xi(t)$ as in Definition 1.1. Then, for any $\sigma \in]0,1[$, there exists a non decreasing, positive function $G(t)$ such that*

$$\int_{B_r} G\left(\frac{I_{p,\varphi^\sigma}(f^p)}{\|f\|_p^p}\right)V(x)\,\mathrm{d}x \leqslant \xi(r) \tag{2.1}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where B_r is a ball with radius r containing the support of f. Also

$$\lim_{t\to\infty}\frac{G(t)}{t} = +\infty. \tag{2.2}$$

Proof. Let $\varepsilon > 0$ and $0 < \sigma < 1$, then

$$I_{p,\varphi^\sigma}(f^p)(x) = \int_{|x-y|\leqslant\varepsilon}\frac{|f(y)|^p\varphi(|x-y|)}{|x-y|^{n-p}\varphi^\sigma(|x-y|)\varphi(|x-y|)}\,\mathrm{d}y + \int_{|x-y|>\varepsilon}\frac{|f(y)|^p}{|x-y|^{n-p}\varphi^\sigma(|x-y|)}\,\mathrm{d}y \leqslant \varphi^{1-\sigma}(\varepsilon)I_{p,\varphi}(f^p) + \frac{1}{\varepsilon^{n-p}\varphi^\sigma(\varepsilon)}\|f\|_p^p. \tag{2.3}$$

Letting $\varepsilon^{n-p}\varphi(\varepsilon) = \Phi(\varepsilon)$, we choose

$$\varepsilon = \Phi^{-1}\left(\frac{\|f\|_p^p}{I_{p,\varphi}(f^p)}\right),$$

a choice which makes the two terms on the right hand side of (2.3) equal. Here (as we will do in the following) we denote the inverse of a function f with f^{-1}.

From (2.3), we have

$$\frac{I_{p,\varphi^\sigma}(f^p)}{\|f\|_p^p} \leqslant \frac{2}{\left[\Phi^{-1}\left(\frac{\|f\|_p^p}{I_{p,\varphi}(f^p)}\right)\right]^{n-p} \varphi^\sigma \left[\Phi^{-1}\left(\frac{\|f\|_p^p}{I_{p,\varphi}(f^p)}\right)\right]}.$$

If

$$\psi(t) = \frac{2}{\left[\Phi^{-1}\left(\frac{1}{t}\right)\right]^{n-p} \varphi^\sigma \left[\Phi^{-1}\left(\frac{1}{t}\right)\right]}$$

and

$$G(t) = \psi^{-1}(t),$$

we obtain

$$G\left(\frac{I_{p,\varphi^\sigma}(f^p)}{\|f\|_p^p}\right) \leqslant \frac{I_{p,\varphi}(f^p)}{\|f\|_p^p}.$$

Finally, using Fubini's theorem

$$\begin{aligned}
\int_{B_r} G\left(\frac{I_{p,\varphi^\sigma}(f^p)(x)}{\|f\|_p^p}\right)\big|V(x)\big|\,\mathrm{d}x &\leqslant \frac{1}{\|f\|_p^p}\int_{B_r} I_{p,\varphi}\big(f^p\big)(x)\big|V(x)\big|\,\mathrm{d}x \\
&= \frac{1}{\|f\|_p^p}\int_{B_r}\left(\int_{\mathbb{R}^n}\frac{\big|f(y)\big|^p}{|x-y|^{n-p}\varphi\big(|x-y|\big)}\,\mathrm{d}y\right)\big|V(x)\big|\,\mathrm{d}x \\
= \frac{1}{\|f\|_p^p}\int_{\mathbb{R}^n}&\left(\int_{B_r}\frac{\big|V(x)\big|}{|x-y|^{n-p}\varphi\big(|x-y|\big)}\,\mathrm{d}x\right)\big|f(y)\big|^p\,\mathrm{d}y \leqslant \xi(r).
\end{aligned}$$

We now prove (2.2).

(2.2) is easily seen to be equivalent to

$$\lim_{s\to 0}\frac{\left[\Phi^{-1}(s)\right]^{n-p}\varphi^\sigma\left[\Phi^{-1}(s)\right]}{s} = +\infty. \tag{2.4}$$

Letting $H(t) = t^{n-p}\varphi^\sigma(t)$, (2.4) can be rewritten as

$$\lim_{s\to 0}\frac{H\left(\Phi^{-1}(s)\right)}{s} = +\infty. \tag{2.5}$$

Since $\lim\limits_{s\to 0}\frac{\Phi(s)}{H(s)} = \lim\limits_{s\to 0}\varphi^{1-\sigma}(s) = 0$ we obtain (2.5). $\square$

Lemma 2.1. *Let* $h\colon\,]0,+\infty[\,\to\,]0,+\infty[$ *such that* $\int_0^1 \frac{[h(t)]^{p'/p}}{t}\,\mathrm{d}t < +\infty$ $\Bigl(p'\colon \frac{1}{p}+\frac{1}{p'}=1\Bigr)$. *Then*

$$I_1(f) \leqslant C(n,p,\operatorname{diam}(\operatorname{spt} f),h)\,[I_{p,h}(f^p)]^{\frac{1}{p}}$$

for all $f\in C_0^\infty(\mathbb{R}^n)$.

P r o o f. Using Hölder inequality, we have

$$\begin{aligned} I_1(f) &= \int_{\mathbb{R}^n} \frac{|f(y)|h^{\frac{1}{p}}(|x-y|)}{|x-y|^{n-1}h^{\frac{1}{p}}(|x-y|)}\,\mathrm{d}y \\ &\leqslant [I_{p,h}(f^p)]^{\frac{1}{p}}\Bigl(\int_{B_r}\frac{h^{\frac{p'}{p}}(|x-y|)}{|x-y|^n}\,\mathrm{d}y\Bigr)^{\frac{1}{p'}} \end{aligned}$$

where $B_r\supseteq \operatorname{spt} f$. □

Corollary 2.1. *In the same hypotheses of Theorem* 2.1 *and assuming also* $\int_0^1 \frac{[\varphi(t)]^{\frac{\sigma p'}{p}}}{t}\,\mathrm{d}t<+\infty$, *we have*

$$\int_{B_r} G\Bigl(\frac{|u|^p}{\|\nabla u\|_p^p}\Bigr)V(x)\,\mathrm{d}x \leqslant C(n,p,\operatorname{diam}(\operatorname{spt} u),\varphi)\xi(r), \tag{2.6}$$

for all $u\in C_0^\infty(\mathbb{R}^n)$, *where* $B_r\supseteq\operatorname{spt} u$ *and* G *is as in Theorem* 2.1.

P r o o f. (2.6) follows by Lemma 2.1, Theorem 2.1 and the well-known inequality

$$|u|\leqslant C(n)I_1\bigl(|\nabla u|\bigr).$$

□

Remark 2.1. The previous results are also true if we substitute the function $\varphi^\sigma(t)$ with a more general non decreasing function $\delta\colon\,]0,+\infty[\,\to\,]0,+\infty[$ such that

$$\lim_{t\to 0}\delta(t)=0,\quad \lim_{t\to 0}\frac{\varphi(t)}{\delta(t)}=0,$$

$\frac{\varphi(t)}{\delta(t)}$ is non decreasing, where $\varphi(t)$ is as in Definition 1.1.

We now collect the results of the previous Theorem 2.1, Lemma 2.1 and Corollary 2.1 in the following proposition, obtaining a non algebraic form of Adams' inequality.

Proposition 2.1. *Let $V \in S_p$, $V \geqslant 0$, $\sigma \in \,]0,1[\,$, $\vartheta = \frac{1}{\frac{\sigma p'}{p}+1}$ and assume*

$$\int_0 \frac{[\eta(t)]^{1-\vartheta}}{t}\,\mathrm{d}t < +\infty. \tag{2.7}$$

Then

$$V \in S_{p,\eta^\vartheta} \tag{2.8}$$

and there exists a non decreasing positive function $G(t)$ such that

$$\int_{B_r} G\left(\frac{|u|^p}{\|\nabla u\|_p^p}\right) V(x)\,\mathrm{d}x \leqslant C(n,p,\eta)\mu(r) \tag{2.9}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$, where $B_r \supseteq \operatorname{spt} u$ and

$$\mu(r) = \frac{2}{C}\int_0^r t^{-1}\eta^{1-\vartheta}(t)\,\mathrm{d}t. \tag{2.10}$$

Moreover, if V has compact support, there exists a positive constant M such that

$$G(t) \geqslant Mt \qquad \forall t > 0. \tag{2.11}$$

Proof. (2.8) has been already proved in Lemma 1.3.

(2.9) follows by Corollary 2.1 assuming $\varphi(t) = \eta^\vartheta(t)$ and σ an arbitrary number in $]0,1[\,$.

In (2.9) $G(t)$ is, as in Theorem 2.1, the inverse of the function

$$\psi(t) = \frac{2}{\left[\tilde{\Phi}^{-1}\left(\frac{1}{t}\right)\right]^{n-p}\eta^{\sigma\vartheta}\left[\tilde{\Phi}^{-1}\left(\frac{1}{t}\right)\right]},$$

where

$$\tilde{\Phi}(t) = t^{n-p}\eta^\vartheta(t).$$

Finally to prove (2.11) we equivalently prove

$$\psi(Mt) \leqslant t \qquad \forall t > 0. \tag{2.12}$$

Let B_{r_0} be a ball containing the support of V, then it is easily seen that $\eta^\vartheta = C =$ const. for $r > 3r_0$.

Then there exists $\delta_1 > 0$ such that

$$\frac{\psi(t)}{t} = 2C^{1-\sigma} = L$$

for $0 < t < \delta_1$.

Furthermore by (1.1) we have that there exists $\delta_2 > 0$ such that

$$\frac{\psi(t)}{t} < L$$

for $t > \delta_2$.

Finally, observing that $\frac{\psi(t)}{t}$ is bounded in $[\delta_1, \delta_2]$ it follows that there exists a positive constant M such that

$$\frac{\psi(t)}{t} \leqslant \frac{1}{M}, \qquad \forall t > 0.$$

Setting $\frac{t}{M} = s$ we obtain (2.12). □

Example. We give now an example of a function f which satisfies the assumptions of Proposition 2.1, but such that $f \notin L^{1,\lambda}$ for $\lambda > n-2$.

Let $B = B_\delta(0)$ the ball centered in 0 and radius $\delta = e^{-3}$ and

$$f(x) = \frac{1}{|x|^2 \big|\log|x|\big|^6} \chi_B(x),$$

where $\chi_B(y)$ is the characteristic function of B, we have that the function

$$\eta(r) = \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{f(y)}{|x-y|^{n-2}} \,\mathrm{d}y,$$

is such that

(i) $\lim\limits_{r\to 0} \eta(r) = 0$

(ii) $\int_0^r \frac{\eta^{1/4}(\varrho)}{\varrho} \,\mathrm{d}\varrho < +\infty$.

Proof. For $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\begin{aligned}
&\int_{|x-y|<r} \frac{1}{|y|^2|x-y|^{n-2}\big|\log|y|\big|^6} \chi_B(y) \,\mathrm{d}y \\
&\qquad = \int_{|y|<|x-y|<r} \frac{1}{|y|^2|x-y|^{n-2}\big|\log|y|\big|^6} \chi_B(y) \,\mathrm{d}y \\
&\qquad\quad + \int_{\substack{\{|x-y|<r\}\cap\\ \{|x-y|<|y|<\delta\}}} \frac{1}{|y|^2|x-y|^{n-2}\big|\log|y|\big|^6} \chi_B(y) \,\mathrm{d}y = I_1 + I_2.
\end{aligned}$$

Let $\sigma = \min(r, \delta)$ we have for I_1

$$\begin{aligned} I_1 &= \int_{|y|<|x-y|<r} \frac{1}{|y|^2|x-y|^{n-2}\big|\log|y|\big|^6}\,\chi_B(y)\,\mathrm{d}y \\ &\leqslant \int_{|y|<r} \frac{1}{|y|^n\big|\log|y|\big|^6}\chi_B(y)\,\mathrm{d}y = C(n)\frac{1}{(-\log\sigma)^5}, \end{aligned}$$

and for I_2 [1]

$$\begin{aligned} I_2 &= \int_{\substack{\{|x-y|<r\}\cap \\ \cap\{|x-y|<|y|<\delta\}}} \frac{1}{|y|^2|x-y|^{n-2}\big|\log|y|\big|^6}\,\chi_B(y)\,\mathrm{d}y \\ &= \int_{\substack{\{|x-y|<r\}\cap \\ \{|x-y|<|y|<\delta\}}} \frac{1}{|y|^2|x-y|^{n-2}\big|\log|y|\big|^6}\,\mathrm{d}y \\ &\leqslant \int_{\{|z|<r\}\cap\{|z|<\delta\}} \frac{dz}{|z|^n(-\log|z|)^6} = C(n)\frac{1}{(-\log\sigma)^5}. \end{aligned}$$

Then we have

$$\eta(r) \leqslant L(r) \equiv 2C(n)\frac{1}{(-\log\sigma)^5}.$$

Because

$$\lim_{r\to 0} L(r) = 0 \quad \text{and} \quad \eta(r) \leqslant L(r),$$

we obtain (i).

To prove (ii) we only consider $r < \delta$.

We have

$$\begin{aligned} \int_0^r \frac{\eta^{\frac{1}{4}}(\varrho)}{\varrho}\,\mathrm{d}\varrho \leqslant \int_0^r \frac{L^{\frac{1}{4}}(\varrho)}{\varrho}\,\mathrm{d}\varrho &= (2C(n))^{\frac{1}{4}} \int_0^r \frac{(-\log\varrho)^{-\frac{5}{4}}}{\varrho}\,\mathrm{d}\varrho \\ &= (2C(n))^{\frac{1}{4}}\,\frac{4}{(-\log r)^{\frac{1}{4}}} < +\infty. \end{aligned}$$

We now prove that the function $f \notin L^{1,\lambda}$ for $\lambda > n-2$.

Indeed letting, for $\varepsilon > 0$, $\lambda = n-2+\varepsilon$ we have

$$\begin{aligned} \frac{1}{r^{n-2+\varepsilon}} \int_{B_r(0)} \frac{\chi_B(y)}{|y|^2\big|\log|y|\big|^6}\,\mathrm{d}y &= \frac{C(n)}{r^{n-2+\varepsilon}} \int_0^r \frac{\varrho^{n-1}}{\varrho^2(-\log\varrho)^6}\,\mathrm{d}\varrho \\ &> \frac{C(n)}{2^{n-2}r^{\varepsilon}} \int_{\frac{r}{2}}^r \frac{\mathrm{d}\varrho}{(-\log\varrho)^6\varrho} = \frac{1}{5}\,\frac{C(n)}{2^{n-2}r^{\varepsilon}}\left[\frac{1}{(-\log r)^5} - \frac{1}{(-\log(\frac{1}{2}r))^5}\right], \end{aligned}$$

and the last quantity is unbounded. $\square$

[1] The function $\frac{1}{t^2(-\log t)^6}$ is decreasing in $]0, e^{-3}[$.

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